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(*) $[-1,1]$, $\quad Q_3(x)=c_0+c_1x+c_2x^2+c_3x^3$

$f(x)=x^2\ln|x|$, $\quad x=0: f(0)=0$.

$c_0, c_1, c_2, c_3, L = ?$ $\quad\Rightarrow\quad$ 5 tačaka ČA

f parna: $\quad f(-x)=(-x)^2\cdot\ln|-x|=f(x)$

$\Rightarrow Q$ parno: $\quad Q(x)=c_0+c_2x^2$

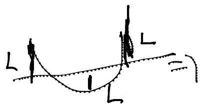
Smena: $t=x^2$, $x=\sqrt{t}$, $x\in[-1,1]$:

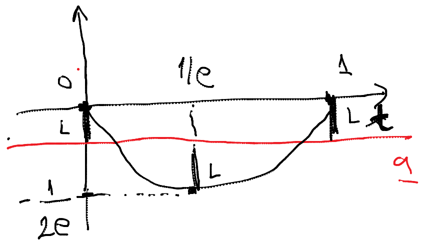
$$\begin{cases} q(t)=c_0+c_2t \\ F(t)=t\cdot\ln\sqrt{t}=t\cdot\ln t^{1/2}=\frac{1}{2}t\ln t \\ t\in[0,1] \end{cases}$$

F konveksna ili konkavna?

$F'(t)=\frac{1}{2}\left(\ln t+t\cdot\frac{1}{t}\right)=\frac{1}{2}(\ln t+1)$

$F''(t)=\frac{1}{2}\cdot\frac{1}{t}=\frac{1}{2t}>0 \quad (t\in[0,1])$

$\Rightarrow F$ konveksna  $\Rightarrow$ a i b su tačke ČA



$F(0)=0$

$F(1)=1/2\cdot 1\cdot\ln 1=0$

$F'(t)=\frac{1}{2}(\ln t+1)=0$

$\ln t=-1 \Rightarrow t=e^{-1}$

$F\left(\frac{1}{e}\right)=\frac{1}{2}\cdot\frac{1}{e}\cdot\ln\frac{1}{e}=-\frac{1}{2e}$

$q=-\frac{1}{4e}=c_0$, $\quad L=\frac{1}{4e}$

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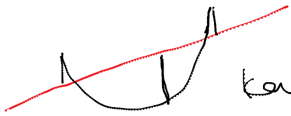
(*) $[0,1]$, $Q_1(x) = c_0 + c_1 x$

$f(x) = \sqrt{1+x^2}$. Računati na 4 decimale.

$f(x)$ jeste parna, ali $[0,1]$ nije simetričan ;

$c_0, c_1, L \overset{?}{=} \Rightarrow$ 3 tčA.
x

$f'(x) =$
$f''(x) = \dfrac{1}{(1+x^2)\sqrt{1+x^2}} > 0 \Rightarrow$  konveksna

$\Rightarrow a = 0$ i $b = 1$ TČA
$d \overset{?}{=}$ treća TČA

$$f(x_i) - Q(x_i) = (-1)^i \alpha L$$

4 jed. 4 nepoz. c_0, c_1, L, d

$$\left.\begin{array}{l} f(a) - Q(a) = \alpha L \\ f(d) - Q(d) = -\alpha L \\ f(b) - Q(b) = \alpha L \\ \left(f(x) - Q(x)\right)'_{x=d} = 0 \end{array}\right\}$$

$a = 0$, $b = 1$:

$$\left.\begin{array}{l} \sqrt{1+0^2} - c_0 = \alpha L \\ \sqrt{1+d^2} - c_0 - c_1 \cdot d = -\alpha L \\ \sqrt{1+1^2} - c_0 - c_1 \cdot 1 = \alpha L \\ \left(\sqrt{1+x^2} - c_0 - c_1 x\right)'_{x=d} = 0 \end{array}\right\}$$

$$\left.\begin{array}{l} 1 - c_0 = \alpha L \\ \sqrt{1+d^2} - c_0 - c_1 d = -\alpha L \\ \sqrt{2} - c_0 - c_1 = \alpha L \\ \dfrac{2d}{2\sqrt{1+d^2}} - c_1 = 0 \end{array}\right\}$$ nelinearan

$1 - c_0 = \alpha L$
$\sqrt{2} - c_0 - c_1 = \alpha L$ $\ominus$

$1 - \sqrt{2} + c_1 = 0$

$\sqrt{2} - c_0 - c_1 = c_2$ $\ominus$

$1 - \sqrt{2} + c_1 = 0$

$\Rightarrow \quad c_1 = \sqrt{2} - 1 = \boxed{0{,}4142 = c_1}$

$$\frac{\cancel{2}d}{\cancel{2}\sqrt{1+d^2}} - a = 0 \quad , \quad a = 0.4142$$

$$\frac{\cancel{x}d}{\cancel{x}\sqrt{1+d^2}} - 0.4142 = 0$$ (nelin. na sa 1 prom.) UNM

$$d_{(n)} = 0.4142 \cdot \sqrt{1+d^2_{(n-1)}} \quad , \quad d_0 = 0$$

$$d_1 = 0.4142 \cdot \sqrt{1+0^2} = 0.4142$$

$$d_2 = 0.4142 \cdot \sqrt{1+0.4142^2} = 0.4483$$

$$d_3 = 0.4539$$

$$d_4 = 0.4549$$

$$d_5 = 0.4550$$

$$d_6 = 0.4551$$

$$d_7 = 0.4551 = d_6 \quad \Rightarrow \boxed{d = 0.4551}$$

Sad linearno $\Rightarrow$

$$\boxed{c_0 = 0.9551} \quad , \quad \alpha L = 0.0449$$

$$\alpha = 1 \, , \, \boxed{L = 0.0449}$$

$$Q(x) = q_1 + q_2 x + q_3 x^2 + \dots + q_{n-1} x^{n-2} \qquad n = \text{length}(\text{alter})$$

$$f(x_i) - Q(x_i) = (-1)^i \alpha L \quad , \quad f(x_i) = Q(x_i) + (-1)^i \alpha L$$

$$\underbrace{\begin{pmatrix} 1 & x_1^{n-2} & \dots & x_1 & 1 \\ -1 & x_2^{n-2} & \dots & x_2 & 1 \\ +1 & x_3^{n-2} & \dots & x_3 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ (-1)^i & x_n^{n-2} & \dots & x_n & 1 \end{pmatrix}}_{A} \begin{pmatrix} \alpha L \\ q_{n-1} \\ q_{n-2} \\ \vdots \\ q_1 \end{pmatrix} = \underbrace{\begin{pmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}}_{b}$$

REMEZOV ALG.

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$f(x) \rightsquigarrow Q_n(x)$, $[a,b]$, n+2 TČA

$\boxed{K0}$ $x_0 < x_1 < \dots < x_{n+1} \in [a,b]$ init()

$\boxed{K1}$ $b_0 + b_1 x_i + b_2 x_i^2 + \dots + b_n x_i^n + (-1)^i E = f(x_i)$

$i = 0, \dots, n+1$

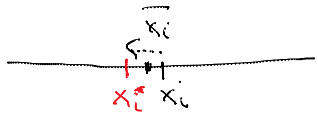
n+2 jne lin. po b_i, E

⇒ riješimo

$\boxed{K2}$ $b_0, b_1, \dots, b_n, E$

$p(x) = b_0 + b_1 x + \dots + b_n x^n$

$\boxed{K3}$ $p(x) - f(x) = ?$ x_i



$p(\bar{x}_i) - f(\bar{x}_i) = +E$

↳ tražimo max $\bar{x}_i$ u okolini x_i

$p(\bar{x}_i) - f(\bar{x}_i) = -E$

↳ min $\bar{x}_i$

$p - f$ n+1 nula

$[a,b]$ n+2 intervala

$[a, x_0, x_1, \dots, b]$

$\boxed{K4}$ n+2 takve $\bar{x}_i$, $i = 0, \dots, n+1$

$z_i = p(\bar{x}_i) - f(\bar{x}_i)$ ⇝ L

TČA 0,5

0,4 | 0,6

Krit. zaus: $\max_i |z_i| - \min_i |z_i| < tol$ → kraj

$> tol$ → $\boxed{K1}$

0.5

0,49 | 0,51

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(*) $f(x)=\sqrt{1+x^2}$, $[0,1]$

$Q_1(x)=c_0+c_1x$, 4 decimale

$\boxed{K0}$ $n=1 \Rightarrow 3$ TČA : $x_0=0$, $x_1=\frac{1}{2}$, $x_2=1$

$\boxed{K1}$ $b_0+b_1\cdot x_i+(-1)^i E=f(x_i)$

$$\left.\begin{array}{l} b_0+b_1\cdot x_0+E=f(x_0) \\ b_0+b_1\, x_1-E=f(x_1) \\ b_0+b_1\, x_2+E=f(x_2) \end{array}\right\} \left.\begin{array}{l} b_0+b_1\cdot 0+E=\sqrt{1+0^2} \\ b_0+b_1\cdot\frac{1}{2}-E=\sqrt{1+\frac{1}{2}^2} \\ b_0+b_1\cdot 1+E=\sqrt{2} \end{array}\right\} \begin{array}{l} b_0=0.9554 \\ b_1=0.4142 \\ E=0.0446 \end{array}$$

$\boxed{K2}$ $p(x)=b_0+b_1x=0.9554+0.4142x$

$\boxed{K3}$ $p(x_i)-f(x_i)=\pm E$?

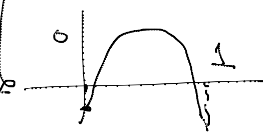
x_0: $p(x_0)-f(x_0)=0{,}9554-1=-0.0446=-E$

$\Rightarrow$ min $p(x)-f(x)$ u okolini $x_0=0$

$$F(x)=p(x)-f(x)=0{,}9554+0.4142x-\sqrt{1+x^2}$$

$$F'(x)=0.4142-\frac{2x}{2\sqrt{1+x^2}}$$

$$F''(x)=\frac{-1}{(1+x^2)\sqrt{1+x^2}}<0$$



min se dostiže na rubu $\overline{x_0=0}$

x_1: $p(x_1)-f(x_1)\overset{?}{=}\pm E$

$p(x_1)-f(x_1)=+E \Rightarrow$ max $F(x)$ u okolini $\frac{1}{2}$

$$F'(x)=0=\underbrace{0.4142-\frac{x}{\sqrt{1+x^2}}}$$ nelin. jna sa 1 nep. (UNM)

$\Rightarrow \overline{x_1=0.4551}$

x_2: $p(x_2)-f(x_2)\overset{?}{=}\pm E$

$p(x_2)-f(x_2)=-E \Rightarrow$ min $F(x)$ u okolini $x_2=1$

$\Rightarrow \overline{x_2=1}$

$\boxed{K4}$ $z_0 = p(\bar{x}_0) - f(\bar{x}_0) = -0.0446$

$z_1 = p(\bar{x}_1) - f(\bar{x}_1) = 0.0452$

$z_2 = p(\bar{x}_2) - f(\bar{x}_2) = -0.0446$

$\max|z_i| = 0.0452$ $\quad$ $\max|z_i| - \min|z_i| < 10^{-4}$

$\min|z_i| = 0.0446$ $\quad$ $0{,}0006 > 10^{-4}$

$\Rightarrow x_0 = \bar{x}_0, \; x_1 = \bar{x}_1, \; x_2 = \bar{x}_2$ na $\boxed{K1}$

II iteracija

$\boxed{K1}$ ~~$b_0 \neq b_1$~~ sistem $\Rightarrow b_0 = 0.9551, \; b_1 = 0.4142$

$E = 0.0449$

$\boxed{K2}$ $p(x) = 0.9551 + 0.4142x$

$\boxed{K3}$ x_0 : min $\bar{x}_0 = 0$

x_1 : max $\bar{x}_1 = 0.4551$

x_2 : min $\bar{x}_2 = 1$

isto kao i u I iteraciji

$\boxed{K4}$ $z_0 = \ldots =$

$z_1 = \ldots =$ $\quad \pm 0.0449$

$z_2 = \ldots =$

$\max|z_i| - \min|z_i| < 10^{-4}$ ✓ $\rightarrow$ kraj

$\Rightarrow x_0 = 0, \; x_1 = 0.4551, \; x_2 = 1$ TČA

$Q = p(x)$

$L = E = 0.0449$

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1) zadato n, Q_n, $L = ?$

2) zadato $L = tol$, $n, Q_n = ?$

Vajerštras $\Rightarrow \exists Q_n$
Haar $\Rightarrow$ jedinstven

$n = 0, 1, 2, 3, \ldots$

čeb. t. Q_0, L, $L < tol$ ↯

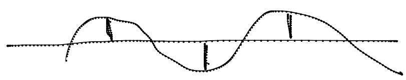
čeb. t. Q_1, L_1 $L < tol$ ✓

Zad 4.55, 4.56

spec. slučaj 2) ako je $f(x) =$ polinom

$$\max_{x\in[a,b]} |\underbrace{\underbrace{f(x)}_{p(x)} - Q(x)}_{\text{polinom}}| = L$$

Čebiševljevi poly



(NA1a)

- ortog. sistem poly. na $[-1,1]$, $p(x) = \frac{1}{\sqrt{1-x^2}}$
- 1) $T_n(x) = \cos(n \arccos x)$

  2) $T_n(x) = \ldots$

  3) $T_n(x) = \ldots$
- koeficijent u x^n u $T_n(x)$: 2^{n-1}
- Nule : $x_k = \cos \frac{(2k+1)\pi}{2n}$
- Ekstremumi : $x_k = \cos \frac{k\pi}{n}$, $T_n(x_k) = (-1)^k$, $k = 0..n$
- Poly. najmanjeg odstupanja od nule
  (normiran / moničan)

  $$\overline{T_n}(x) = 2^{1-n} \cdot T_n(x)$$

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Lema: $P_n(x)$ moničan poly. stepena n

$$\Rightarrow \max_{x\in[-1,1]} |P_n(x)| \geq \max_{x\in[-1,1]} |\overline{T}_n(x)| = 2^{1-n}$$

Dokaz: - - - -

$$- \quad \underset{t\in}{[a,b]} \neq \underset{x\in}{[-1,1]}, \quad t = \frac{b+a}{2} + \frac{b-a}{2}x$$

$$x = \frac{2t-(b+a)}{b-a}$$

$$\underset{\text{moničan}}{\overline{T}_n(x)} \rightsquigarrow \overline{T}_n\left(\frac{2t-(b+a)}{b-a}\right)$$

nije moničan, $\left(\frac{2}{b-a}\right)^n$ uz t^n

$$\overline{T}^{[a,b]}(t) = \underbrace{\left(\frac{b-a}{2}\right)^n \cdot \overline{T}_n\left(\frac{2t-(b+a)}{b-a}\right)}_{\text{moničan}}$$

$\downarrow$

poly najmanjeg odstupanja od 0 na $[a,b]$ sa koef. $\underline{1}$ uz najviši stepen

$$= \left(\frac{b-a}{2}\right)^n \cdot 2^{1-n} \cdot \underbrace{T_n\left(\frac{2t-(b+a)}{b-a}\right)}_{\text{čeb. poly}}$$

$$= (b-a)^n \cdot 2^{1-2n}\, T_n\left(\frac{2t-(b+a)}{b-a}\right), \quad t\in[a,b]$$

$$\Rightarrow \max_{x\in[a,b]} |P_n(x)| \geq \max_{x\in[a,b]} |\overline{T}_n^{[a,b]}(x)|$$

$$\parallel$$

$$\max_{x\in[a,b]} |(b-a)^n \cdot 2^{1-n}| \cdot |\underbrace{T_n\left(\frac{2t-(b+a)}{b-a}\right)}_{\pm 1}|$$

$$\parallel$$

$$(b-a)^n \cdot 2^{1-n} \cdot \underline{1}$$

(*) Da li $\exists$ polinom $P_2(x)$ t.d. je

$$\max_{x\in[-1,1]} |x^4 - P_2(x)| < 0.1$$

1) $f(x) = x^4$, $\underset{L}{P_2} = c_0 + c_1 x + c_2 x^2 \Big\} = ?$ $\quad L < 0.1$?

2) preko Čeb. poly.

$$\max_{x\in[-1,1]} |x^4 - c_2 x^2 - c_1 x - c_0| < 0.1 \ ?$$

$f(x) = x^4 \Rightarrow f(-x) = f(x) \Rightarrow$ parna

$\Rightarrow P$ parno $\Rightarrow P_1 = c_0 + c_2 x^2$

$$\max_{x\in[-1,1]} |\underbrace{x^4 - c_2 x^2 - c_0}_{\text{polinom 4. stepena}}| < 0.1 \ ?$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$\overline{T}_4(x) = \frac{1}{8} \cdot T_4(x) = x^4 - x^2 + \frac{1}{8}$$

$$x^4 - c_2 x^2 - c_0 = x^4 - x^2 + \frac{1}{8} \Rightarrow c_2 = 1, \quad c_0 = -\frac{1}{8}$$

$$P_2(x) = x^2 - \frac{1}{8} \rightarrow \equiv Q_0$$

$$\max_{x\in[-1,1]} |x^4 - x^2 + \frac{1}{8}| = \max_{x\in[-1,1]} |\overline{T}_4(x)| = \max_{x\in[-1,1]} |\frac{1}{8} T_4(x)|$$

$$= \frac{1}{8} \cdot \underbrace{\max_{x\in[-1,1]} |T_4(x)|}_{1} = \frac{1}{8} = 0.125 \not< 0.1$$

$$L = 0.125$$

$\Rightarrow \nexists \ P_2(x)$