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$\{g_k\}: 1, \cos kx, \sin kx \dots$

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

$$e^{\pm ikx} = \cos kx \pm i \cdot \sin kx$$

$$\sin kx = \frac{1}{2i}\left(e^{ikx} - e^{-ikx}\right)$$

$$\cos kx = \frac{1}{2}\left(e^{ikx} + e^{-ikx}\right)$$

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cdot \frac{e^{ikx} + e^{-ikx}}{2} + b_k \frac{e^{ikx} - e^{-ikx}}{2i} \cdot \frac{i}{i}$$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \frac{e^{ikx} + e^{-ikx}}{2} + b_k \frac{i\left(e^{ikx} - e^{-ikx}\right)}{-2}$$

$$= \underbrace{\frac{a_0}{2}}_{c_0} + \sum_{k=1}^{\infty} e^{ikx} \cdot \underbrace{\frac{a_k - i \cdot b_k}{2}}_{c_{-k}} + e^{-ikx} \underbrace{\frac{a_k + i \cdot b_k}{2}}_{c_k}$$

$$= \frac{a_0}{2} + \sum \quad + \sum$$

$$= c_0 + \underbrace{\sum_{k=1}^{\infty} c_{-k} \cdot e^{ikx}}_{\sum_{-\infty}^{-1} c_k e^{-ikx}} + \sum_{k=1}^{\infty} c_k \cdot e^{-ikx}$$

$$c_0 = c_0 \cdot e^0$$

$$\boxed{f(x) \approx \sum_{k=-\infty}^{\infty} c_k \cdot e^{-ikx}}$$

KOMPLEKSNI ZAPIS FURIJEOVOG REDA

$\{e^{-ikx}\}$ $k = 0, \pm 1, \pm 2 \dots$ ortogonalan na $[-\pi, \pi]$ potpun sistem

$$\left(e^{-ikx}, e^{-imx}\right) = \int_{-\pi}^{\pi} e^{-ikx} \cdot \overline{e^{-imx}}\, dx = \int_{-\pi}^{\pi} e^{-ikx} \cdot e^{+imx}\, dx$$

$$= \int_{-\pi}^{\pi} e^{i(m-k)x}\, dx$$

$$= \begin{cases} k = m: & \int_{-\pi}^{\pi} e^0\, dx = \int_{-\pi}^{\pi} dx = 2\pi \\ k \neq m: & \int_{-\pi}^{\pi} e^{i(m-k)x}\, dx = \dots \frac{1}{i(m-k)} e^{i(m-k)x} \Big|_{-\pi}^{\pi} = \dots = 0 \end{cases}$$

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$$C_k = \frac{(f, e^{-ikx})}{(e^{-ikx}, e^{-ikx})} = \frac{1}{2\pi} \cdot (f, e^{-ikx}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cdot e^{+ikx} dx$$

$\{e^{-ikx}\}$ sopstvene fje

$$\frac{d}{dx} \underbrace{e^{ikx}} = \underbrace{i \cdot k}_{\text{const}} \cdot \underbrace{e^{ikx}}$$

$$\Delta e^{ikx_j} = e^{ik(x_j+h)} - e^{ikx_j} = e^{ikx_j}(e^{ikh} - 1)$$

$$\Delta e^{ikx} = \frac{e^{ikh} - 1}{h} \cdot e^{ikx}$$

Ako period nije $\underbrace{2\pi}_{x}$ nego $\underbrace{T}_{t}$

$\Rightarrow$ smena $x = \frac{2\pi}{T} \cdot t$

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(*) $f(x) = x \quad , \quad x \in [-\pi, \pi]$

a) Razviti u Furier red u trigonom. formi

b) —"— kompleksnoj

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{-ikx}$$

f parna - $b_k = 0$
- $c_k \in \mathbb{R}$ (Re)

f neparna - $a_k = 0$
c_k imaginarni (Im)

f ni parna ni neparna - $a_k, b_k \neq 0$
c_k (Re + Im)

a) f neparna $\Rightarrow a_k = 0$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \sin kx \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \sin kx = \left\{ \begin{array}{ll} u = x & , dv = \sin kx \\ du = dx & v = -\frac{1}{k}\cos kx \end{array} \right\}$$

$$= \ldots = \frac{\pi}{k}(-1)^{k+1}$$

$$f(x) = \sum_{k=1}^{\infty} \frac{\pi}{k}(-1)^{k+1} \cdot \sin kx$$

b) $$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cdot e^{-ikx} \, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x \cdot \underbrace{e^{-ikx}}_{\cos kx - i \cdot \sin kx} dx$$

$$= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} x \cdot \cos kx \, dx - \int_{-\pi}^{\pi} i \cdot x \cdot \sin kx \, dx\right) = \ldots = \frac{(-1)^{k+1}}{i \cdot k}$$

$$f(x) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(-1)^{k+1}}{i \cdot k} \cdot e^{-ikx}$$

$k = 0, \; c_0 = 0$

$$c_0 = \frac{1}{2\pi}\left(\frac{x^2}{2}\right)\Big|_{-\pi}^{\pi} = \frac{1}{2\pi}\left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right) = 0$$

(*) a) Odrediti Furijeov red f-je $f(x) = \begin{cases} -1, & x \in [-\pi, 0) \\ 1, & x \in (0, \pi] \end{cases}$

b) Dokazati da je $1 + \frac{1}{9} + \frac{1}{25} + \dots + \frac{1}{(2n+1)^2} + \dots = \frac{\pi^2}{8}$

a) $a_k = \dots = 0$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin kx \, dx = \frac{1}{\pi} \left(\int_0^{\pi} 1 \cdot \sin kx \, dx + \int_{-\pi}^{0} (-1) \cdot \sin kx \, dx \right)$$

$$= \dots = \frac{1}{k\pi} \Big(\underbrace{-\cos k\pi}_{\pm 1 = (-1)^k} + \underbrace{\cos 0}_{1} + \underbrace{\cos 0}_{1} - \underbrace{\cos(-k\pi)}_{\pm 1 = (-1)^k} \Big)$$

$$= \frac{2(1 - (-1)^k)}{k\pi}$$

$$f(x) = \cancel{\frac{a_0}{2}}^{0} + \sum \cancel{a_k} \cdot \cos kx + b_k \sin kx$$

$$= \sum_{k=1}^{\infty} \frac{2(1-(-1)^k)}{k\pi} \cdot \sin kx = \sum_{n=0}^{\infty} \frac{2(1+1)}{(2n+1)\pi} \cdot \sin(2n+1)x$$

k - parno $\frac{2(1-1)}{k\pi} = 0$ $\qquad = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \cdot \sin(2n+1)x$

k - neparno
$= 2n+1$ $\quad k=1 \Rightarrow n=0$
$k=3 \Rightarrow n=1$
$k=5 \Rightarrow n=2$

b) $$\|f\|^2 = (f, f) = \sum_{k=1}^{\infty} |c_k|^2 = \left(\frac{a_0}{2}\right)^2 + \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$ važi ako je ortonormiran

$$f(x) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \cdot \underbrace{\sin(2n+1)x}_{\text{ortonormiran?}}$$

$$\|\sin kx\|_2 = \sqrt{(\sin kx, \sin kx)} = \sqrt{\int_{-\pi}^{\pi} \sin^2 kx \, dx} = \left\{ \sin^2\alpha = \frac{1-\cos 2\alpha}{2} \right\}$$

$$= \dots = \sqrt{\pi} \neq 1$$

$$f(x) = \sum_{n=0}^{\infty} \underbrace{\frac{4}{\pi} \cdot \frac{1}{2n+1} \cdot \sqrt{\pi}}_{c_k} \cdot \underbrace{\frac{\sin(2n+1)x}{\sqrt{\pi}}}_{\text{ortonormiran}}$$

$$\|f\|^2 = \int_0^{\pi} 1\,dx + \int_{-\pi}^{0} (-1)\,dx = \dots = 2\pi$$

$$\|f\|^2 = \sum |c_k|^2$$

$$2\pi = \sum \frac{4^2}{\pi^2}\cdot\frac{1}{(2n+1)^2}\cdot\cancel{\pi}$$

$$\sum \frac{1}{(2n+1)^2} = 2\pi\cdot\frac{\pi}{16} = \frac{\pi^2}{8}$$ ✓

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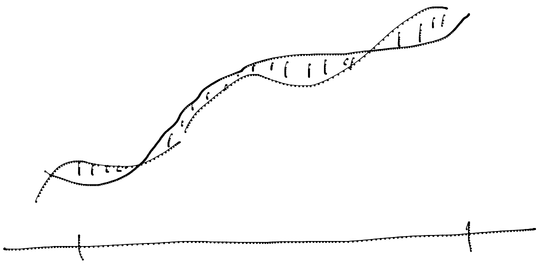
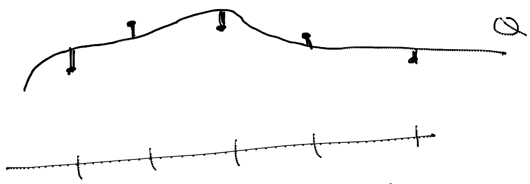
f - tablično

x	$\dots$
f	$\dots$

$(f, g_k) = \int \dots ?$

Metoda najmanjih kvadrata





$$(f,g) = \sum_{i=0}^{n} p_i \cdot f(x_i)\cdot g(x_i), \quad p_i > 0, \quad x_i \in [a,b]$$

$$\|f\|^2 = (f,f) = \sum_{i=1}^{n} p_i \,(f(x_i))^2$$

n+1 tački, $g_0, \dots, g_m$, $m \le n$

$$Q_0(x) = \sum_{i=0}^{m} c_i^0 g_i(x)$$

$$E^2(f) = \|f - Q_0\|^2 = \min_Q \|f - Q\|^2$$

$$\sum_{i=0}^{n} p_i\cdot(f(x_i) - Q_0(x_i))^2$$

$$\sum_{i=0}^{m} c_i^0 (g_i, g_j) = (f, g_j), \quad j = 0, \dots, m$$

$m = n$: $\|f - Q_0\|^2 = 0$, $f(x_k) = Q_0(x_k)$

Q_0 int. poly

$m < n$: preodređen sistem jn. na

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f - periodicka, tablicno

$0 \leq x_1 < x_2 < \dots < x_n \leq 2\pi$

$1, \cos x, \sin x, \dots, \cos mx, \sin mx \quad , \quad n \geq 2m+1$

$$Q_0(x) = \frac{a_0}{2} + \sum_{k=1}^{m} a_k \cos kx + b_k \sin kx$$

x_k - ekvidistantni $[0, 2\pi)$

$x_1 = \alpha, \quad x_2 = 2\alpha, \dots, \quad x_n = n \cdot \alpha \quad , \quad \alpha = \frac{2\pi}{n}$

$$e^{\pm ikx} = \cos kx \pm i \sin kx$$

$$\sum_{j=1}^{n} e^{ikx_j} = \sum_{j=1}^{n} \cos kx_j + i \sum_{j=1}^{n} \sin kx_j$$

$$\sum_{j=1}^{n} e^{ikx_j} = \sum_{j=1}^{n} e^{ikj\cdot\alpha} = \sum_{j=1}^{n} \left(\underbrace{e^{ik\alpha}}\right)^j \quad , \quad \sum_{k=0}^{n-1} a \cdot r^k = a \frac{r^n - 1}{r-1}$$

$$= e^{ik\alpha} \frac{\left(e^{ik\alpha}\right)^n - 1}{e^{ik\alpha} - 1} = e^{ik\alpha} \frac{\left(e^{ik\frac{2\pi}{n}}\right)^n - 1}{e^{ik\alpha} - 1}$$

$$e^{ik2\pi} = \underbrace{\cos 2k\pi}_{1} + i \cdot \underbrace{\sin 2k\pi}_{0}$$

$$= e^{ik\alpha} \frac{1-1}{e^{ik\alpha} - 1} = 0$$

$e^{ik\alpha} \neq 1$

$k\alpha \neq 0$

$k\alpha \neq C \cdot 2\pi \quad , \quad C = 0, \pm 1, \dots$

$k \cdot \frac{2\pi}{n} \neq C \cdot 2\pi$

$k \neq C \cdot n$

$$\Rightarrow \sum_{j=1}^{n} \cos kx_j + i \cdot \sum_{j=1}^{n} \sin kx_j = 0$$

$$\Rightarrow \sum_{j=1}^{n} \cos kx_j = 0 \quad \wedge \quad \sum_{j=1}^{n} \sin kx_j = 0$$

$$(\cos kx, \cos lx) = \sum_{j=1}^{n} \cos kx_j \cdot \cos lx_j$$

$$= \begin{cases} k=l=0: & \sum \cos 0 \cdot \cos 0 = \sum_{j=1}^{n} 1 = n \\ k=l\neq 0: & \sum \cos^2 kx_j = \dots = \frac{n}{2} \\ k\neq l & \sum \cos kx \cdot \cos lx = \dots = 0 \end{cases}$$

$$(\sin kx, \sin lx) = \sum \sin kx \cdot \sin lx$$

$$= \begin{cases} k=l, & \frac{n}{2} \\ k\neq l, & 0 \end{cases}$$

$$(\sin kx, \cos lx) = \sum \sin kx \cdot \cos lx = \begin{cases} k=l: & 0 \\ k\neq l: & 0 \end{cases}$$

$\Rightarrow$ jeste ortogonalan

$$\Rightarrow \quad c_i^0 = \frac{(f, g_i)}{(g_i, g_i)}$$

$$a_k = \frac{(f, g_k)}{(g_k, g_k)} = \frac{(f, \cos kx_j)}{(\cos kx_j, \cos kx_j)} = \frac{2}{n} \sum_{j=1}^{n} f(x_j) \cdot \cos kx_j \qquad k=0,\dots,m$$

$$b_k = \frac{(f, g_k)}{(g_k, g_k)} = \frac{(f, \sin kx_j)}{(\sin kx_j, \sin kx_j)} = \frac{2}{n} \sum_{j=1}^{n} f(x_j) \cdot \sin kx_j$$

Zad 4.20

$$Q = c_0 + c_1 x$$

$i=0: \quad c_0 + c_1 \cdot x_0 = f_0 \qquad c_0 + c_1 \cdot 12 = 8{,}85$

$i=1: \quad c_0 + c_1 \cdot x_1 = f_1 \qquad c_0 + c_1 \cdot 14 = 8{,}31$

$\vdots$

$i=5: \quad c_0 + c_1 x_5 = f_5 \qquad c_0 + c_1 \cdot 22 = \underbrace{5{,}68}_{b}$

pr. sistem, 2 nep., 6 jedn.

$$A = \begin{bmatrix} 12 & 1 \\ 14 & 1 \\ 16 & 1 \\ 18 & 1 \\ 20 & 1 \\ 22 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} c_1 \\ c_0 \end{bmatrix}$$

$A' /$

$$\underbrace{A}_{6\times 2} \cdot \underbrace{C}_{2\times 1} = \underbrace{b}_{6\times 1}$$

$$\underbrace{A'A}_{2\times 2} \cdot \underbrace{C}_{2\times 1} = \underbrace{A'b}_{2\times 1}$$

2 jne 2 nep

$c_0 = \dots \quad c_1 = \dots$

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(*) Q 3 stepena

$$\begin{bmatrix} x_i^3 & x_i^2 & x_i & 1 \end{bmatrix}_{6\times 4} \cdot \begin{bmatrix} c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} f \end{bmatrix}$$

(*) Zad. 4.19.

(*) Primer 7 (Skripta)

$$(x_i - c_1)^2 + (y_i - c_2)^2 = r^2 \quad , i = 1, \dots, 10$$

$$x_i^2 - 2x_i c_1 + c_1^2 + y_i^2 - 2y_i c_2 + c_2^2 = r^2 \quad , i = 1, \dots, 10$$

(c_1, c_2) , r ?

nelinearan :(

10 jednačina
3 nepoznate

$$c_3 = r^2 - c_1^2 - c_2^2$$

linearan

$$2x_i c_1 + 2y_i c_2 + c_3 = x_i^2 + y_i^2$$

$$A = \begin{pmatrix} 2x_1 & 2y_1 & 1 \\ \vdots & \vdots & \vdots \\ 2x_{10} & 2y_{10} & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} x_1^2 + y_1^2 \\ \vdots \\ x_{10}^2 + y_{10}^2 \end{pmatrix}$$

Diskretna Furijeova transf.

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$$f(x) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{-ikx}$$

$$x_j = \frac{2\pi j}{n} \quad , j = 0, 1, \dots, n-1 \quad , \quad [0, 2\pi)$$

$$f(x_j) = \sum_{k=0}^{n-1} c_k \cdot e^{-ikx_j} \quad ; c_k = ?$$

$$f = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{pmatrix} \quad , \quad f_j = f(x_j)$$

$$e^{-ikx_j} = e^{-ik\frac{2\pi j}{n}} = \big(\underbrace{e^{-i\frac{2\pi}{n}}}\big)^{kj} = \overline{\omega}^{\,kj}$$

$$f_j = \sum_{k=0}^{n-1} \overline{\omega}^{\,kj} \cdot c_k \quad , j = 0, \dots, n-1$$

$$f = F^* \cdot c$$

$$F = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ \vdots & \omega^2 & \omega^4 & \dots & \omega^{2(n-1)} \\ & \vdots & & & \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & \omega^{(n-1)^2} \end{pmatrix} \begin{array}{l} j=0 \\ j=1 \\ j=2 \\ \\ j=n-1 \end{array} \\ k=0 \quad k=1 \quad k=2 \qquad\quad k=n-1 \end{array}$$

Furijeova mat

Lema: $F^* \cdot F = F \cdot F^* = n \cdot I$

k - vrsta, j - kolona F^*F

$$1 + \overline{\omega}^k \cdot \omega^j + \overline{\omega}^{2k} \cdot \omega^{2j} + \dots + \overline{\omega}^{(n-1)k} \cdot \omega^{(n-1)j}$$

(dijag.) $k = j \Rightarrow 1 + (\underbrace{\overline{\omega} \cdot \omega}_{e^0 = 1})^k + (\underbrace{\overline{\omega} \cdot \omega}_{1})^{2k} + \dots + (\underbrace{\overline{\omega} \cdot \omega}_{1})^{(n-1)k} = n$

(van dijag.) $k \neq j \Rightarrow r = \overline{\omega}^k \omega^j = e^{-i\frac{2\pi}{n}\cdot k} \cdot e^{i\frac{2\pi}{n}\cdot j} = e^{i\frac{2\pi}{n}(j-k)}$

$$= 1 + r + r^2 + r^3 + \dots + r^{(n-1)}$$

$$= \frac{r^n - 1}{r - 1} = \frac{1 - 1}{r - 1} = 0$$

$$r^n = \left(e^{i\frac{2\pi}{\cancel{n}}(j-k)}\right)^{\cancel{n}} = 1$$

☒

$$F^* \cdot F = F \cdot F^* = n \cdot I \quad / : n$$

$$\left(\frac{1}{\sqrt{n}} F\right) \cdot \left(\frac{1}{\sqrt{n}} \cdot F^*\right) = I \Rightarrow \frac{1}{\sqrt{n}} F \text{ je unitarna}$$

$(F^*)^{-1}$) $f = F^* \cdot c \Rightarrow c = \underbrace{(F^*)^{-1}} \cdot f \Rightarrow \boxed{c = \frac{1}{n} \cdot F \cdot f}$

$$F \cdot F^* = n \cdot I \quad / \cdot (F^*)^{-1}$$

$$F = n \cdot (F^*)^{-1}$$

$$(F^*)^{-1} = \frac{1}{n} \cdot F$$

$$c_k = \frac{1}{n} \sum_{j=0}^{n-1} f_j \cdot \omega^{kj}$$

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(*) $f = [1 \ 0 \ 1 \ 0]^T$ dft ?

$$n = 4, \quad \omega = e^{+i\frac{2\pi}{4}} = e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i \cdot \sin\frac{\pi}{2} = i$$

$$F = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

$$c = \frac{1}{4} \cdot F \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix}$$

Brza Fourierova transf. (FFT)

DFT reda n = 2 DFT reda $\frac{n}{2}$

$$\begin{tikzcd}[row sep=tiny] & & & 1 \\ & & 2 \arrow[ur] \arrow[dr] & \\ & 4 \arrow[ur] \arrow[dr] & & 1 \\ & & 2 \arrow[r] \arrow[dr] & 1 \\ 8 \arrow[uur] \arrow[ddr] & & & 1 \\ & & 2 \arrow[r] \arrow[dr] & 1 \\ & 4 \arrow[ur] \arrow[dr] & & 1 \\ & & 2 \arrow[r] \arrow[dr] & 1 \\ & & & 1 \end{tikzcd}$$

$n = 2m$