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(*) Naći najbolju aprox. vektora $y=[3,5,2]^T \in \mathbb{R}^3$ vektorima $x_1=[1,0,0]^T$ i $x_2=[0,1,0]^T$ ako je rastojanje definisano:

a) $\|v\|_2 = \sqrt{\sum v_i^2}$ b) $\|v\|_1 = \sum |v_i|$, c) $\|v\|_\infty = \max|v_i|$

$$Q = c_1 x_1 + c_2 x_2$$

$$E_n(Q) = \inf \|y - Q\| = \inf_{c_1,c_2} \|y - c_1 x_1 - c_2 x_2\|$$

$$y - c_1x_1 - c_2x_2 = \begin{pmatrix}3\\5\\2\end{pmatrix} - c_1\begin{pmatrix}1\\0\\0\end{pmatrix} - c_2\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}3-c_1\\5-c_2\\2\end{pmatrix}$$

a) $E_n = \inf \sqrt{\underbrace{(3-c_1)^2}_{\geq 0} + \underbrace{(5-c_2)^2}_{\geq 0} + 2^2} = 2$

$c_1 = 3$ $\quad c_2 = 5$

$$Q = 3\cdot x_1 + 5\cdot x_2 = \begin{bmatrix}3\\5\\0\end{bmatrix}$$

b) $E_n = \inf(\underbrace{|3-c_1|}_{\geq 0} + \underbrace{|5-c_2|}_{\geq 0} + |2|) = 2$

$c_1 = 3$ $\quad c_2 = 5$

$$Q = 3x_1 + 5x_2$$

c) $E_n = \inf(\max(\underbrace{|3-c_1|}_{\leq 2}, \underbrace{|5-c_2|}_{\leq 2}, |2|)) = 2$

$3 - c_1 \leq 2 \Rightarrow$ $\boxed{1 \leq c_1 \leq 5}$

$5 - c_2 \leq 2 \Rightarrow$ $\boxed{3 \leq c_2 \leq 7}$

$$Q = c_1 x_1 + c_2 x_2$$

nije jedinstveno

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(*) Neka je u prostoru linearnih funkcija $f(x)=c_1x+c_0$ norma definisana sa $\|f\|=|f(0)|+|f(1)|$.
Dokazati da ovo jeste norma, a zatim odrediti konstantu c koja najbolje aproksimira f-ju $f(x)=x$.

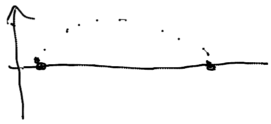
1) $\|f\|\geq 0$?

$$\|f\| = \underbrace{|f(0)|}_{\geq 0}+\underbrace{|f(1)|}_{\geq 0}\geq 0 \quad ✓$$

2) $\|f\|=0 \iff f\equiv 0$

$$f\equiv 0 \Rightarrow \|f\|=\underbrace{|f(0)|}_{0}+\underbrace{|f(1)|}_{0}=0$$

$$\|f\|=0 \Rightarrow \|f\|=0=\underbrace{|f(0)|}_{\geq 0}+\underbrace{|f(1)|}_{\geq 0} \Rightarrow f(0)=0 \wedge f(1)=0.$$



$\Rightarrow f\equiv 0$

3) $\|\alpha f\|=|\alpha|\cdot\|f\|$

$$\|\alpha f\|=|\alpha f(0)|+|\alpha f(1)|=|\alpha|\,\underbrace{(|f(0)|+|f(1)|)}=|\alpha|\cdot\|f\|$$

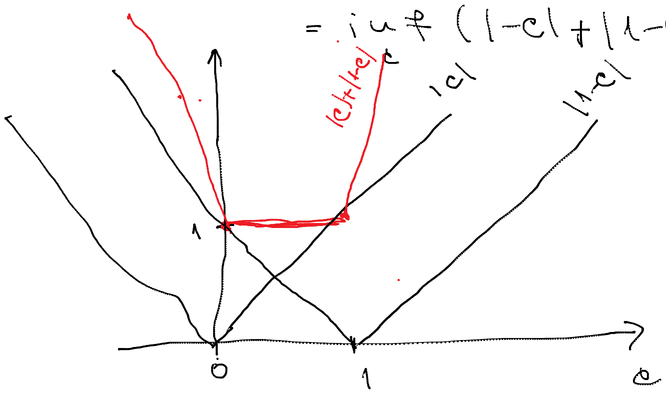
4) $\|f+g\|\leq\|f\|+\|g\|$

$$\|f+g\|=|(f+g)(0)|+|(f+g)(1)|=$$
$$=|f(0)+g(0)|+|f(1)+g(1)|$$
$$\leq|f(0)|+|g(0)|+|f(1)|+|g(1)|$$
$$=\|f\|+\|g\|$$

$Q=c$, $f(x)=x$

$$\|f-Q\|=\inf_c\|f-c\|=\inf_c\|x-c\|=\inf_c(|\underbrace{f(0)}_{0}-c|+|\underbrace{f(1)}_{1}-c|)$$

$$=\inf(|-c|+|1-c|)$$



$c\in[0,1]$

$Q=c$ nije jedinstven

$E_n=1$

Najbolja aprox u Hilbertovom prostoru

$$\|f\|^2=(f,f)$$

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$$\left(E_n(f)\right)^2=\|f-\sum_{i=1}^{n}c_i g_i\|^2=(f-\sum_{i=1}^{n}c_i g_i,\ f-\sum_{i=1}^{n}c_i g_i)$$

Da li je Q jedinstveno u Hilbertovom prostoru?

(T) Hilbertov prostor je strogo normiran

$\|x+y\|=\|x\|+\|y\|$, x,y - lin. zavisni

* $x=0 \vee y=0 \Rightarrow$ ✓

* $x\neq 0 \wedge y\neq 0$

$$\|x+y\|^2=(\|x\|+\|y\|)^2=\|x\|^2+2\cdot\|x\|\cdot\|y\|+\|y\|^2$$

$$\|x+y\|^2=(x+y,x+y)=(x,x)+(x,y)+(y,x)+(y,y)$$

$$=\|x\|^2+\|y\|^2+2\operatorname{Re}(x,y)$$

=

(*) $\cancel{2}\|x\|\cdot\|y\|=\cancel{2}\operatorname{Re}(x,y)$

$(x,y)=a$, $a\neq 0$

$(z,y)=0$, $z=x-by \Rightarrow x=z+by$

$$(z,y)=(x-by,y)=(x,y)-b(y,y)=a-b\|y\|^2=0$$

$$\Rightarrow b=\frac{a}{\|y\|^2} \ ✓ \ {}_{\neq 0}$$

* $z=0 \Rightarrow x-by=0 \Rightarrow x=by$ ✓

* $z\neq 0 \Rightarrow$ $\|x\|^2=\|z+by\|^2=(z+by,z+by)$

$$=\underbrace{\|z\|^2}_{>0}+\|by\|^2$$

$$>\|by\|^2$$

$$\operatorname{Re}(x,y)=\operatorname{Re}(z+by,y)=\operatorname{Re}\underbrace{(z,y)}_{=0}+\operatorname{Re}(by,y)=\operatorname{Re}(b)\cdot\|y\|^2 \overset{(*)}{=} \|x\|\cdot\cancel{\|y\|}$$

$$\operatorname{Re}(b)\cdot\cancel{\|y\|}=\|x\|>\|by\|=|b|\cdot\cancel{\|y\|}$$

$\operatorname{Re}(b)>|b|$ ↯ $\rightarrow$

z mora biti $0 \Rightarrow$ x,y lin. zav. ⊠

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$\mathcal{R}$ - Hilbertov

$\mathcal{H} \subset \mathcal{R}$, $g_1, \dots, g_n \in \mathcal{H}$ lin. nez.

$f \in \mathcal{R}$, $Q_0 = \sum_{i=1}^{n} c_i^0 g_i \in \mathcal{H}$

<u>Lema 1</u>: Neka je Q_0 element najbolje aproksimacije za f iz $\mathcal{H}$. Tada je $f - Q_0$ ortogonalna na svim elementima potprostora $\mathcal{H}$, tj. Q_0 je ortogonalna projekcija f na $\mathcal{H}$.

$$(f - Q_0, Q) = 0, \ \forall Q \in \mathcal{H}$$

pps. $\exists Q_1 \in \mathcal{H}$ t.d. $(f - Q_0, Q_1) = \alpha \neq 0$

pp. $\|Q_1\| = 1$ $\quad \left(\frac{Q_1}{\|Q_1\|} \right)$

$$Q_2 = Q_0 + \alpha Q_1 \Rightarrow Q_2 \in \mathcal{H}$$

$$\begin{aligned}
\|f - Q_2\|^2 &= (f - Q_2, f - Q_2) \\
&= (f - Q_0 - \alpha Q_1, f - Q_0 - \alpha Q_1) \\
&= (f - Q_0, f - Q_0) - \bar{\alpha} \underbrace{(f - Q_0, Q_1)}_{\alpha} - \alpha \underbrace{(Q_1, f - Q_0)}_{\bar{\alpha}} + \alpha \bar{\alpha} (Q_1, Q_1) \\
&= \|f - Q_0\|^2 - \bar{\alpha}\alpha - \cancel{\alpha \bar{\alpha}} + \cancel{\alpha \bar{\alpha}} \underbrace{\|Q_1\|^2}_{1} \\
&= \|f - Q_0\|^2 - \bar{\alpha}\alpha \\
&= \|f - Q_0\|^2 - \underbrace{|\alpha|^2}_{>0} \quad (\alpha \neq 0) \\
&< \|f - Q_0\|^2 \\
&= E_n(f)
\end{aligned}$$

$$\|f - Q_2\|^2 < \|f - Q_0\|^2 = E_n(f) \quad ↯ \quad \boxtimes$$

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Lema 2: Ako $(f-Q_0, Q)=0$ za proizvoljni element $Q\in\mathcal{H}$ onda je Q_0 element najbolje aprox iz $\mathcal{H}$ za f.

$Q\in\mathcal{H}$, $Q\neq Q_0$

$$\|f-Q\|^2 = (f-Q, f-Q)$$
$$= ((f-Q_0)+(Q_0-Q), (f-Q_0)+(Q_0-Q))$$
$$= (f-Q_0, f-Q_0) + \underbrace{(f-Q_0, \underbrace{Q_0-Q}_{\in\mathcal{H}})}_{=0} + \underbrace{(\underbrace{Q_0-Q}_{\in\mathcal{H}}, f-Q_0)}_{=0} + (Q_0-Q, Q_0-Q)$$
$$= \|f-Q_0\|^2 + \underbrace{\|Q_0-Q\|^2}_{>0} \quad (\text{jer } Q_0\neq Q)$$
$$> \|f-Q_0\|^2$$

$\|f-Q\|^2 > \|f-Q_0\|^2 = E_n(f) \Rightarrow Q_0$ je element najbolje aprox $\boxtimes$

$(f-Q_0, Q)=0$, $\forall Q\in\mathcal{H}$

$(f-Q_0, g_j)=0$, $j=1,\dots,n$, $g_j\in\mathcal{H}$

$Q_0=\sum_{i=1}^n c_i^\circ g_i$

$(f-\sum_{i=1}^n c_i^\circ g_i, g_j)=0$, $j=1,\dots,n$

$(f, g_j) - (\sum_{i=1}^n c_i^\circ g_i, g_j) = 0$, $j=1,\dots,n$

$\boxed{\sum_{i=1}^n c_i^\circ (g_i, g_j) = (f, g_j), \quad j=1,\dots,n}$ sistem n jednačina sa n nepoznatih $c_1,\dots,c_n$

$G(g_1,\dots,g_n) = \left|[(g_i, g_j)]\right|$ GRAMOVA determinanta

$G\neq 0 \Rightarrow \exists!$ rešenje ✓

$G=0$ ako $g_1,\dots,g_n$ lin. zavisni ↯

Problem: matrica može da bude loše uslovljena!

1) ne pravite promene (zaokruživanja) ⟶ računa na papiru
2) koristimo ~~ortonormirane~~ ortogonalne sisteme

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$$(g_i, g_j) = \begin{cases} *, & i=j \\ 0, & i \neq j \end{cases}$$

ORTOGONALNI

$$(g_i, g_j) = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

ORTONORMIRANI

Ortonormirani

$$\sum_{i=1}^{n} c_i^\circ \underbrace{(g_i, g_j)}_{\delta_{ij}} = (f, g_j) \quad , j = 1, \dots, n$$

$$\boxed{c_j^\circ = (f, g_j) \quad , j = 1, \dots, n}$$

$$Q_0 = \sum_{i=1}^{n} (f, g_i) \cdot g_i$$

$E_n(f)$ u slučaju ortonormiranih:

$$\|f - Q\|^2 = \|f - Q_0\|^2 + \|Q - Q_0\|^2 \qquad \text{(iz leme 2)} \quad , Q, Q_0 \in H$$

$$Q = 0$$

$$\|f\|^2 = \|f - Q_0\|^2 + \|Q_0\|^2$$

$$E_n(f) = \|f - Q_0\|^2 = \|f\|^2 - \|Q_0\|^2$$

$$= (f, f) - \left(\sum_i c_i^\circ g_i, \sum_j c_j^\circ g_j \right)$$

$$= (f, f) - \sum_i \sum_j c_i^\circ \overline{c_j^\circ} \underbrace{(g_i, g_j)}_{\delta_{ij}}$$

$$= (f, f) - \sum_i |c_i^\circ|^2$$

$$= (f, f) - \sum_i |(f, g_i)|^2$$

$$E_n(f) = \underbrace{\|f - Q_0\|^2}_{\geq 0} = \underline{\|f\|^2 - \sum_{i=1}^{n} |(f, g_i)|^2 \geq 0}$$

$$\therefore \quad \|f\|^2 \geq \sum_i |(f, g_i)|^2 = \sum_i |c_i^\circ|^2$$

$$\|f\|^2 \geq \sum_{i=1}^{n} |c_i^\circ|^2 \quad \text{Besselova nejednakost}$$

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$g_1, \dots, g_n, \dots \quad , n \to \infty$

Ako je ortonormirana : $c_i^\circ = (f, g_i)$ Furijeovi koef.

$$f \sim \underbrace{c_1^\circ g_1 + c_2^\circ g_2 + \dots + c_n^\circ g_n + \dots}_{\text{Go - Furijeov red}}$$

Beselova nejednakost : $\sum_{i=1}^{\infty} |c_i^\circ|^2 \le \|f\|^2$

(T) Furijeov red elementa f po ortonormiranom sistemu $\{g_k\}$ je konvergentan.

(D) $f \sim c_1^\circ g_1 + \dots + c_n^\circ g_n + \dots \quad , \quad c_i^\circ = (f, g_i)$

$\{g_k\} : \delta_{ij} = (g_i, g_j)$

$S_n = \sum_{i=1}^{n} c_i^\circ g_i$ — n-ta parc. suma reda

Red (K) ako $\{S_n\}$ (K)

$\{S_n\}$ košijev ? u kompletnom svaki košijev je (K)

Banahov je po def. kompletan

Hilbertov je Banahov sa skal. proiz.

$m<n$: $\|S_n - S_m\|^2 = \| \sum_{i=1}^{n} c_i^\circ g_i - \sum_{j=1}^{m} c_j^\circ g_j \|^2 = \| \sum_{i=m+1}^{n} c_i^\circ g_i \|^2$

$$= \left(\sum_{i=m+1}^{n} c_i^\circ g_i , \sum_{j=m+1}^{n} c_j^\circ g_j \right)$$

$$= \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} c_i^\circ \overline{c_j^\circ} \underbrace{(g_i, g_j)}_{\delta_{ij}}$$

$$= \sum_{i=m+1}^{n} |c_i|^2$$

$\sum_{i=1}^{\infty} |c_i^\circ|^2$ (pozitivni članovi, ograničen odozgo (Besel))

$\Rightarrow$ konvergentan

$\sum_{i=m+1}^{\infty} |c_i^\circ|^2$ - ostatak reda $\to 0$

$\|S_n - S_m\|^2 \to 0 \quad , m, n \to \infty$

$\Rightarrow \{S_n\}$ košijev $\Rightarrow$ (K) $\quad \exists \lim_{n\to\infty} S_n = S, \quad S \in H$ $\boxtimes$

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Ⓣ U Hilbertovom prostoru $\mathcal{R}$, Furijeov red proizvoljnog elementa po potpunom ortonormiranom sistemu elemenata konvergira ka tom elementu.

Ⓓ $S = \sum_{i=1}^{n} c_i^\circ g_i \quad , \quad f = S \; ?$

$(f - S, g_k) = 0 \quad ?$

$(f - S, g_k) = (f - S_n + S_n - S, g_k)$

$= (f, g_k) - (S_n, g_k) + (S_n - S, g_k)$

$n > k: \quad (S_n, g_k) = \left(\sum_{i=1}^{n} c_i^\circ g_i, g_k\right) = \sum_{i=1}^{n} c_i^\circ \underbrace{(g_i, g_k)}_{\delta_{ik}}$

$\overset{i=k}{=} c_k^\circ$

$(f, g_k) = c_k^\circ$

$(f - S, g_k) = \cancel{c_k^\circ} - \cancel{c_k^\circ} + (S_n - S, g_k)$

$0 \le |(f - S, g_k)| = |(S - S_n, g_k)| \le \underbrace{\|S - S_n\|}_{\to 0} \cdot \|g_k\| \xrightarrow{n \to \infty} 0$

Koši-Švarc: $|(x, y)| \le \|x\| \cdot \|y\|$

$\lim_{n \to \infty} S_n = S$

ne zavisi od n

$\Rightarrow \quad \underbrace{(f - S}, \underbrace{g_k}_{\neq 0}) = 0 \quad , \forall k$

zbog potpunosti $\{g_k\}$ $\quad f - S = 0$.

$f = S$ ✓ ⊠

$\|f - Q_0\|^2 = \|f\|^2 - \sum_{i=1}^{n} |c_i^\circ|^2$ (iz leme) $\quad n \to \infty$, $Q_0 = S_n$, $f = S$

$\Rightarrow \underbrace{\|f - S_n\|^2}_{\xrightarrow{n \to \infty} 0} = \|f\|^2 - \sum_{i=1}^{\infty} |c_i^\circ|^2$

$\Rightarrow \boxed{\|f\|^2 = \sum_{i=1}^{\infty} |c_i^\circ|^2}$

PARSEVALOVA JEDNAKOST

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$$(f - S, g_k) = (f - \sum_{i=1}^{\infty} c_i^{\circ} g_i, g_k)$$

$$= (f, g_k) - \sum_{i=1}^{\infty} c_i^{\circ} \underline{(g_i, g_k)}$$

$$= c_k^{\circ} - c_k^{\circ}$$

$$= 0 \quad :)$$

SREDNJEKVADRATNA APROKSIMACIJA

$\mathcal{H}$ - Hilbertov $\quad \mathcal{H} = L_2[a,b]$

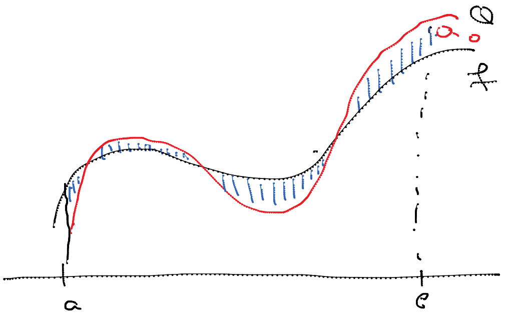
$$\|f\|^2 = (f, f) = \int_a^b f^2 \, dx \quad , \; f \in L[a,b]$$

Q_0 - element najbolje srednje kvadratne aprox.

$$(f, g) = \int_a^b p(x) f(x) \cdot g(x) \, dx, \quad p(x) > 0$$

$$Q_0 = \sum_{i=1}^{n} c_i^{\circ} g_i$$

$$E_n(f) = \|f - Q_0\| = \inf_Q \|f - Q\| = \inf_Q \sqrt{\int_a^b p(x) (f(x) - Q(x))^2 dx}$$



Ortogonalan sistem

$$\sum_{i=1}^{n} c_i^{\circ} \underbrace{(g_i, g_j)}_{* \; i=j} = (f, g_j), \quad j = 1, \dots, n$$

$$\Rightarrow \boxed{c_j^{\circ} = \frac{(f, g_j)}{(g_j, g_j)}, \quad j = 1, \dots, n}$$

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(*) Naći najbolju srednjekv. aprox. polinomom 2. stepena funkcije $f(x)=\sqrt{x}$ na $[0,1]$.

$$Q_0 := c_0 + c_1 x + c_2 x^2$$

$$g_0(x) = 1, \quad g_1(x) = x, \quad g_2(x) = x^2$$

$$\sum_{i=0}^{2} c_i (g_i, g_j) = (f, g_j), \quad j = 0,1,2$$

$$(g_k, g_j) = \int_0^1 x^k \cdot x^j \, dx = \frac{x^{k+j+1}}{k+j+1} \Big|_0^1 = \frac{1}{k+j+1}$$

$$(f, g_j) = \int_0^1 \sqrt{x} \cdot x^j \, dx = \int_0^1 x^{j+\frac{1}{2}} \, dx = \frac{x^{j+3/2}}{j+3/2} \Big|_0^1 = \frac{1}{j+3/2}$$

$$= \frac{2}{2j+3}$$

$$\begin{array}{llll}
 & i=0 & i=1 & i=2 \\
j=0: & c_0 \cdot 1 & + \; c_1 \cdot \frac{1}{2} & + \; c_2 \cdot \frac{1}{3} = \frac{2}{3} \\
j=1: & c_0 \cdot \frac{1}{2} & + \; c_1 \cdot \frac{1}{3} & + \; c_2 \cdot \frac{1}{4} = \frac{2}{5} \\
j=2: & c_0 \cdot \frac{1}{3} & + \; c_1 \cdot \frac{1}{4} & + \; c_2 \cdot \frac{1}{5} = \frac{2}{7}
\end{array}$$

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

Hilbertova mat.

$$\mathrm{cond}(A) = \|A\| \cdot \|A^{-1}\| = 524, \ldots$$

Ne smo zaokruživanje .

Rešavamo sa razlomcima:

$$c_0 = \frac{6}{35}, \quad c_1 = \frac{48}{35}, \quad c_2 = -\frac{4}{7}$$

$$Q_0(x) = \frac{6}{35} + \frac{48}{35} \cdot x - \frac{4}{7} x^2$$

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(*) Naći najbolju srednjekv. aprox. polinomom $\underline{V}$ stepena f-je $f(x)=|x|$ na $[-1,1]$ u slučajevima kada je težinska f-ja jednaka

a) $p(x)=1$ b) $p(x)=\frac{1}{\sqrt{1-x^2}}$

a) Ležandrovi poly.

$$L_n(x)=\frac{1}{2^n\cdot n!}\cdot\frac{d^n}{dx^n}\left((x^2-1)^n\right)$$

$$(L_i,L_j)=\begin{cases}0, & i\neq j\\ \frac{2}{2j+1}, & i=j\end{cases} \qquad c_j=\frac{(f,L_j)}{(L_j,L_j)}$$

Lež. poly parnog stepena su parne f-je
neparnog neparne

$$c_j=\frac{2j+1}{2}\cdot(f,L_j)$$

$$(f,L_0)=\int_{-1}^{1}f(x)\cdot\underbrace{L_0(x)}_{1}dx=\int_{-1}^{1}|x|\cdot 1\cdot dx=2\int_0^1 x\,dx=\frac{x^2}{2}\Big|_0^1\cdot 2$$

$$=\frac{1}{2}\cdot 2=1$$

$$c_0=\frac{2\cdot 0+1}{2}\cdot(f,L_0)=\frac{1}{2}$$

$$(f,L_1)=\int_{-1}^{1}f(x)\cdot\underbrace{L_1(x)}_{x}dx=2\int_0^1 \;\dots\; =0$$

$$c_1=0$$

$$(f,L_2)=2\int_0^1 x\cdot L_2\,dx=2\int_0^1\frac{x(3x^2-1)}{2}dx \;\underset{\text{apr. 5 čvora}}{\overset{\text{Simpson KF}}{=}}\;\dots$$

$$L_2=\dots=\frac{1}{2}(3x^2-1)$$

$$c_2=\frac{2\cdot 2+1}{2}\cdot(f,L_2)=0{,}625$$

$$c_3=0$$

$$c_4=\frac{2\cdot 4+1}{2}\cdot 2\int_0^1 x\cdot L_4\,dx=\dots=-0.1875$$

$$L_4=\frac{1}{8}\left(35x^4-30x^2+3\right)$$

$$c_5=0$$

$$Q_5(x)=c_0\cdot L_0+c_1\cdot L_1+c_2L_2+c_3L_3+c_4L_4+c_5L_5$$
$$=\frac{1}{2}\cdot 1+0{,}625\cdot\frac{1}{2}(3x^2-1)-0.1875\cdot\frac{1}{8}(35x^4-30x^2+3)$$