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$L_n(x_i, y_j) = f(x_i, y_j)$, $0 \le i+j \le n$

$L_n(x,y) - L_{n-1}(x,y)$ stepen $\le n$

$0 \le i+j \le n$ $\quad 0 \le i+j < n$ nule (x_i, y_j) $i+j<n$

$a_{n0}(x-x_0)(x-x_1)\dots(x-x_{n-1}) +$

$+ a_{n-1,1}(x-x_0)\cdot\dots\cdot(x-x_{n-2})\cdot(y-y_0) +$

$+ a_{n-2,2}(x-x_0)\cdots(x-x_{n-3})(y-y_0)(y-y_1) + \dots$

$\dots + a_{0n}(y-y_0)\cdots(y-y_{n-1})$

$$\Rightarrow L_n(x,y) = \underbrace{L_{n-1}(x,y)} + \sum_{i=0}^{n} a_{n-i,i}(x-x_0)\cdot\dots\cdot(x-x_{n-i-1})\cdot(y-y_0)\cdots(y-y_{i-1})$$

$x_{-1} \stackrel{def}{=} 1 \stackrel{def}{=} y_{-1}$

$$L_{n-1}(x,y) = \underbrace{L_{n-2}(x,y)} + \sum \dots$$

$\dots$

L_0

$$\Rightarrow L_n(x,y) = a_{00} + a_{10}(x-x_0) + a_{01}(y-y_0)$$
$$+ a_{20}(x-x_0)(x-x_1) + a_{11}(x-x_0)(y-y_0) + a_{02}(y-y_0)(y-y_1)$$
$$+ \dots +$$
$$+ a_{n0}(x-x_0)\cdots(x-x_{n-1}) + a_{n-1,1}(x-x_0)\cdots(x-x_{n-2})(y-y_0) +$$
$$+ \dots + a_{0n}(y-y_0)\cdots(y-y_{n-1})$$

$x=x_0, y=y_0$: $\quad L_n(x_0, y_0) = \boxed{a_{00} = f(x_0, y_0)}$

$x=x_1, y=y_0$: $\quad L_n(x_1, y_0) = a_{00} + a_{10}(x_1 - x_0) = f(x_1, y_0)$

$$\Rightarrow a_{10} = \frac{f(x_1,y_0) - a_{00}}{x_1 - x_0} = \frac{f(x_1,y_0) - f(x_0,y_0)}{x_1 - x_0}$$

$$a_{10} = f[x_0, x_1; y_0]$$

$x=x_0, y=y_1: \; L_n(x_0,y_1) = a_{00} + \underline{a_{01}}(y_1-y_0) = f(x_0,y_1)$

$$a_{01} = \frac{f(x_0,y_1) - a_{00}}{y_1-y_0} = \frac{f(x_0,y_1) - f(x_0,y_0)}{y_1-y_0}$$

$$a_{01} = f[x_0; y_0, y_1]$$

$y=y_0: \; L_n(x,y_0) = a_{00} + a_{10}(x-x_0) + \dots + a_{n0}(x-x_0)\cdots(x-x_{n-1})$

$L_n(x_i,y_0) = f(x_i,y_0)$

$\boxed{a_{i0} = f[x_0,\dots,x_i; y_0]}$, $i=0,\dots,n$

$x=x_0:$ . . . . .

$\boxed{a_{0j} = f[x_0; y_0,\dots,y_j]}$

$$\begin{aligned} y=y_1: \; L_n(x,y_1) = &\left(a_{00}+a_{01}(y_1-y_0)\right) \\ &+ \left(a_{10}+a_{11}(y_1-y_0)\right)(x-x_0) \\ &+ \left(a_{20}+a_{21}(y_1-y_0)\right)(x-x_0)(x-x_1) \\ &+ \dots + \\ &+ \left(a_{n-1,0}+a_{n-1,1}(y_1-y_0)\right)(x-x_0)\cdots(x-x_{n-2}) \\ &+ \underbrace{a_{n0}(x-x_0)\cdots(x-x_{n-1})}_{=0} \end{aligned}$$

za $x=x_k$

$L_n(x_k,y_1) = f(x_k,y_1)$

$\underline{a_{i0}} + \underline{a_{i1}}(y_1-y_0) = f[x_0,\dots,x_i; y_1]$

$$a_{i1} = \frac{f[x_0,\dots,x_i; y_1] - a_{i0}}{y_1-y_0} = \frac{f[x_0,\dots,x_i; y_1] - f[x_0,\dots,x_i; y_0]}{y_1-y_0}$$

$\boxed{a_{i1} = f[x_0,\dots,x_i; y_0, y_1]}$

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$$\boxed{L_n(x,y) = \sum_{k=0}^{n} \sum_{i+j=k} f[x_0 \ldots x_i; y_0 \ldots y_j] \cdot (x-x_0) \cdots (x-x_{i-1}) \cdot (y-y_0) \cdots (y-y_{j-1})}$$

NNM sa podijeljenim za dvodim. int.

$x_i - x_{i-1} = h, \quad y_j - y_{j-1} = k$ ekvidistantna

$$\Delta^{1+0} f(x_i, y_j) = f(x_{i+1}, y_j) - f(x_i, y_j)$$

$f(x_i, y_j)$
f_{ij}

$$\Delta^{0+1} f(x_i, y_j) = f(x_i, y_{j+1}) - f(x_i, y_j)$$

$$\Delta^{2+0} f(x_i, y_j) = \Delta^{1+0}(\Delta^{1+0} f_{ij}) = \Delta^{1+0}(f_{i+1,j} - f_{ij})$$

$$= \Delta^{1+0} f_{i+1,j} - \Delta^{1+0} f_{ij}$$

$$= f_{i+2,j} - f_{i+1,j} - f_{i+1,j} + f_{ij}$$

$$= f_{i+2,j} - 2 f_{i+1,j} + f_{ij}$$

$$\Delta^{0+2} f_{ij} = \Delta^{0+1}(\Delta^{0+1} f_{ij}) = \Delta^{0+1}(f_{i,j+1} - f_{ij})$$

$$= \Delta^{0+1} f_{i,j+1} - \Delta^{0+1} f_{ij}$$

$$= f_{i,j+2} - f_{i,j+1} - f_{i,j+1} + f_{ij}$$

$$= f_{i,j+2} - 2 f_{i,j+1} + f_{ij}$$

$$\Delta^{1+1} f_{ij} = \Delta^{1+0}(\Delta^{0+1} f_{ij})$$

$$= \Delta^{0+1}(\Delta^{1+0} f_{ij})$$

$$f[x_i \ldots, x_{i+k}] = \frac{\Delta^k f_i}{h^k \cdot k!} \qquad (\text{UNM})$$

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$$f[x_0, x_1; y_0] = \frac{1}{h^1 \cdot 1!} \cdot \Delta^{1+0} f_{00}$$

$$f[x_0; y_0, y_1] = \frac{1}{k} \cdot \Delta^{0+1} f_{00}$$

$$f[x_0, x_1, x_2; y_0] = \frac{1}{h^2 \cdot 2!} \cdot \Delta^{2+0} f_{00}$$

$$f[x_0; y_0, y_1, y_2] = \frac{1}{k^2 \cdot 2!} \cdot \Delta^{0+2} f_{00}$$

$$f[x_0, x_1; y_0, y_1] = \frac{1}{h \cdot k} \cdot \Delta^{1+1} f_{00}$$

$$\vdots$$

$$f[x_0, \ldots, x_i; y_0, \ldots, y_j] = \binom{m}{i} \cdot \frac{1}{m! \cdot h^i k^j} \Delta^{i+j} f_{00}$$

$$i+j = m$$

$$(x, y) \quad , \quad p = \frac{x - x_0}{h} , \quad q = \frac{y - y_0}{k}$$

$$L_n(x,y) = f_{00} + p \cdot \Delta^{1+0} f_{00} + q \cdot \Delta^{0+1} f_{00}$$
$$+ \frac{1}{2}\left(p(p-1) \cdot \Delta^{2+0} f_{00} + 2pq \Delta^{1+1} f_{00} + q(q-1) \cdot \Delta^{0+2} f_{00}\right)$$
$$+ \frac{1}{3!}\left(p(p-1)(p-2) \Delta^{3+0} f_{00} + 3p(p-1) \cdot q \cdot \Delta^{2+1} f_{00} + \right.$$
$$\left. + 3pq(q-1) \Delta^{1+2} f_{00} + q(q-1)(q-2) \Delta^{0+3} f_{00}\right)$$
$$+ \cdots$$

NNtn sa konačnim razlikama

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(*) Data je f-ja $f(x,y)$ tablicno:

$y \backslash x$	x_0 0.2	x_1 0.5	x_2 0.8
$y_0 = 0.4$	0.185	0.169	0.159
$y_1 = 0.5$	0.171	0.162	0.157
y_2 0.6	0.148	0.151	0.155

$$f(\overset{\in(x_0,x_1)}{0.253}, \overset{\in(y_0,y_1)}{0.413}) = ?$$

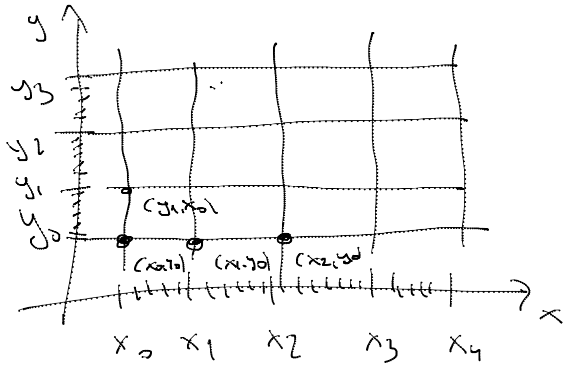
$h = 0{,}3, \quad k = 0{,}1$

$$p = \frac{x - x_0}{h} = 0.177, \qquad q = \frac{y - y_0}{k} = 0.130$$

$$f(x,y) \approx L_2(x,y) = f_{00} + p \cdot \Delta^{1+0} f_{00} + q \cdot \Delta^{0+1} f_{00} + \frac{1}{2!} p(p-1) \cdot \Delta^{2+0} f_{00}$$
$$+ \frac{1}{2} \cdot 2pq \Delta^{1+1} f_{00} + \frac{1}{2!} \cdot q(q-1) \Delta^{0+2} f_{00}$$

$$\Delta^{1+0} f_{00} = f_{10} - f_{00} = 0.169 - 0.185 = -0.016$$
$$\Delta^{0+1} f_{00} = f_{01} - f_{00} = 0.171 - 0.185 = -0.014$$
$$\Delta^{2+0} f_{00} = \Delta^{1+0} f_{10} - \Delta^{0+1} f_{00} = f_{20} - f_{10} - \Delta^{0+1} f_{00}$$
$$= 0.159 - 0.169 + 0.014 = 0.006$$
$$\Delta^{0+2} f_{00} = \Delta^{0+1} f_{01} - \Delta^{0+1} f_{00} = f_{02} - f_{01} - \Delta^{0+1} f_{00}$$
$$= 0.148 - 0.171 + 0.014 = -0.009$$
$$\Delta^{1+1} f_{00} = \Delta^{1+0} f_{01} - \Delta^{1+0} f_{00} = f_{11} - f_{01} - \Delta^{1+0} f_{00}$$
$$= 0.162 - 0.171 + 0.016 = 0.007$$

$$f(0.253, 0.413) \approx 0.185 + 0.177(-0.016) + 0.13 \cdot (-0.014)$$
$$+ \frac{1}{2} 0.177(0.177 - 1) \cdot 0.006 + \frac{1}{2} \cdot 2 \cdot 0.177 \cdot 0.13 \cdot 0.007$$
$$+ \frac{1}{2} 0.13 (0.13 - 1) \cdot (-0.009) = \underline{0.186}$$



$$F = \begin{array}{ccc} f(x_0,y_0) & f(x_1,y_0) & \\ \end{array}$$

$$F = \begin{bmatrix} * & * & * \\ & & \\ * & & \end{bmatrix}$$

$$X = \begin{bmatrix} 0.2 & 0.5 & 0.8 \\ 0.2 & 0.5 & 0.8 \\ 0.2 & 0.5 & 0.8 \end{bmatrix} \qquad Y = \begin{bmatrix} 0.4 & 0.4 & 0.4 \\ 0.5 & 0.5 & 0.5 \\ 0.6 & 0.6 & 0.6 \end{bmatrix}$$

$$z = x^2 + y^2 \qquad [-1,1] \times [-1,1]$$

APROKSIMACIJA

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Najbolja aprox u Banahovom prostoru (lin. norm.)

$f \in R \quad , \quad \|\cdot\|$

$g_1, \dots, g_n \in \mathcal{H} \subset R \quad$ - lin. nez.

$Q = \sum_{i=1}^{n} c_i \cdot g_i$

$\underbrace{Q_0 = \sum_{i=1}^{n} c_i^0 g_i}_{\text{element najbolje aprox}} \quad , \quad E_n(f) = \|f - Q_0\| = \|f - \sum_{i=1}^{n} c_i^0 g_i\|$

$= \inf_{c_1, \dots, c_n} \|f - \sum_{i=1}^{n} c_i g_i\| = \inf_{Q} \|f - Q\|$

(T) U lin. normiranom prostoru $\exists$ element najbolje aproksimacije

$F_f(c_1, \dots, c_n) = \|f - \sum_{i=1}^{n} c_i g_i\|$

F_f neprekidna?

$|F_f(c_1^1, \dots, c_n^1) - F_f(c_1^2, \dots, c_n^2)| = \left| \|f - \sum_{i=1}^{n} c_i^1 g_i\| - \|f - \sum_{i=1}^{n} c_i^2 g_i\| \right|$

$|\|a\| - \|b\|| \leq \|a - b\|$

$\leq \|(f - \sum_{i=1}^{n} c_i^1 g_i) - (f - \sum_{i=1}^{n} c_i^2 g_i)\|$

$= \|\sum_{i=1}^{n} (c_i^2 - c_i^1) \cdot g_i\|$

$\leq \sum_{i=1}^{n} |c_i^2 - c_i^1| \cdot \|g_i\|$

$\Rightarrow F_f$ neprekidna po c_i (za bilo koje f)

$F_f(c_1, \dots, c_n) \geq \heartsuit \quad ?$

$\|c_1 g_1 + \dots + c_n g_n - f\| \geq \|\underbrace{c_1 g_1 + \dots + c_n g_n}_{?}\| - \|f\|$

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$f=0$: $F_0(c_1,\dots,c_n) = \|\sum c_i g_i\|$ neprekidna

$\|c\|_2 = 1$ $\quad (\|c\|_2^2 = \sum c_i^2)$ ograničen

$\Rightarrow \exists (\tilde c_1,\dots,\tilde c_n) \in$ sferi t.d. $\underbrace{F_0(\tilde c_1,\dots,\tilde c_n)}_{\tilde F} = \min_{c_i} F_0(c_1,\dots,c_n)$

$\tilde F = 0$?

$\tilde F = \|c_1 g_1 + \dots + c_n g_n\| \overset{?}{=} 0$

$(\|x\| = 0 \iff x = 0)$

$c_1 g_1 + \dots + c_n g_n = 0$

1) $c_i = 0$ ↯ $\sum c_i^2 = 1$

2) $\exists c_i \neq 0 \Rightarrow g_i$ lin. zav. ↯

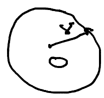
$\Rightarrow \tilde F \neq 0$

$(c_1,\dots,c_n) \neq (0,\dots,0)$:

$$F_0(c_1,\dots,c_n) = \|c_1 g_1 + \dots + c_n g_n\|$$
$$= \|c\|_2 \cdot \left\| \frac{c_1}{\|c\|_2} \cdot g_1 + \dots + \frac{c_n}{\|c\|_2} g_n \right\|$$
$$= \underbrace{\|c\|_2 \cdot F_0\Big(\underbrace{\frac{c_1}{\|c\|_2},\dots,\frac{c_n}{\|c\|_2}}_{= \sum \frac{c_i^2}{\|c\|_2} = 1}\Big)}_{\geq \tilde F}$$
$$\geq \|c\|_2 \cdot \tilde F$$

$(c_1,\dots,c_n) = (0,\dots,0)$:

$F_f(0,\dots,0) = \|f - \sum 0 g_i\| = \|f\|$



neprekidna i u okolini 0, $\|c\|_2 \le r$, $r \geq \frac{2\|f\|}{\tilde F}$

ograničeno kuglom sa poluprečnikom r

$\Rightarrow \exists (c_1^0,\dots,c_n^0) \in$ kugli $\underbrace{F_f(c_1^0,\dots,c_n^0)}_{F^0} = \min_{\|c\| \le r} F_f(c_1,\dots,c_n)$

$\|f\| = F_f(0,\dots,0) \geq F^0$

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$$F_f(a_1, \dots, a_n) = \| a_1 g_1 + \dots + a_n g_n - f \|$$
$$\geq \| a_1 g_1 + \dots + a_n g_n \| - \| f \|$$
$$\geq \|c\|_2 \cdot \tilde{F} - \|f\|$$
$$\geq \delta \cdot \tilde{F} - \|f\|$$
$$\geq \frac{2\|f\|}{\tilde{F}} \cdot \tilde{F} - \|f\|$$
$$= \|f\|$$
$$\geq F^0$$

$\boxtimes$

(T) Ako je prostor R strogo normiran lin. prostor onda je Q_0 jedinstven.

strogo normiran: $\|x + y\| = \|x\| + \|y\| \iff x = \lambda y,\ \lambda \geq 0$
x, y lin. zav.

(D) pps. $\exists\, Q_0^1 \neq Q_0^2$, $\quad Q_0^j = \sum_{i=1}^{n} c_i^j g_i$, $j = 1, 2$

$$E_n(f) = \|f - Q_0^1\| = \|f - Q_0^2\|$$

$E_n(f) \neq 0$: $\quad f - Q_0^1 = f - Q_0^2 = 0 \Rightarrow Q_0^1 = Q_0^2$ ↯

$$\left\| f - \underbrace{\frac{Q_0^1 + Q_0^2}{2}} \right\| = \left\| \frac{f - Q_0^1}{2} + \frac{f - Q_0^2}{2} \right\|$$
$$\leq \frac{1}{2}\|f - Q_0^1\| + \frac{1}{2}\|f - Q_0^2\| \qquad (1)$$
$$= \frac{1}{2} \cdot E_n(f) + \frac{1}{2} E_n(f)$$
$$= E_n(f)$$

$$\frac{Q_0^1 + Q_0^2}{2} = \sum \frac{c_i^1 + c_i^2}{2} g_i = \frac{1}{2} \sum \underbrace{(c_i^1 + c_i^2)}_{c_i^3} g_i = Q_0^3$$

$$\|f - Q_0^3\| \geq E_n(f) = \inf \|f - \sum c_i g_i\|$$

$\Rightarrow E_n \leq \|f - Q_0^3\| \leq E_n \Rightarrow \|f - Q_0^3\| = E_n \qquad (2)$

$\Rightarrow Q_0^3$ također el. najb. aprox

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$$\underbrace{\tfrac{1}{2}\|f-Q_0^1\| + \tfrac{1}{2}\|f-Q_0^2\|}_{} \overset{(1)}{=} E_n(f) \overset{(2)}{=} \left\|f - \frac{Q_0^1+Q_0^2}{2}\right\|$$

$$= \left\|\tfrac{1}{2}(f-Q_0^1) + \tfrac{1}{2}(f-Q_0^2)\right\|$$

Ovo je moguće samo ako je

$$\frac{f-Q_0^1}{2} = \lambda \cdot \frac{f-Q_0^2}{2} \qquad \text{(stroga normiranost)}$$

$\lambda = 1:\quad f - Q_0^1 = f - Q_0^2 \Rightarrow Q_0^1 = Q_0^2 \quad ↯$

$\lambda \neq 1:\quad f - Q_0^1 = \lambda(f - Q_0^2)$

$$= \lambda f - \lambda Q_0^2$$

$$(1-\lambda) f = Q_0^1 - \lambda Q_0^2$$

$$f = \frac{Q_0^1 - \lambda Q_0^2}{1-\lambda} = \sum_{i=1}^{n} \underbrace{\frac{c_i^1 - \lambda c_i^2}{1-\lambda}}_{\text{const}} g_i$$

f je lin. komb. g_i

$\Rightarrow E_n(f) = 0 \quad ↯ \qquad \boxtimes$