SPLAJN

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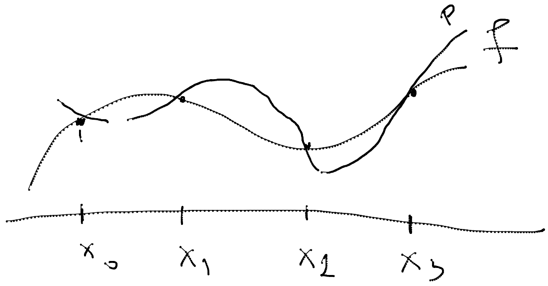
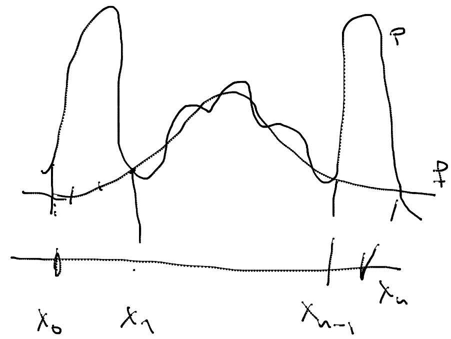
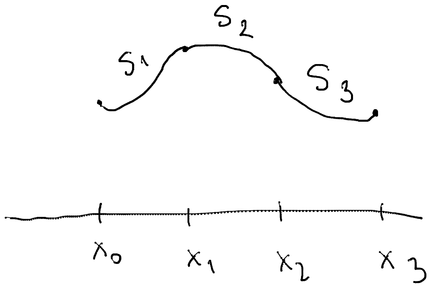


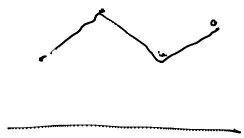




$P_n \quad [x_0, x_n]$

$\downarrow$

$[x_i, x_{i-1}] \quad S_m$



$$\Delta = \{a = x_0 < x_1 < \dots < x_n = b\}$$

Def: Splajn reda m definisan podelom Δ je realna fja na $[a,b]$ koja ima sledeće osobine:

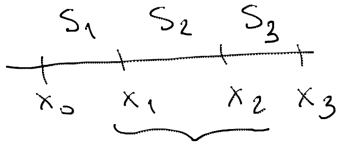
1) $S_\Delta^m(f; x) \in C^{m-1}[a,b]$

2) S_Δ^m je polinom stepena m na svakom od intervala $[x_i, x_{i+1}]$, $i = 0, \dots, n-1$ podele Δ.

x_i:

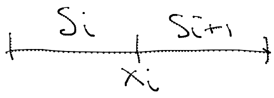
$$\left.\begin{array}{l} S_1(x_i) = S_2(x_i) = f(x_i) \\ S_1'(x_i) = S_2'(x_i) \quad \neq f'(x_i) \\ \vdots \\ S_1^{m-1}(x_i) = S_2^{m-1}(x_i) \end{array}\right\} \forall \text{ unutrašnju tačku}$$

$m = 3$



n + 1 tačka

n intervala



Uslovi interpolacije : x_i : $S_i(x_i) = S_{i+1}(x_i) = f(x_i)$

$$\underset{\text{tačka}}{(n-1)} \cdot \underset{\text{uslov}}{2}$$

x_0, x_n : $\left.\begin{array}{l} S_1(x_0) = f(x_0) \\ S_n(x_n) = f(x_n) \end{array}\right\}$ 2 uslova

$2(n-1)+2 = 2n$ uslova interpolacije

Uslovi glatkosti: x_i unutrašnja: $\left.\begin{array}{l} S_i'(x_i) = S_{i+1}'(x_i) \\ S_i''(x_i) = S_{i+1}''(x_i) \end{array}\right\} 2(n-1)$

Ukupno: $2n + 2(n-1) = \underline{4n-2}$

$S_i(x) = c_{i0} + c_{i1}x + c_{i2}x^2 + c_{i3}x^3$

4 nepoznate

Ukupno nepoznatih: $\underline{\underline{4 \cdot n}}$

nije jedinstveno rnje !

T Neka je podelom Δ intervala $[a,b]$ definisan int. kubni splajn funkcije $f(x) \in C^2[a,b]$ koji zadovoljava jedan od uslova na granici:

a) $S''_\Delta(f;a) = S''_\Delta(f;b) = 0$ pri rodni

b) $f(x)$ i $S_\Delta(f;x)$ su periodične na $[a,b]$ periodičan

c) $f'(a) = S'_\Delta(f;a)$ i $f'(b) = S'_\Delta(f;b)$.

Tada je splajn $S_\Delta(f;x)$ jednoznačno određen i važi

$$|f|^2 \geq |S|^2$$ svojstvo minimalnosti

tj. $|f-S|^2 = |f|^2 - |S|^2 \geq 0$ $|\cdot|$ polunorma !!!

$f=0 \not\Leftrightarrow \|f\|=0$

gde je $|f|^2 = \int_a^b (f''(x))^2 dx$.

$f \not\equiv 0 \Rightarrow |f| = 0$

$y = y(x)$ kriva $K = \dfrac{\|y''\|}{(1+y'^2)^{3/2}}$

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Lema: Ako je $f \in C^2[a,b]$ i S kubni splajn određen podelom Δ

onda je:

$$|f-S|^2 = |f|^2 - |S|^2 - 2\Big[(f'(x)-S'(x))\cdot S''(x)\Big|_a^b - \sum_{i=1}^{n} \underbrace{(f(x)-S(x))}_{\forall x_i = 0,\ f(x_i)=S(x_i)} \cdot S'''(x)\Big|_{x_{i-1}^+}^{x_i^-}\Big]$$

S'''

gde je $g(x)\Big|_{x_{i-1}^+}^{x_i^-} = \lim_{x \to x_i^-} g(x) - \lim_{x \to x_{i-1}^+} g(x)$

Dokaz:

$$|f-S|^2 = \int_a^b [f(x)-S(x)]^2\,dx$$

$$= \underbrace{\int_a^b [f''(x)]^2\,dx}_{|f|^2} - 2\int_a^b f''(x)\cdot S''(x)\,dx + \int_a^b [S''(x)]\,dx$$

$$- \underbrace{\int_a^b [S''(x)]^2\,dx}_{|S|^2} + \int_a^b [S''(x)]^2\,dx$$

$$= |f|^2 - |S|^2 - 2\int_a^b S''(x)\,(f''(x)-S''(x))\,dx$$

$\downarrow$

$x \in [x_{i-1}, x_i]$, $i = 1, \dots, n$

$$\int_{x_{i-1}}^{x_i} (f''-S'')\cdot S''\,dx = \left\{\begin{array}{ll} u = S'', & dv = f''-S'' \\ du = S''', & v = f'-S' \end{array}\right\}$$

$$= (f'-S')\cdot S''\Big|_{x_{i-1}}^{x_i} - \underbrace{\int_{x_{i-1}}^{x_i} (f'-S')\cdot S'''\,dx}_{} = \left\{\begin{array}{ll} u = S''', & dv = f'-S' \\ du = S^{IV}, & v = f-S \end{array}\right.$$

$$= (f'-S')\cdot S''\Big|_{x_{i-1}}^{x_i} - \Big[\underbrace{(f-S)\cdot S'''\Big|_{x_{i-1}}^{x_i}}_{\substack{f(x_i)-S(x_i)=0 \\ f(x_{i-1})-S(x_{i-1})=0}} - \underbrace{\int_{x_{i-1}}^{x_i} (f-S)\cdot \underbrace{S^{IV}}_{}\,dx}_{=0}\Big]$$

$$= (f'-S')\cdot S''\Big|_{x_{i-1}}^{x_i}$$

$$\int_a^b (f'-S')\,S''\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (f'-S')\,S''\,dx = \sum_{i=1}^{n} (f'-S')\cdot S''\Big|_{x_{i-1}}^{x_i}$$

$$= (f'-S')\cdot S''\Big|_a^b \quad \boxtimes$$

Dokaz (T):

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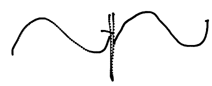
Iz Leme: $\| f-S\|^2 = |f|^2 - |S|^2 - 2\,(f'-S')\cdot S''\Big|_a^b$

a) $S''(a) = S''(b) = 0 \Rightarrow \quad = 0$

b) $f(a) = f(b)$, $S'''(a) = S'''(b)$ } vrednosti prvi izvodi } isti



c) $f'(a) = S'(a)$, $f'(b) = S'(b)$

$f' - S' = 0$

$$\Rightarrow |f-S|^2 = |f|^2 - |S|^2 \geq 0 \Rightarrow |f|^2 \geq |S|^2 \quad ✓$$

$$\underbrace{\int_a^b (f-S)^2\,dx}_{\geq 0}$$

jedinstvenost:

pps. $\exists\; S_1^*(x) \neq S_2^*(x)$

$$\left.\begin{aligned} |S_1^* - S_2^*|^2 &= |S_1^*|^2 - |S_2^*|^2 \geq 0 \\ |S_2 - S_1|^2 &= |S_2|^2 - |S_1|^2 \geq 0 \end{aligned}\right\} \ominus$$

$$0 = 2|S_1|^2 - 2|S_2|^2 \Rightarrow |S_1|^2 = |S_2|^2$$

$$|S_1 - S_2|^2 = \underbrace{|S_1|^2 - |S_2|^2}_{=} = 0$$

$$\int_a^b \underbrace{[S_1'' - S_2'']^2}_{\geq 0} dx = 0 \Rightarrow S_1'' - S_2'' = 0$$

$$\Rightarrow S_1'' = S_2'' \Rightarrow S_1 = S_2 + \underbrace{px + q}$$

$\big(f(a) = S_1(a) = S_2(a)$, $f(b) = S_1(b) = S_2(b)$

$$\left.\begin{aligned} x=a:&\quad f(a) = f(a) + p\cdot a + q \\ x=b:&\quad f(b) = f(b) + p\cdot b + q \end{aligned}\right\} \quad \left.\begin{aligned} pa + q &= 0 \\ p\cdot b + q &= 0 \end{aligned}\right\} \quad \begin{vmatrix} a & 1 \\ b & 1 \end{vmatrix} = a - b \neq 0$$

$\Rightarrow \exists!$ rešenje $\Rightarrow$ trivijalno je jedino $p = q = 0$

$$S_1 = S_2 + 0 + 0$$

$$\boxed{S_1 = S_2} \quad ↯ \qquad \boxtimes$$

Egzistencija

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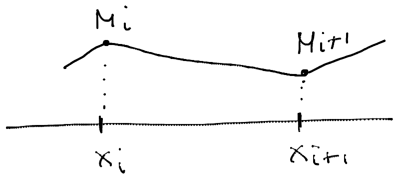
$$\Delta = \{ a = x_0 < \dots < x_n = b \}$$

$$x \in [x_i, x_{i+1}] : \quad S_i = c_{0i}x^3 + c_{1i}x^2 + c_{2i}x + c_{3i} \qquad , i = 0, \dots, n-1$$

$$h_i = x_i - x_{i-1} \qquad , i = 1, \dots, n$$

$$M_i = S''(x_i)$$ MOMENTI splajna



$$S''(x) = M_i \cdot \frac{x_{i+1} - x}{h_{i+1}} + M_{i+1} \frac{x - x_i}{h_{i+1}} \quad , x \in [x_i, x_{i+1}] \qquad / \int$$

$$S'(x) = -M_i \frac{(x_{i+1} - x)^2}{2h_{i+1}} + M_{i+1} \frac{(x - x_i)^2}{2h_{i+1}} + A_i \qquad / \int$$

$$S(x) = M_i \frac{(x_{i+1} - x)^3}{6 \cdot h_{i+1}} + M_{i+1} \frac{(x - x_i)^3}{6h_{i+1}} + A_i(x - x_i) + B_i$$

$$x_i : \quad S(x_i) = f(x_i) = M_i \frac{(x_{i+1} - x_i)^3}{6h_{i+1}} + 0 + 0 + B_i$$

$$= M_i \cdot \frac{h_{i+1}^2}{6} + B_i \quad \Rightarrow \boxed{B_i = f(x_i) - M_i \frac{h_{i+1}^2}{6}}$$

$$x_{i+1} : \quad S(x_{i+1}) = f(x_{i+1}) = 0 + M_{i+1} \frac{(x_{i+1} - x_i)^3}{6h_{i+1}} + A_i(x_{i+1} - x_i) + B_i$$

$$= M_{i+1} \frac{h_{i+1}^2}{6} + h_{i+1} \cdot A_i + B_i$$

$$\Rightarrow A_i = \frac{1}{h_{i+1}} \left(f(x_{i+1}) - M_{i+1} \frac{h_{i+1}^2}{6} - B_i \right)$$

$$= \frac{1}{h_{i+1}} \left(f(x_{i+1}) - M_{i+1} \frac{h_{i+1}^2}{6} - f(x_i) + M_i \frac{h_{i+1}^2}{6} \right)$$

$$\boxed{A_i = \frac{f(x_{i+1}) - f(x_i)}{h_{i+1}} - \frac{h_{i+1}^{\cancel{2}}}{6} (M_{i+1} - M_i)}$$