FFT

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$n = 2m$, $\omega_n^2 = \omega_m$, $\omega_n = e^{i\frac{2\pi}{n}}$

$y = F_n \cdot x$, $y^e = F_m \cdot x^e$, $y^o = F_m \cdot x^o$

$x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$, $x^e = \begin{pmatrix} x_0 \\ x_2 \\ \vdots \\ x_{n-2} \end{pmatrix}$ $x^o = \begin{pmatrix} x_1 \\ x_3 \\ \vdots \\ x_{n-1} \end{pmatrix}$

$$y_j = \sum_{k=0}^{n-1} \omega_n^{kj} \cdot x_k = \sum_{k=0}^{m-1} \overset{\omega_m}{\overset{\shortparallel}{\omega_n^{2}}}{}^{kj} \cdot x_{2k} + \sum_{k=0}^{m-1} \omega_n^{(2k+1)j} \cdot x_{2k+1}$$

$$= \sum_{k=0}^{m-1} \omega_m^{kj} \cdot x^e + \sum_{k=0}^{m-1} \omega_n^{2kj} \cdot \omega_n^j \cdot x^o$$

$$= \underbrace{\sum_{k=0}^{m-1} \omega_m^{kj} \cdot x^e}_{} + \omega_n^j \cdot \underbrace{\sum_{k=0}^{m-1} \omega_m^{kj} \cdot x^o}_{} \leftarrow$$

$$= y_j^e + \omega_n^j \cdot y_j^o \quad , \; j = 0, \dots, m-1$$

$$y_{j+m} = \sum_{k=0}^{m-1} \underbrace{\omega_m^{k(j+m)}}_{} \cdot x^e + \underbrace{\omega_n^{j+m}}_{} \cdot \sum_{k=0}^{m-1} \underbrace{\omega_m^{k(j+m)}}_{} \cdot x^o \quad , \; j = 0, \dots, m-1$$

$$\omega_m^{k(j+m)} = \omega_m^{kj} \cdot \omega_m^{km} = \omega_m^{kj} \cdot \underbrace{e^{i\frac{2\pi}{m} \cdot km}}_{1} = \omega_m^{kj}$$

$$\omega_n^{j+m} = \omega_n^{j+\frac{n}{2}} = \omega_n^j \cdot \omega_n^{n/2} = \omega_n^j \cdot \underbrace{e^{i\frac{2\pi}{n} \cdot \frac{n}{2}}}_{-1} = \omega_n^j \cdot (-1) = -\omega_n^j$$

$$y_{j+m} = \sum_{k=0}^{m-1} \omega_m^{kj} \cdot x^e - \omega_n^j \cdot \sum_{k=0}^{m-1} \omega_m^{kj} \cdot x^o$$

$$= y_j^e - \omega_n^j y_j^o$$

$$c = \frac{1}{n} \cdot y$$

$$n = 2^\ell$$

Množenje polinoma

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$p_1(x) = 1 + x$, $p_2(x) = 2 + 3x + x^2$, $p(x) = p_1(x) \cdot p_2(x)$

$f = [1, 1]$ $\qquad$ $g = [2, 3, 1]$ $\qquad$ $c = [c_0, c_1, c_2, c_3] \xrightarrow{DFT} \hat{c}$

$f = [1, 1, 0, 0] \xrightarrow{DFT} \hat{f}$, $g = [2, 3, 1, 0] \xrightarrow{DFT} \hat{g}$

DFT f: $\quad \omega_n = e^{i\frac{2\pi}{n}}$, $n = 4 \Rightarrow \omega_n = i$, $F_4 = (\ldots)$

$$\hat{f} = \frac{1}{n} F_n \cdot f = \frac{1}{4} \cdot F_4 \cdot f = \left[\frac{1}{2}, \frac{1}{4}(1+i), 0, \frac{1}{4}(1-i)\right]^T$$

$$\hat{f}_k = \frac{1}{n} \cdot \sum_{j=0}^{n-1} \omega_n^{jk} \cdot f_j \quad , \quad p_1(x) = \sum_{j=0}^{n-1} x^j \cdot f_j \Rightarrow \hat{f}_k = \frac{1}{n} \cdot p_1(\omega_n^k) \quad (*)$$

DFT g: $\quad \hat{g} = \frac{1}{n} \cdot F_n \cdot g = \frac{1}{4} F_4 \cdot g = \left[\frac{3}{2}, \frac{1}{4}(1+3i), 0, \frac{1}{4}(1-3i)\right]^T$

$$\hat{g}_k = \frac{1}{n} \sum_{j=0}^{n-1} \omega_n^{jk} g_j \quad , \quad p_2(x) = \sum_{j=0}^{n-1} x^j \cdot g_j \Rightarrow \hat{g}_k = \frac{1}{n} \cdot p_2(\omega_n^k) \quad (**)$$

DFT c: $\quad \hat{c} = \frac{1}{n} \cdot F_n \cdot c = \frac{1}{4} \cdot F_4 \cdot c$

$$\hat{c}_k = \frac{1}{n} \sum_{j=0}^{n-1} \omega_n^{jk} \cdot c_j \quad , \quad p(x) = \sum_{j=0}^{n-1} x^j \cdot g_j \Rightarrow \hat{c}_k = \frac{1}{n} \cdot p(\omega_n^k) \quad (***)$$

$$\hat{f} \overset{(\cdot)}{*} \hat{g} = \left[\frac{3}{4}, -\frac{1}{8} + \frac{1}{4}i, 0, -\frac{1}{8} - \frac{1}{4}i\right]^T$$

koord. po koord.

$$(\hat{f} .* \hat{g})_k = \hat{f}_k \cdot \hat{g}_k \overset{(*),(**)}{=} \frac{1}{n} \cdot p_1(\omega_n^k) \cdot \frac{1}{n} \cdot p_2(\omega_n^k)$$

$$= \frac{1}{n^2} \cdot \underbrace{p_1(\omega_n^k) \cdot p_2(\omega_n^k)}_{p(\cdot)}$$

$$= \frac{1}{n^2} \cdot \underline{p(\omega_n^k)}$$

$$\overset{(***)}{=} \frac{1}{n^2} \cdot n \cdot \hat{c}_k$$

$$= \frac{1}{n} \hat{c}_k$$

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koord. po koord.

$$\hat{f} \cdot \hat{g} = \frac{1}{n} \hat{c}$$

$$\hat{c} = \frac{1}{n} F_n \cdot c \;\Rightarrow\; F_n^* / \; \hat{f} \cdot \hat{g} = \frac{1}{n} \cdot \frac{1}{n} F_n \cdot c = \frac{1}{n^2} \cdot F_n \cdot \underset{?}{\underline{c}}$$

$$\Rightarrow F_n^* (\hat{f} \cdot \hat{g}) = \frac{1}{n^2} \underbrace{F_n^* F_n}_{n \cdot I} \cdot c$$

$$\Rightarrow c = \underbrace{n \cdot \underbrace{F_n^* (\hat{f} \cdot \hat{g})}_{\text{inverzna DFT vektora } \hat{f}\hat{g}}}_{\text{konvolucija } (f * g)}$$

$$n = 4, \quad F_4^*, \quad \overline{\omega}_4 = e^{-i\frac{2\pi}{4}}, \quad \overline{\omega} = -i \qquad F_4^* = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

$$c = n \cdot F_n^* (\hat{f} \cdot \hat{g}) = 4 \cdot F_4^* \cdot [\; \ldots \;]$$

$$= [2, 5, 4, 1]^T$$

$$p(x) = p_1(x) \cdot p_2(x) = 2 + 5x + 4x^2 + x^3$$

Ravnomerna aproksimacija

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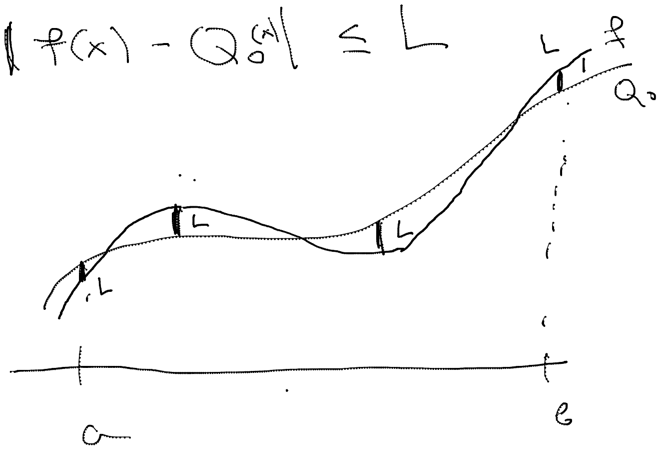
$f \rightsquigarrow Q$

$|f(x) - Q_0(x)| \leq L$



$C[a,b]$, $\|f\| = \sup\limits_{x\in[a,b]} |f(x)|$

$E_n(f) = \|f - Q_0\| \leq \|f - Q\|$, $\forall Q = \sum\limits_{j=0}^{n} c_j \cdot g_j(x)$

$E_n(f) = \|f - Q_0\| = \inf\limits_{Q} \|f - Q\| = \inf\limits_{Q} \sup\limits_{x} |f(x) - Q(x)|$

$C[a,b] \overset{T \text{ sa početka lin. norm.}}{\Longrightarrow} \exists$ element najbolje (ravn.) aprox.

$C[a,b]$ nije strogo normiran $\not\Rightarrow$ jedinstvenost

Teorema Haar: Da bi za proizvoljno zadatu f-ju $f \in C[a,b]$ postojao JEDINSTVENI generalisani (lin. kom.) polinom najbolje aprox, potrebno je i dovoljno da f-je $g_0, \dots, g_n$ obrazuju Čebiševljev sistem f-ja (generalisani polinom nema na $[a,b]$ više od n različitih nula).

$\{g_k\}: 1, x, x^2, \dots, x^n$

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Teorema de La Vallée Poussin

Neka $\exists$ $n+2$ tačke odsečka $[a,b]$, $x_0 < \dots < x_{n+1}$, takve da je

$$\operatorname{sign}\left[(f(x_i) - Q(x_i))(-1)^i\right] = \text{const}$$

(pri prelazu od x_i ka x_{i+1}, $f(x) - Q(x)$ menja znak)

pri čemu je $Q(x)$ polinom stepena n.

Tada je: $E_n(f) \geq \mu = \min_{i=0,\dots,n+1} |f(x_i) - Q(x_i)|$.

D) $\mu = 0$: $E_n(f) = \|f - Q_0\| \geq 0$ ✓

$\mu \neq 0$: pps. $E_n(f) < \mu$

$$\operatorname{sign}[Q(x_i) - Q_0(x_i)] = \operatorname{sign}[\underbrace{Q(x_i) - f(x_i)}_{|\cdot| \geq \mu} + \underbrace{f(x_i) - Q_0(x_i)}_{|\cdot| \leq E_n(f) < \mu}]$$

$$\underline{|Q(x_i) - f(x_i)|} > |f(x_i) - Q_0(x_i)|$$

$$\| \quad \operatorname{sign}[Q(x_i) - f(x_i)], \quad i = 0, \dots, n+1$$

$Q - Q_0$ je poly stepena $\leq n$, menja znak $n+2$ puta

$\Rightarrow$ ~~ima~~ $n+1$ nula ↯

∎

Čebiševljeva teorema: Da bi polinom stepena n $Q(x)$ bio polinom najbolje ravnomerne aproksimacije ($E_n(f) = \|f - Q\|$) fje $f(x)$ potrebno je i dovoljno da na $[a,b]$ postoji bar $n+2$ tačke $x_0 < \dots < x_{n+1}$ takve da je

$$f(x_i) - Q(x_i) = \alpha \cdot (-1)^i \|f - Q\|, \quad i = 0, \dots, n+1$$

pri čemu je $\alpha = \pm 1$ istovremeno za sve i.

Tačke $x_0 \dots x_{n+1}$ nazivaju se Tačkama Čebiševljeve alternanse.

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$\boxed{\Leftarrow}$ $\exists\, n+2$ tačke $x_0, \dots, x_{n+1}$ t.d. $|f(x_i) - Q(x_i)| = \alpha L$

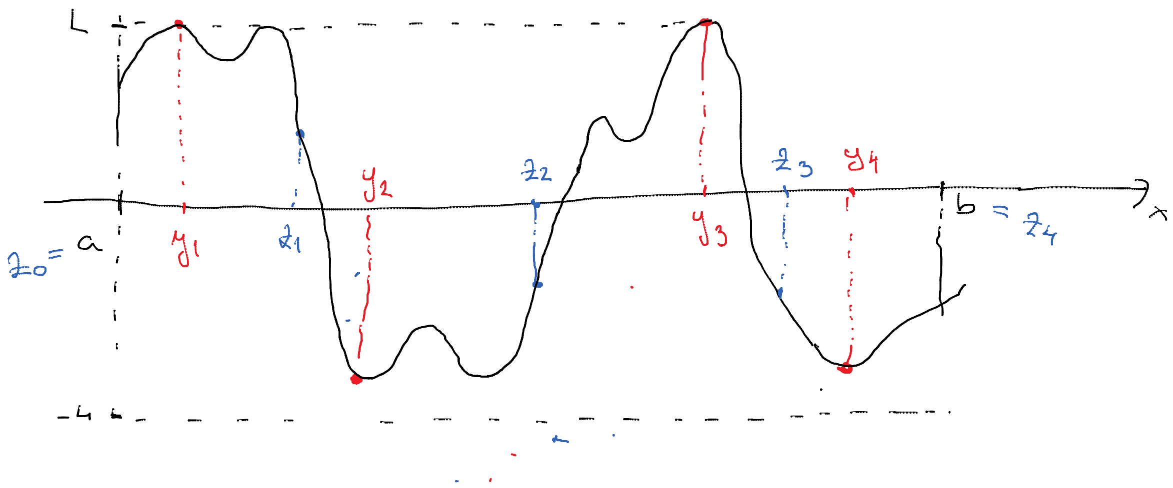
$$L = \| f - Q \|$$

Pretp. t. : $\mu = \min |f(x_i) - Q(x_i)| = L \leq E_n(f)$ $\quad (*)$

Sa druge strane: $L = \| f - Q \| \geq \| f - Q_0 \| = E_n(f)$ $\quad (**)$

$$\left.\begin{array}{ll} (*): & L \leq E_n(f) \\ (**) & L \geq E_n(f) \end{array}\right\} \; L = E_n(f) \Rightarrow Q = Q_0$$

$\boxed{\Rightarrow}$ $E_n(f) = \| f - Q \| \Rightarrow f(x_i) - Q(x_i) = \alpha (-1)^i \cdot \| f - Q \|$, $i = 0, \dots, n+1$



y_1 - donja granica $x \in [a, b]$ t.d. $|f(x) - Q(x)| = L$

$$f(y_1) - Q(y_1) = +L$$

y_2 - donja granica $x \in [y_1, b]$ t.d. $f(x) - Q(x) = -L$

y_{k+1} - donja granica $x \in (y_k, b]$. t.d. $f(x) - Q(x) = (-1)^k \cdot L$

y_m - // - $\quad \to y_m = b$

$\quad \to \nexists\, x \in (y_m, b]$ $\quad f(x) - Q(x) = (-1)^m \cdot L$

$\downarrow$

Tačke Čeb. alter

$m \geq n+2$ ✓

$m < n+2$:

$m < n+2$:

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$z_k: \quad |f(z_k) - Q(z_k)| < L \quad , \; z_k < y_{k+1}$

$[z_{k-1}, z_k] \;;\; k = 1, \dots, m \; : \quad \exists$ tačke $\quad f(x) - Q(x) = (-1)^{k-1} \cdot L$

$\nexists$ tačka $\quad f(x) - Q(x) = (-1)^k L$

Tačaka z_k ima $m+1$, najviše $n+2$

$m+1 < n+2+1 = n+3$

$P(x) = \prod_{j=1}^{m-1} (z_j - x) \qquad$ poly. stepena $\underset{\le n}{m-1 < n+2-1 = n+1}$

$Q_\varepsilon(x) = Q(x) + \varepsilon \cdot P(x) \quad , \; \varepsilon > 0 \quad , \; st(Q_\varepsilon) \le n$

$f(x) - Q_\varepsilon(x) = f(x) - Q(x) - \varepsilon \cdot P(x) \quad , [z_{k-1}, z_k]$
$k = 1, \dots, m$

$x \in [z_0, z_1)$: $\quad x < z_1 \Rightarrow P(x) > 0$

$$f(x) - Q_\varepsilon(x) = \underbrace{f(x) - Q(x)}_{\le L} - \underbrace{\varepsilon}_{>0} \cdot \underbrace{P(x)}_{>0} < L \qquad (*)$$

$x = z_1$: $\quad P(z_1) = 0$

$|f(z_1) - Q(z_1)| < L \quad \checkmark$

$x \in [z_0, z_1]$: $\quad f(x) - Q(x) > -L$

$$f(x) - Q_\varepsilon(x) = \underbrace{f(x) - Q(x)}_{> -L} - \underbrace{\varepsilon}_{>0} \cdot \underbrace{P(x)}_{>0} > -L \qquad (**)$$

$\varepsilon P(x) < f(x) - Q(x) + L$

$\varepsilon < \dfrac{f(x) - Q(x) + L}{P(x)}$

$\varepsilon < \varepsilon_0 = \dfrac{\min |f(x) - Q(x) + L|}{\max |P(x)|}$

$(*) \text{ i } (**) \Rightarrow x \in [z_0, z_1] \quad |f(x) - Q_\varepsilon(x)| < L \quad , \; \varepsilon < \varepsilon_0$

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$x \in [z_1, z_2]$: isto (obrnuto) ε_1

$\vdots$

$x \in [z_{k-1}, z_k]$ ε_k

$x \in [a, b]$: $|f^{(x)} - Q_\varepsilon^{(x)}| < L$, $\varepsilon = \min\{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{m-1}\}$

$$\Rightarrow \|f - Q_\varepsilon\| < L = \|f - Q\|$$

$\Rightarrow Q_\varepsilon$ ima manju grešku od $Q\ (\equiv Q_0)$ ⚡

$$\Rightarrow m \geq n+2$$

☒

(T) Polinom najbolje ravn. aprox. neprekidne f-je je jedinstven.

(D) pps. $\exists Q_1(x) \neq Q_2(x)$ istog stepena

$$\|f - Q_1\| = \|f - Q_2\| = E_n(f)$$

$$\left\|f - \frac{Q_1+Q_2}{2}\right\| = \left\|\frac{1}{2}(f - Q_1) + \frac{1}{2}(f - Q_2)\right\|$$

$$\leq \frac{1}{2}\underbrace{\|f - Q_1\|}_{E_n} + \frac{1}{2}\underbrace{\|f - Q_2\|}_{E_n} \qquad (*)$$

$$= E_n(f)$$

$\frac{Q_1+Q_2}{2}$ je stepena $n \Rightarrow \left\|f - \frac{Q_1+Q_2}{2}\right\| \geq E_n(f)$ $(*,*)$

$(*)$ i $(**) \Rightarrow \left\|f - \frac{Q_1+Q_2}{2}\right\| = E_n(f)$

$\Rightarrow \frac{Q_1+Q_2}{2}$ je takođe polinom najbolje rav. ap.

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Čeb. t: $\exists x_0 \ldots x_{n+1}$:

$$\left| f(x_k) - \frac{Q_1(x_k)+Q_2(x_k)}{2} \right| = E_n(f) \quad | \cdot 2$$

$$\Big| \underbrace{f(x_k) - Q_1(x_k)}_{\le \|f-Q_1\| = E_n} + \underbrace{f(x_k) - Q_2(x_k)}_{\le \|f-Q_2\| = E_n} \Big| = 2E_n(f)$$

$$| f(x_k) - Q_j(x_k) | \le \sup_x | f(x) - Q_j(x) | = E_n(f) \quad , j=1,2$$

moguće ako:

$$f(x_k) - Q_1(x_k) = f(x_k) - Q_2(x_k) = \pm E_n(f)$$

$$\Rightarrow Q_1(x_k) = Q_2(x_k) \quad , \quad k = 0, \ldots, n+1$$

Q_1 i Q_2 stepena n (različiti) su jednaki u $n+2$ različite tačke ↯

$\Rightarrow Q_1 = Q_2$ $\boxtimes$

Vajerštrasova t: Ako je $f \in C[a,b]$ tada za proizvoljno $\varepsilon > 0$, $\exists\, Q(x)$ t.d. $|f(x) - Q(x)| < \varepsilon$, $\forall x \in [a,b]$

$Q_0 = ?$

$x_0, \ldots, x_{n+1}$: $f(x_i) - Q_0(x_i) = \alpha(-1)^i \cdot L$, $L = \|f - Q_0\|$

L: 1 nepoznata

$Q_0(x_i)$: $c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n \rightarrow n+1$ nepoznatih

$n+2$ jednačine

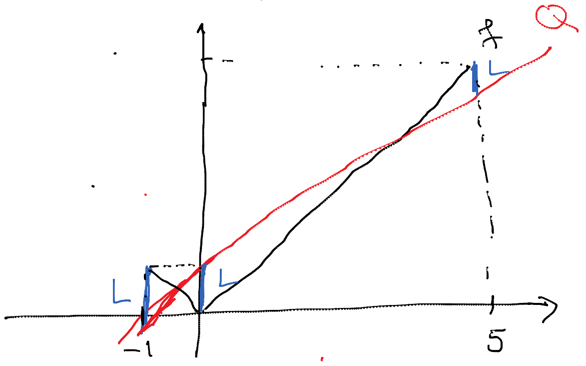
$$\left(f(x) - Q(x) \right)'_{x=x_i} = 0$$

(*) $Q_1(x) = ?$, $f(x) = |x|$, $x \in [-1, 5]$

$Q_1(x) = c_0 + c_1 x$

$n = 1 \Rightarrow 3$ točke Č.A.

$c_0, c_1, L = ?$



Točke čeb. alt.:

$x_0 = -1, \quad x_1 = 0, \quad x_2 = 5$

$$\left.\begin{aligned} f(x_0) - Q(x_0) &= \alpha \cdot L \\ f(x_1) - Q(x_1) &= -\alpha L \\ f(x_2) - Q(x_2) &= \alpha L \end{aligned}\right\} \quad \left.\begin{aligned} 1 - c_0 - c_1(-1) &= \alpha L \\ 0 - c_0 - c_1 \cdot 0 &= -\alpha L \\ 5 - c_0 - c_1 \cdot 5 &= \alpha L \end{aligned}\right\} \quad \begin{aligned} c_0 &= \frac{5}{6} \\ c_1 &= \frac{2}{3} \\ \alpha L &= \frac{5}{6} \end{aligned}$$

$Q_0(x) = \frac{5}{6} + \frac{2}{3} x$, $L = \frac{5}{6}$, $\alpha = +1$