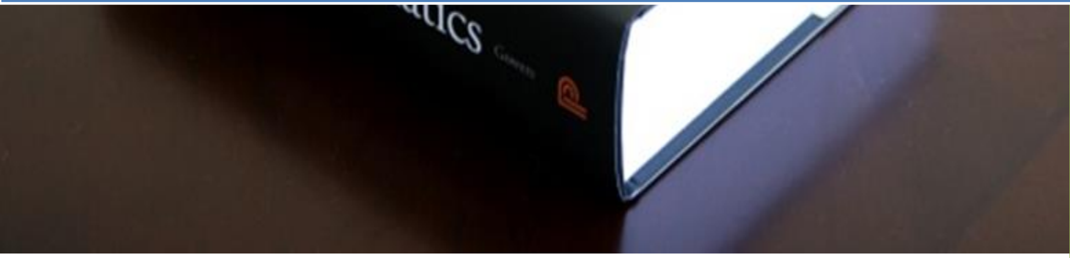
ENGLISH FOR MATHEMATICIANS



**The intensive course for the senior students of
The Institute of Mathematics, Economics and Mechanics**

MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

Odessa I.I.Mechnikov national university
Faculty of Romance and Germanic Philology

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ENGLISH FOR MATHEMATICIANS

(the intensive course for the senior students of The Institute of Mathematics, Economics and Mechanics)

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Р 865 Англійська мова для математиків (інтенсивний курс для студентів математичних спеціальностей Інституту математики, економіки і механіки) = English for mathematicians (the intensive course for the students-mathematicians of The Institute of Mathematics, Economics and Mechanics) / О.А. Румянцева, ОНУ імені І.І. Мечникова. – Одеса, 2015. – 145 с.

Навчальний посібник є сучасним, розробленим з урахуванням досвіду роботи, варіантом інтенсивного курсу англійської мови для студентів і аспірантів, що продовжують вивчення професійної англійської мови в магістратурі або аспірантурі університету. Робота з навчальним посібником дозволяє опанувати англійську мову на рівні, достатньому для практичного використання в професійній діяльності.

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CONTENTS

PREFACE

This course is intended for students of non-English major in the Department of Mathematics. The course aims at developing students' language skills in English context of mathematics with emphasis on reading, translating, speaking and writing. The language content, mainly focuses on: firstly, key points of mathematical terminology and key functions appropriate to this level; secondly, language vocabulary and models that are important for decoding and translating mathematical texts; thirdly, language skills developed as outlined below.

This textbook contains 16 units with a Glossary handbook of mathematical terms and abbreviations designed to provide 34 hours of learning for the second term of senior and graduate students.

Course structural organization. Each unit contemplates the following:

1) Text presentation: the mathematical topics are provided in original context. Thus the first part of the unit includes professionally oriented texts taken from English speaking sources. After the text there is represented a list of mathematical terminology to be learnt.
2) Comprehension questions: Students are guided to the understanding of the professional language, and directed to mastering rules for their own benefit.
3) Practice: Speaking, reading and writing skills as well as grammar exercises are provided to consolidate the active language.
4) Professional skills development: Language is used for realistic purposes. The information given in the texts coincides with the information presented during Calculus studies.
5) Reading and speaking: The texts in units are intergrated with various free speaking activities. They contain the basic data about the history of mathematics and the most prominent mathematicians.
6) Writing: The book is supplied with writing activities. It contains grammar exercises to revise and develop different grammatical aspects.
7) Translating: The translation will encourage students to review their performance and to decide about the priorities for their own future self-study.

Mathematics is the gate and key to science
/Roger Bacon/

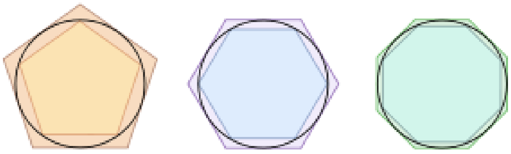
UNIT 1

Text 1. Introduction to Mathematical analysis

Mathematical analysis is a branch of mathematics that includes the theories of differentiation, integration, measure, limits, infinite series, and analytic function.

These theories are usually studied in the context of real and complex numbers and functions. Analysis evolved from calculus, which involves the elementary concepts and techniques of analysis. Analysis may be distinguished from geometry; however, it can be applied to any space of mathematical objects that has a definition of nearness (a topological space) or specific distances between objects (a metric space).

History



Archimedes used the method of exhaustion to compute the area inside a circle by finding the area of regular polygons with more and more sides. This was an early but informal example of a limit, one of the most basic concepts in mathematical analysis.

Mathematical analysis formally developed in the 17th century during the Scientific Revolution, but many of its ideas can be traced back to earlier mathematicians. Early results in analysis were implicitly present in the early days of ancient Greek mathematics. For instance, an infinite geometric sum is implicit in Zeno's paradox of the dichotomy.

Later, Greek mathematicians such as Eudoxus and Archimedes made more explicit, but informal, use of the concepts of limits and convergence when they used the method of exhaustion to compute the area and volume of regions and solids. The explicit use of infinitesimals appears in Archimedes' *The Method of Mechanical Theorems*, a work rediscovered in the 20th century. In Asia, the Chinese mathematician Liu Hui used the method of exhaustion in the 3rd century AD to find the area of a circle. Zu Chongzhi established a method that would later be called Cavalieri's principle to find the volume of a sphere in the 5th century. The Indian mathematician Bhāskara II gave examples of the derivative and used what is now known as Rolle's theorem in the 12th century.

In the 14th century, Madhava of Sangamagrama developed infinite series expansions, like the power series and the Taylor series, of functions such as sine, cosine, tangent and arctangent. Alongside his development of the Taylor series of the trigonometric functions, he also estimated the magnitude of the error terms created by truncating these series and gave a rational approximation of an infinite series. His followers at the Kerala school of astronomy and mathematics further expanded his works, up to the 16th century.

The modern foundations of mathematical analysis were established in 17th century Europe. Descartes and Fermat independently developed analytic geometry, and a few decades later Newton and Leibniz independently developed infinitesimal calculus, which grew, with the stimulus of applied work that continued through the 18th century, into analysis topics such as the calculus of variations, ordinary and partial differential equations, Fourier analysis, and generating functions. During this period, calculus techniques were applied to approximate discrete problems by continuous ones.

In the 18th century, Euler introduced the notion of mathematical function. Real analysis began to emerge as an independent subject when Bernard Bolzano introduced the modern definition of continuity in 1816, but Bolzano's work did not become widely known until the 1870s. In 1821, Cauchy began to put calculus on a firm logical foundation by rejecting the principle of the generality of algebra widely used in earlier work, particularly by Euler. Instead, Cauchy formulated calculus in terms of geometric ideas and infinitesimals. Thus, his definition of continuity required an infinitesimal change in x to correspond to an infinitesimal change in y. He also introduced the concept of the Cauchy sequence, and started the formal theory of complex analysis. Poisson, Liouville, Fourier and others studied partial differential equations and harmonic analysis. The contributions of these mathematicians and others, such as Weierstrass, developed the (ε, δ) - definition of limit approach, thus founding the modern field of mathematical analysis.

In the middle of the 19th century Riemann introduced his theory of integration. The last third of the century saw the arithmetization of analysis by Weierstrass, who thought that geometric reasoning was inherently misleading, and introduced the "epsilon-delta" definition of limit. Then, mathematicians started worrying that they were assuming the existence of a continuum of real numbers without proof. Dedekind then constructed the real numbers by Dedekind cuts, in which irrational numbers are formally defined, which serve to fill the "gaps" between rational numbers, thereby creating a complete set: the continuum of real numbers, which had already been developed by Simon Stevin in terms of decimal expansions. Around that time, the attempts to refine the theorems of Riemann integration led to the study of the "size" of the set of discontinuities of real functions.

Also, "monsters" (nowhere continuous functions, continuous but nowhere differentiable functions, space-filling curves) began to be investigated. In this context, Jordan developed his theory of measure, Cantor developed what is now called naive set theory, and Baire proved the Baire category theorem. In the early 20th century, calculus was formalized using an axiomatic set theory. Lebesgue solved the problem of measure, and Hilbert introduced Hilbert spaces to solve integral equations. The idea of normed vector space was in the air, and in the 1920s Banach created functional analysis.

Important concepts

Metric spaces. In mathematics, a metric space is a set where a notion of distance (called a metric) between elements of the set is defined.

Much of analysis happens in some metric space; the most commonly used are the real line, the complex plane, Euclidean space, other vector spaces, and the integers. Examples of analysis without a metric include measure theory (which describes size rather than distance) and functional analysis (which studies topological vector spaces that need not have any sense of distance).

Formally, A metric space is an ordered pair (M, d) where M is a set and d is a metric on M, i.e., a function $d\colon M \times M \to \mathbb{R}$
such that for any $x, y, z \in M$, the following holds:

1. $d(x,y) = 0$ iff $x = y$ (*identity of indiscernibles*),
2. $d(x,y) = d(y,x)$ (*symmetry*) and
3. $d(x,z) \le d(x,y) + d(y,z)$ (*triangle inequality*) .

By taking the third property and letting $z = x$, it can be shown that $d(x,y) \ge 0$ (*non-negative*).

Sequences and limits

A sequence is an ordered list. Like a set, it contains members (also called *elements, or terms*). Unlike a set, order matters, and exactly the same elements can appear multiple times at different positions in the sequence. Most precisely, a sequence can be defined as a function whose domain is a countable totally ordered set, such as the natural numbers.

One of the most important properties of a sequence is *convergence*. Informally, a sequence converges if it has a *limit*. Continuing informally, a (singly-infinite) sequence has a limit if it approaches some point *x*, called the limit, as *n* becomes very large. That is, for an abstract sequence (a_n) (with *n* running from 1 to infinity understood) the distance between a_n and *x* approaches 0 as $n \to \infty$, denoted

$$\lim_{n\to\infty} a_n = x.$$

Mathematical terminology

differentiation - дифференцирование, отыскание производной
integration – интегрирование, вычисление интеграла
measure – мера; показатель; критерий; масштаб; делитель
infinite series - бесконечный ряд
calculus - 1) исчисление 2) математический анализ (учебная дисциплина, раздел высшей математики)
mathematical object – математический объект
topological space - топологическое пространство; **metrical space** – метрическое пространство
method of exhaustion – метод последовательных элиминаций
regular polygon – правильный многоугольник
limit - 1) предел; граница 2) pl. интервал значений
Zeno's paradox of the dichotomy – парадокс дихотомии Зенона или апория «Дихотомия» (последовательное деление целого на две части).
Zeno of Elea – Зено́н Эле́йский (490 до н. э. - 430 до н. э.), древнегреческий философ, ученик Парменида. Родился в Элее, Лукания. Знаменит своими апори́ями, которыми он пытался доказать противоречивость концепций движения, пространства и множества. Научные дискуссии, вызванные этими парадоксальными рассуждениями, существенно углубили понимание таких фундаментальных понятий, как роль дискретного и непрерывного в природе, адекватность физического движения и его математической модели и др.
Liu Hui – Лю Хуэй известен своими комментариями на «Математику в девяти книгах», которая представляет собой сборник решений математических задач из повседневной жизни. Лю Хуэй опубликовал «Цзю чжан суаньшу» в 263 году со своими комментариями, это старейшая сохранившаяся публикация книги. Самые известные труды Лю Хуэя:

Расчёт числа π методом вписанных правильных многоугольников.
Решение систем линейных уравнений методом, названным впоследствии именем Гаусса.
Расчёт объёма призмы, пирамиды, тетраэдра, цилиндра, конуса и усечённого конуса;
метод неделимых.

Cavalieri's principle – Принцип Кавальери, наиболее полное выражение и теоретическое обоснование метод неделимых получил в работе итальянского математика Бонавентуры Кавальери в ***современном виде:***
Для плоскости: Площади двух фигур с равными по длине хордами всех их общих секущих, параллельных прямой, по одну сторону от которой они лежат, равны.
Для пространства: Объёмы двух тел над плоскостью, с равными по площади сечениями всех общих секущих их плоскостей, параллельных данной плоскости, равны.
Принцип Кавальери явился одним из первых шагов на пути к интегральному исчислению. В частности, используя обозначения бесконечно малых, он доказал теорему, эквивалентную современной формуле: $\int_0^a x^n\,dx = \frac{a^{n+1}}{n+1}.$
Taylor series–ряды Те́йлора, разложение функции в бесконечную сумму степенных функций.
infinite series expansions – разложение бесконечных рядов
power series - степенной ряд
Rolle's theorem – Теорема Ро́лля (теорема о нуле производной): если вещественная функция, непрерывная на отрезке [a; b] и дифференцируемая на интервале (a; b), принимает на концах этого интервала одинаковые значения, то на этом интервале найдётся хотя бы одна точка, в которой производная функции равна нулю.
infinitesimal [ɪnfɪnɪ'tesɪm(ə)l] - бесконечно малая величина
infinitesimal calculus - анализ бесконечно малых величин
sine [saɪn], cosine ['kəusaɪn], tangent ['tænʤ(ə)nt], arctangent – синус, косинус, тангенс, арктангенс
derivative – производная, производная функция

Newton and Leibniz; Descartes and Fermat – Ньютон и Лейбниц; Декарт и Ферма
calculus of variations - вариационное исчисление
Fourier analysis - гармонический анализ, Фурье-анализ
generating function - порождающая функция, производящая функция
Cauchy sequence - фундаментальная последовательность (Коши)
theory of complex analysis – теория комплексного анализа
Siméon Denis Poisson – Симео́н Дени́ Пуассо́н (21 июня 1781 - 25 апреля 1840), знаменитый французский математик, механик и физик. Число научных трудов Пуассона превосходит 300. Они относятся к разным областям чистой математики, математической физики, теоретической и небесной механики.
Joseph Liouville – Жозеф Лиувилль (24 марта 1809 – 8 сентября 1882), французский математик. Систематически исследовал разрешимость ряда задач, дал строгое определение понятию элементарной функции и квадратуры. В частности, исследовал возможность интегрирования заданной функции, алгебраической или трансцендентной, в элементарных функциях, и разрешимость в квадратурах линейного уравнения 2-го порядка.
Jean Baptiste Joseph Fourier – Жан Батист Жозеф Фурье (21 марта 1768 – 16 мая 1830), французский математик и физик. Доказал теорему о числе действительных корней алгебраического уравнения, лежащих между данными пределами (Теорема Фурье 1796). Исследовал, независимо от Ж. Мурайле, вопрос об условиях применимости разработанного Исааком Ньютоном метода численного решения уравнений (1818). Нашёл формулу представления функции с помощью интеграла, играющую важную роль в современной математике. Доказал, что всякую произвольно начерченную линию, составленную из отрезков дуг разных кривых, можно представить единым аналитическим выражением. Его имя внесено в список величайших учёных Франции, помещённый на первом этаже Эйфелевой башни.
harmonic analysis - гармонический анализ
(ε, δ) - definition of limit - "epsilon-delta definition of limit"
theory of integration – теория интегрирования
discontinuities - нарушение последовательности; прерывность;
continuum of real numbers – континуум действительных чисел
Julius Wilhelm Richard Dedekind - Ю́лиус Вильге́льм Ри́хард Дедеки́нд (6 октября 1831 - 12февраля 1916) - немецкий математик, известный работами по общей алгебре и основаниям вещественных чисел.
Dedekind cuts - Дедекиндово сечение
a complete set – полное множество
Simon Stevin - Си́мон Сте́вин (1548 - 1620), фламандский математик, механик и инженер.
decimal expansions - представление [многоразрядного числа или дроби] в десятичной форме примеры: преобразование простой дроби (common fraction) в десятичную, особенно если в результате получается 0,(n), как в случае 1/3 = 0,333333...
Riemann integration – Интегра́л Ри́мана (одно из важнейших понятий математического анализа. Введён Бернхардом Риманом в1854 году, и является одной из первых формализаций понятия интеграла.
arithmetization of analysis – арифметизация анализа
Karl Theodor Wilhelm Weierstrass – Карл Те́одор Вильге́льм Ве́йерштрасс (31 октября 1815 - 19 февраля 1897) - немецкий математик, «отец современного анализа»
limit – лимит, предел
discontinuities of real functions – разрыв непревывности вещественных функций
nowhere continuous function – всюду разрывная функция
nowhere differentiable functions (Weierstrass functions) - функция Вейерштрасса – пример непрерывной функции, нигде не имеющей производной
space-filling curve – заполняющая пространство кривая (is a curve whose range contains the entire 2-dimensional unit square (or more generally an n-dimensional hypercube)

Мари́ Энмо́н Ками́ль Жорда́н (5января 1838 – 22января 1922) – французский математик, известный благодаря своим фундаментальным работам в теории групп и «Курсу анализа».

theory of measure – теория мер (в математическом анализе мера Жордана используется для построения интеграла Римана)

Georg Ferdinand Ludwig Philipp Cantor – Гео́рг Ка́нтор (3 марта 1845, Санкт-Петербург – 6 января 1918, Галле (Заале)) – немецкий математик. Он наиболее известен как создатель теории множеств, ставшей краеугольным камнем в математике. Кантор ввёл понятие взаимно-однозначного соответствиямежду элементами множеств, дал определения бесконечного и вполне-упорядоченного множеств и доказал, что действительных чисел «больше», чем натуральных. Теорема Кантора, фактически, утверждает существование «бесконечности бесконечностей». Он определил понятия кардинальных и порядковых чисел и их арифметику. Его работа представляет большой философский интерес, о чём и сам Кантор прекрасно знал.

naive set theory - наи́вная тео́рия мно́жеств (раздел математики, в котором изучаются общие свойства множеств)

René-Louis Baire – Рене́-Луи́ Бэр, французский математик. Является одним из создателей современной теории вещественных функций и дескриптивной теории множеств. Одной из важнейших работ математика стала теорема Бэра. Бэр также разработал классификацию разрывных функций.

Henri Léon Lebesgue - Анри Леон Лебе́г (28 июня 1875, Бове, департамент Уаза – 26 июля 1941, Париж) – французский математик, член Парижской АН (1922), профессор Парижского университета (с 1910). Наиболее известен как автор теории интегрирования. Интеграл Лебега нашёл широкое применение в теории вероятностей.

David Hilbert – Дави́д Ги́льберт (23 января 1862 – 14 февраля 1943), немецкий математик-универсал, внёс значительный вклад в развитие многих областей математики. В 1910—1920-е годы (после смерти Анри Пуанкаре) был признанным мировым лидером математиков. Гильберт разработал широкий спектр фундаментальных идей во многих областях математики, в том числе теорию инвариантов и аксиоматику евклидовой геометрии. Он сформулировал теорию гильбертовых пространств, одной из основ современного функционального анализа.

Stefan Banach – Сте́фан Ба́нах (30 марта 1892, Краков – 31 августа 1945, Львов) – польский математик, профессор Львовского университета (1924), декан физико-математического факультета этого университета (1939). Член Польской АН и член-корреспондент АН УССР. Один из создателей современного функционального анализа и львовской математической школы.

normed vector space – нормированное векторное пространство

metric space – метрическое пространство

metric – метрика, т.е. функция, определяющая расстояние между двумя точками пространства или двумя элементами множества

distance – 1) расстояние; дистанция 2) интервал; промежуток

real line – вещественная прямая (ось)

complex plane – комплексная плоскость бесконечная двумерная плоскость, служащая для представления комплексных чисел (complex number); образована перпендикулярными действительной (real axis) и мнимой (imaginary axis) осями, на которых откладываются соответственно действительная и мнимая части комплексного числа

Euclidean space – евклидово пространство пространство, в котором местоположение каждой точки задано и расстояния между точками вычисляются как корень квадратный из суммы квадратов разностей координат по каждому измерению. В математике рассматриваются и неевклидовы пространства (non-Euclidean space), где это правило не выполняется.

vector spaces – векторное пространство; **integer** – целое число

complex plane – комплексная плоскость, плоскость комплексной переменной complex plane = complex number plane

ordered pair – упорядоченная пара

Cavalieri's principle (method of indivisibles) – Принцип Кавальери (Метод неделимых) наиболее полное выражение и теоретическое обоснование метод неделимых получил в работе итальянского математика Бонавентуры Кавальери «Геометрия неделимых непрерывных, выведенная из некоего нового подсчёта» :

- Фигуры относятся друг к другу, как все их линии, взятые по любой регуле [базе параллельных], а тела — как все их плоскости, взятые по любой регуле.
- Если два тела имеют одинаковую высоту, и если сечения тел, равноудалённые и параллельные плоскости, на которой те покоятся, всегда останутся в заданном отношении, то и объёмы тел останутся в этом отношении.

В современном виде **для плоскости:** Площади двух фигур с равными по длине хордами всех их общих секущих, параллельных прямой, по одну сторону от которой они лежат, равны. **Для пространства**: Объёмы двух тел над плоскостью, с равными по площади сечениями всех общих секущих их плоскостей, параллельных данной плоскости, равны.

function – фунция; **iff -** тогда и-только тогда

triangle inequality – аксиома треугольника, неравенство треугольника

totally ordered – вполне упорядоченный; **countable -** исчисляемый

Grammar, Lexical, Translation and Speaking Exercises

Task 1. Answer the questions.

1.1. What main problems does Mathematical analysis deal with?

1.2. What sphere of science did analysis evolve from?

1.3. In what way can analysis be distinguished from geometry and in what way can it be applied to a topological space and to a metric space?

1.4. Name the ancient Greek mathematicians:

- who described the method of exhaustion?
- who represented paradox of the dichotomy?
- who used the concepts of limits and convergence?

1.5. What outstanding discoveries in maths were made by Chinese and Indian scientists?

1.5. Describe the most prominent discoveries of mathematicians in times past and conclude about their influence on current conceptions in Mathematical analysis?

1.6. When were established the modern foundations of mathematical analysis?

1.7. What European mathematicians developed such branches as analytic geometry, infinitesimal calculus?

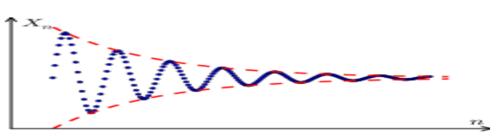
1.8. Give the interpretation and definition to the following notions: the calculus of variations, ordinary and partial differential equations, Fourier analysis and generating functions.

1.9. Name the scientists who introduced the notion of mathematical function, differential equations and harmonic analysis. Expand upon the essence of these mathematical discoveries.

1.10. Give the determinations (in your own words) to the following notions: sequence, limit, complete set, metric spaces.

1.11. Formulate the Rolle's theorem and Cavalieri's principle in modern interpretation.

1.12. What sequence does the given below plot describe? Write the formula of the sequence for real numbers?



Task 2. Translate the definitions into Russianand find the suitable term from the opposite column.

1) a pair of elements a, b having the property that (a, b) = (u, v) if and only if a = u, b = v; 2) a space that has an associated family of subsets that constitute a topology. The relationships between members of the space are mathematically analogous to those between points in ordinary two- and three-dimensional space; 3) a notional line in which every real number is conceived of as represented by a point; 4) a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output. An example is the function that relates each real number *x* to its square (x^2). The output of a function *f* corresponding to an input *x* is denoted by *f*(*x*) (read "*f* of *x*"). In this example, if the input is −3, then the output is 9, and we may write *f*(−3) = 9. The input variable(s) are sometimes referred to as the argument(s) of the function; 5) in logic and mathematics, if and only if is a biconditional logical connective between statements; 6) able to be counted; 7) the positive integers (whole numbers) 1, 2, 3, etc.; 8) a mathematical series whose terms contain ascending positive integral powers of a variable, such as $a_0 + a_1x + a_2x^2 + \dots$	**a) natural numbers** **b) topological space** **c) ordered pair** **d) power series** **e) real line** **f) countable** **g) iff** **h) function**

Task 3. Translate into Russian and make the report on Cavalieri's principle to the group.

In geometry, **Cavalieri's principle**, sometimes called the *method of indivisibles*, named after Bonaventura Cavalieri, is as follows:

2-dimensional case: Suppose two regions in a plane are included between two parallel lines in that plane. If every line parallel to these two lines intersects both regions in line segments (отрезок прямой, линейный сегмент) of equal length, then the two regions have equal areas.

3-dimensional case: Suppose two regions in three-space (solids) are included between two parallel planes. If every plane parallel to these two planes intersects both regions in cross-sections of equal area, then the two regions have equal volumes.

Today Cavalieri's principle is seen as an early step towards integral calculus, and while it is used in some forms, such as its generalization in Fubini's theorem, results using Cavalieri's principle can often be shown more directly via integration. In the other direction, Cavalieri's principle grew out of the ancient Greek method of exhaustion, which used limits but did not use infinitesimals.

Figure 1

Two stacks of coins with the same volume, illustrating Cavalieri's principle in three dimensions

Task 4. Give the lectures to the group on the topics mentioned in the text:

Lecture 1 Series. (Definition, basic properties and example (Zeno's dichotomy and its mathematical representation); **Lecture 2. (ε, δ)-definition of limit**

Task 5. Join the two sentences to make one sentence, beginning with a gerund.

Model: *She's a teacher. It's hard work.*

Being a teacher is hard work / Teaching is hard work.

1. Capital letters are used to name geometrical objects. It is very convenient. 2. You are to classify these quadrilaterals. It requires the knowledge of some properties. 3. We are going to locate this point on the y axis. It will give us the first point on the line. 4. The student intends to divide a circle into a certain number of congruent parts. It will help him to obtain a regular polygon. 5. The base and the altitude of a rectangle are to be multiplied. It will give the product of its dimensions or the area of the rectangle. 6. Don't argue! It's no use. In a crossed quadrilateral, the interior angles on either side of the crossing add up to 720°. 7. Don't deny this fact! It is useless. A square is a quadrilateral, a parallelogram, a rectangle and a rhombus. 8. You are going to divide a heptagon (a 7-sided polygon) into five triangles. Is it any good?

Task 6. Choose the right preposition. Make sensible sentences.

1. Are you interested	on	a. disturbing you.
2. She is very good	of	b. looking after the children.
3. He insisted	to	c. learning foreign languages.
4. I apologize	at	d. having more time for doing things he wants to.
5. The teacher is fed up	in	
6. She succeeded	with	e. understanding this – its too difficult.
7. My friend is keen	for	f. answering our stupid questions.
8. Professor is looking forward		g. studying.
9. This student is not capable		h. considering his solution of the problem.
10. His sister is tired		i. doing sums.
		j. getting good education.

Task 7. Complete the sentences using a gerund as an attribute.

1. I didn't very much like the idea of … .
2. What is the purpose of … ?
3. She had no difficulty (in) … .
4. You have made great progress in … .
5. He was late, and he was afraid of … .
6. Can you imagine the pleasure of … .
7. He always produces the impression of … .
8. I am afraid you do not realize the importance of … .

UNIT 2

Text 2. Main branches of Mathematical Analysis

Real analysis. Real analysis (traditionally, the theory of functions of a real variable) is a branch of mathematical analysis dealing with the real numbers and real-valued functions of a real variable. In particular, it deals with the analytic properties of real functions and sequences, including convergence and limits of sequences of real numbers, the calculus of the real numbers, and continuity, smoothness and related properties of real-valued functions.

Complex analysis, traditionally known as the **theory of functions of a complex variable**, is the branch of mathematical analysis that investigates functions of complex numbers. It is useful in many branches of mathematics, including algebraic geometry, number theory, applied mathematics; as well as in physics, including hydrodynamics, thermodynamics, mechanical engineering, electrical engineering, and particularly, quantum field theory. Complex analysis is particularly concerned with the analytic functions of complex variables (or, more generally, meromorphic functions). Because the separate real and imaginary parts of any analytic function must satisfy Laplace's equation, complex analysis is widely applicable to two-dimensional problems in physics.

Functional analysis. Functional analysis is a branch of mathematical analysis, the core of which is formed by the study of vector spaces endowed with some kind of limit-related structure (e.g. inner product, norm, topology, etc.) and the linear operators acting upon these spaces and respecting these structures in a suitable sense. The historical roots of functional analysis lie in the study of spaces of functions and the formulation of properties of transformations of functions such as the Fourier transform as transformations defining continuous, unitary etc. operators between function spaces. This point of view turned out to be particularly useful for the study of differential and integral equations.

Differential equations. A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. Differential equations play a prominent role in engineering, physics, economics, biology, and other disciplines.

Differential equations arise in many areas of science and technology, specifically whenever a deterministic relation involving some continuously varying quantities (modeled by functions) and their rates of change in space and/or time (expressed as derivatives) is known or postulated. This is illustrated in classical mechanics, where the motion of a body is described by its position and velocity as the time value varies. Newton's laws allow one (given the position, velocity, acceleration and various forces acting on the body) to express these variables dynamically as a differential equation for the unknown position of the body as a function of time. In some cases, this differential equation (called an equation of motion) may be solved explicitly.

Measure theory. A measure on a set is a systematic way to assign a number to each suitable subset of that set, intuitively interpreted as its size. In this sense, a measure is a generalization of the concepts of length, area, and volume. A particularly important example is the Lebesgue measure on a Euclidean space, which assigns the conventional length, area, and volume of Euclidean geometry to suitable subsets of the n-dimensional Euclidean space $\mathbb{R}^n$. For instance, the Lebesgue measure of the interval $[0, 1]$ in the real numbers is its length in the everyday sense of the word – specifically, 1.

Technically, a measure is a function that assigns a non-negative real number or $+\infty$ to (certain) subsets of a set X. It must assign 0 to the empty set and be (countably) additive: the measure of a 'large' subset that can be decomposed into a finite (or countable) number of 'smaller' disjoint subsets, is the sum of the measures of the "smaller" subsets. In general, if one wants to associate a *consistent* size to *each* subset of a given set while satisfying the other axioms of a measure, one only finds trivial examples like the counting measure. This problem was resolved by defining measure only on a sub-collection of all subsets; the so-called *measurable* subsets, which are required to form a σ-algebra. This means that countable unions, countable intersections and complements of measurable subsets are measurable. Non-measurable sets in a Euclidean space, on which the Lebesgue measure cannot be defined consistently, are necessarily complicated in the sense of being badly mixed up with their complement. Indeed, their existence is a non-trivial consequence of the axiom of choice.

Numerical analysis. Numerical analysis is the study of algorithms that use numerical approximation (as opposed to general symbolic manipulations) for the problems of mathematical analysis (as distinguished from discrete mathematics).

Modern numerical analysis does not seek exact answers, because exact answers are often impossible to obtain in practice. Instead, much of numerical analysis is concerned with obtaining approximate solutions while maintaining reasonable bounds on errors. Numerical analysis naturally finds applications in all fields of engineering and the physical sciences, but in the 21st century, the life sciences and even the arts have adopted elements of scientific computations. Ordinary differential equations appear in celestial mechanics (planets, stars and galaxies); numerical linear algebra is important for data analysis; stochastic differential equations and Markov chains are essential in simulating living cells for medicine and biology.

Other topics in mathematical analysis:

- Calculus of variations deals with extremizing functionals, as opposed to ordinary calculus which deals with functions.
- Harmonic analysis deals with Fourier series and their abstractions.
- Geometric analysis involves the use of geometrical methods in the study of partial differential equations and the application of the theory of partial differential equations to geometry.
- Clifford analysis, the study of Clifford valued functions that are annihilated by Dirac or Dirac-like operators, termed in general as monogenic or Clifford analytic functions.

- *p*-adic analysis, the study of analysis within the context of *p*-adic numbers, which differs in some interesting and surprising ways from its real and complex counterparts.
- Non-standard analysis, which investigates the hyperreal numbers and their functions and gives a rigorous treatment of infinitesimals and infinitely large numbers.
- Computable analysis, the study of which parts of analysis can be carried out in a computable manner.
- Stochastic calculus – analytical notions developed for stochastic processes.
- Set-valued analysis – applies ideas from analysis and topology to set-valued functions.
- Convex analysis, the study of convex sets and functions.

Techniques from analysis are also found in other areas such as: physical sciences. The vast majority of classical mechanics, relativity, and quantum mechanics is based on applied analysis, and differential equations in particular. Examples of important differential equations include Newton's second law and the Einstein field equations. Functional analysis is also a major factor in quantum mechanics.

Mathematical terminology

real number – действительное (вещественное) число любое положительное, отрицательное число или нуль; разделяются на рациональные и иррациональные.

function – функция: **exponential function** – экспоненциальная функция; **inverse function** – обратная функция; **linear function** – линейная функция; **trigonometric function** – тригонометрическая функция

sequence - последовательность, ряд

convergence – сближение, конвергенция, схождение в одной точке **Ant: divergence**

sequence of real numbers – последовательность действительных чисел

calculus – 1) исчисление – формальная математическая система, задаваемая множеством базовых символов, множеством синтаксических правил для порождения из базовых элементов произвольных, множеством аксиом (заведомо истинных элементов данного исчисления) и множеством правил вывода (семантических правил), с помощью которых из одних элементов системы порождаются др.; 2) математический анализ (учебная дисциплина, раздел высшей математики)

continuity – непрерывность; преемственность; неразрывность; целостность;

smoothness – гладкость (напр. функции)

real-valued functions – действительная функция

functions of complex numbers – функции комплексных чисел

algebraic geometry – алгебраическая геометрия

number theory – теория чисел, математическая дисциплина, изучающая свойства чисел.

applied mathematics – прикладная математика научная дисциплина, изучающая применение математических методов в других отраслях знаний, в свою очередь делится на ряд направлений

physics, hydrodynamics, thermodynamics – физика, гидродинамика, термодинамика

mechanical engineering and electrical engineering – машиностроение и электротехника

quantum field theory – квантовая теория поля (КТП)

analytic function – аналитическая функция
real and imaginary parts of any analytic function – дествительная и мнимая часть любой аналитической функции
Laplace's equation - уравнение Лапласа
Functional analysis - функциональный анализ
vector spaces - векторное пространство
inner product - скалярное произведение, внутреннее произведение (векторов)
norm - норма вектора (функционал, заданный на векторном пространстве и обобщающий понятие длины вектора или абсолютного значения числа)
topology – топология
linear operators - лине́йный опера́тор (обобщение линейной числовой функции (точнее, функции $y = kx$) на случай более общего множества аргументов и значений
linear map (linear mapping, linear transformation , linear function) - лине́йное отображе́ние
space of functions – функциональное пространство
Fourier transform - Преобразование Фурье ($\mathcal{F}$) — операция, сопоставляющая функции вещественной переменной другую функцию вещественной переменной.
continuous - непрерывный, континуальный, неразрывный
unitary - унитарный; единичный; однократный
differential and integral equations – дифференциальные и интегральные уравнения
mathematical equation – математическое уравнение
variable – переменная, переменная величина
derivatives of various orders – производные различного порядка
engineering – инженерное дело, **economics** – экономика, **biology** – биология
Newton's laws (of motion) - законы (движения) Ньютона
equation of motion – уравнение движения, динамическое уравнение
set – множество, **subset** - подмножество
Lebesgue measure - лебегова мера
Euclidean space - евклидово пространство пространство, в котором местоположение каждой точки задано и расстояния между точками вычисляются как корень квадратный из суммы квадратов разностей координат по каждому измерению. В математике рассматриваются и неевклидовы пространства (non-Euclidean space), где это правило не выполняется.
length - длина; расстояние; отрезок; долгота; **area** – площадь; **volume** – объем
Euclidean geometry – евклидова геометрия
interval – интервал; промежуток времени; отрезок; расстояние
$+\infty$ **and** $-\infty$ (positive infinity and negative infinity) – $+\infty$ (положительная бесконечность) и $-\infty$ (отрицательная бесконечность)
empty set – пустое множество (множество, не содержащее ни одного элемента)
counting measure - считающая мера (мера, сосредоточенная на множестве целых чисел и равная для каждого из них единице)
σ**-algebra (sigma-algebra)** – σ-алгебра (си́гма-а́лгебра), т.е. алгебра множеств, замкнутая относительно операции счётного объединения. Сигма-алгебра играет важнейшую роль в теории меры и интегралов Лебега, а также в теории вероятностей.
countable – исчисляемый
union - объедине́ние мно́жеств (су́мма или соедине́ние) в теории множеств - множество, содержащее в себе все элементы исходных множеств. Объединение двух множеств A и B обычно обозначается $A \cup B$, но иногда можно встретить запись в виде суммы $A + B$.

intersections - пересече́ние мно́жеств в теории множеств - это множество, которому принадлежат те и только те элементы, которые одновременно принадлежат всем данным множествам.
complements - разность двух множеств – это теоретико-множественная операция, результатом которой является множество, в которое входят все элементы первого множества, не входящие во второе множество. Обычно разность множеств A и B обозначается как $A \setminus B$, но иногда можно встретить обозначение $A - B$ и $A \sim B$.
Non-measurable sets – неисчисляемые множества
axiom of choice - аксиомой выбора называется следующее высказывание теории множеств: для всякого семейства X непустых множеств существует функция f, которая каждому множеству семейства сопоставляет один из элементов этого множества. Функция f называется функцией выбора для заданного семейства.
Numerical analysis - численный анализ – научное направление, изучающее алгоритмы решения задач непрерывной математики (в отличие от дискретной математики (discrete mathematics))
algorithm - алгоритм (программа решения математических либо других задач, предписывающая, какие действия и в какой последовательности необходимо предпринять для получения требуемого результата)
approximation – приближение; аппроксимация; приблизительное соответствие
symbolic manipulations (computer algebra, symbolic computation or algebraic computation) - символьные вычисления - это преобразования и работа с математическими равенствами и формулами как с последовательностью символов, компьютерная алгебра (в отличие от численных методов) занимается разработкой и реализацией аналитических методов решения математических задач на компьютере и предполагает, что исходные данные, как и результаты решения, сформулированы в аналитическом (символьном) виде.
discrete mathematics - дискретная математика охватывает такие направления, как комбинаторный анализ, теория графов, теория управляющих систем, теория функциональных систем, криптография, теория кодирования, вероятностные задачи дискретной математики, алгоритмы и анализ их сложности, комбинаторные и вычислительные задачи теории чисел и алгебры
ordinary differential equations - обыкнове́нные дифференциа́льные уравне́ния (ОДУ) — это дифференциальные уравнения для функции от одной переменной.
celestial mechanics - механика небесных тел
numerical linear algebra – линейная алгебра
stochastic differential equation – стохастическое дифференциальное уравнение (СДУ) – дифференциальное уравнение, в котором один член или более имеют стохастическую природу, то есть представляют собой стохастический процесс (т.е. случайный процесс).
Markov chain – це́пь Ма́ркова, т.е. последовательность случайных событий с конечным или счётным числом исходов, характеризующаяся тем свойством, что, говоря нестрого, при фиксированном настоящем будущее независимо от прошлого.
Calculus of variations – вариационное исчисление
extremized function - экстремизованная функция
calculus – 1) исчисление; дифференциальное исчисление; интегральное исчисление; 2) математический анализ (учебная дисциплина, раздел высшей математики)
Harmonic analysis - гармонический анализ
Fourier series - ряд Фурье

Geometric analysis – геометрический анализ
partial differential equation – дифференциальное уравнение в частных производных (частные случаи также известны как уравнения математической физики, УМФ) – дифференциальное уравнение, содержащее неизвестные функции нескольких переменных и их частные производные.
Clifford analysis – анализ Клиффорда
***p*-adic analysis** – р-адический анализ
Non-standard analysis - нестандартный анализ
hyperreal numbers - гипервещественное число
rigorous treatment – точная трактовка
infinitesimals - бесконечно малая величина
Stochastic calculus – стохастическое исчисление
Set-valued analysis – анализ многозна́чных фу́нкций
multivalued function (multifunction, many-valued function, set-valued function, set-valued map, point-to-set map, multi-valued map, multimap) – многозначная функция – обобщение понятия функции, допускающее наличие нескольких значений функции для одного аргумента
Convex analysis – выпуклый анализ
Einstein field equation - уравне́ния Эйнште́йна – уравнение гравитационного поля в общей теории относительности, связывающие между собой метрику искривлённого пространства-времени со свойствами заполняющей его материи.

Grammar, Lexical, Translation and Speaking Exercises

Task 1. Answer the questions.

1. What mathematical notions does the Real analysis deal with?
2. What types of functions does the Complex analysis concerned with?
3. Describe the historical roots of functional analysis.
5. What kind of disciplines do the differential equations play a prominent role in?
6. Referring to the measure theory how can the measure of a 'large' subset be decomposed into?
7. What fields does the Numerical analysis find its applications in?
8. Enumerate the basic forms of Mathematical Analyses and expand on their principles.

Task 2. Ask the special questions.

1. Some properties are established by way of reasoning (how). 2. Geometry is concerned with the properties and relationships of figures in space (what ... with). 3. Some figures such as cubes and spheres have three dimensions (how many). 4. Many discoveries were made in the nineteenth century (when). 5. The truth of non-mathematical propositions in real life is much less certain (where). 6. The given proposition and its converse can be stated as follows (in what way). 7. Pure mathematics deals with the development of knowledge for its own purpose and need (what ... with). 8. Carl Gauss proved that every algebraic equation had at least one

root (who). 9. There are three words having the same meaning (how many). 10. The given definition corresponds to the idea of uniqueness (what).

Task 3. Translate the definitions into Russian and find the suitable term from the opposite column.

1. A number (symbol i) whose square equals a real negative number. These numbers were invented to allow equations to be solved when they have no real roots. For example, 1 has two real square roots, +1 and -1. The equation $x^2 = 1$ thus has two real roots, x = 1 and x = -1. The number -1 has no real square roots, so the equation $x^2 = -1$ has no real roots. However, the 'imaginary' number, denoted by i, allows the equation $x^2 =-1$ to have two imaginary roots, x = i and x =-i. By convention i always precedes any coefficient other than 1 or -1.	a) applied mathematics b) differential calculus and integral calculus c) dot product or scalar product d) number theory e) divergent, convergent f) infinitesimal g) imaginary number
2. In mathematics, the limit of a sequence is the value that the terms of a sequence "tend to". If such a limit exists, the sequence is called ____. A sequence which does not converge is said to be ____.	
3. The branch of mathematics that deals with the properties and relationships of numbers, especially the positive integers is called __________ .	
4. A scalar function of two vectors, equal to the product of their magnitudes and the cosine of the angle between them, also called _________.	
5. The branch of mathematics that deals with the finding and properties of derivatives and integrals of functions, by methods originally based on the summation of infinitesimal differences. The two main types are_________.	
6.The abstract science of number, quantity, and space, either as abstract concepts (pure mathematics), or as applied to other disciplines such as physics and engineering is called ______.	
7. Extremely small. However small a number other than zero may be, it is always possible to find another even closer to zero. The derivative of a continuous function considers the limit to which the ratio between changes in a function and changes in its argument tends as both changes become infinitesimally small.	

Task 4. Translate the sentences according to the models.

Model 1. *There are various ways of evaluating formulae. – Существуют различные способы вычисления формул.*

1. There are a lot of important theorems in this book. 2. There are sets containing no elements. 3. There has been recently developed a new method of proving the theorem. 4. There are many measurements to be made. 5. There weren't any problems with my term paper last year. 6. There will be enough work for everybody at the next conference.

Model 2. *There exist a lot of equivalent relations. – Существует много эквивалентных отношений.*

1. There exists no difference between these two expressions. 2. There exists at least one element in a non-empty set. 3. There exist some important statements in the article. 4. There exist many different ways of defining a circle. 5. There exist no solutions to the problem presented.

Model 3. *To a pair of numbers there corresponds a point in the plane. – Паре чисел соответствует точка на плоскости.*

1. To a linear equation there corresponds a straight line in the Euclidean space. 2. To a point in three dimensional space there correspond its three coordinates. 3. To each number in X there corresponds a unique element in Y. 4. To any two objects a, b there corresponds a new object. 5. If to each member x of a set there corresponds one value of a variable y, then y is a function of x.

Task 5. Substitute the correct mathematical terms and translate the sentences.

In mathematics, a function space is a set of functions of a given kind from a set X to a set Y. Function spaces appear in various areas of mathematics:

In (теория множества), the set of functions from X to Y may be denoted $X \to Y$ or Y^X. As a special case, the (множество всех подмножеств, булеан множества) of a set X may be identified with the set of all functions from X to $\{0, 1\}$, denoted 2^X. The set of (биекции) from X to Y is denoted $X \leftrightarrow Y$. The factorial notation $X!$ may be used for permutations of a single set X.

(set theory, power set, bijections)

In (линейной алгебре) the set of all (линейных преобразований) from (векторного пространста) V to another one, W, over the same (поле) , is itself a vector space (with the natural definitions of 'addition of functions' and 'multiplication of functions by scalars': this vector space is also over the same field as that of V and W.);

(linear algebra, linear transformations, vector space, field)

In (функциональный анализ) the same is seen for continuous linear transformations, including (топология векторного пространства) and many of the major examples are function spaces carrying a topology; the best known examples include (гильбертово пространство и банахаво пространство).

(Hilbert spaces and Banach spaces, functional analysis, topologies on the vector spaces)

In (функциональный анализ) the set of all functions from the (натуральные числа) to some set X is called a (пространство последовательностей) . It consists of the set of all possible (последовательности) of elements of X.

(functional analysis, natural numbers, sequence space, sequences)

Task 6. Complete these sentences by putting the verb in brackets into the Present Simple or the Present Continuous.

To solve the problem of gravitation, scientists …………… (consider) time-space geometry in a new way nowadays.

Quantum rules …………… (obey) in any system.

We …………… (use) Active Server for this project because it ……… (be) Web-based.

Scientists …………… (trace and locate) the subtle penetration of quantum effects into a completely classical domain.

Commonly we ………… (use) C + + and JavaScript.

At the moment we …………… (develop) a Web–based project.

Its domain ………… (begin) in the nucleus and ………… (extend) to the solar system.

Right now I …………. (try) to learn how to use Active Server properly.

Task 7. Put “can”, “can not”, ”could”, ”could not” into the following sentences.

Parents are finding that they ………….. no longer help their children with their arithmetic homework.

The solution for the construction problems …………… be found by pure reason.

The Greeks …………….. solve the problem notbecause they were not clever enough, but because the problem is insoluble under the specified conditions.

Using only a straight-edge and a compass the Greeks …………. easily divide any line segment into any number of equal parts.

Web pages…………. offer access to a world of information about and exchange with other cultures and communities and experts in every field.

Task 8. Answer the questions.

Do you know the adjective of the noun “algebra”?

Can you name a new division of algebra?

What is your favourite field in modern maths?

Why do you like studying maths?

What basic problems do the following fields of algebra – linear algebra, Lie group, Boolean algebra, homological algebra, vector algebra, matrix algebra – deal with?

UNIT 3

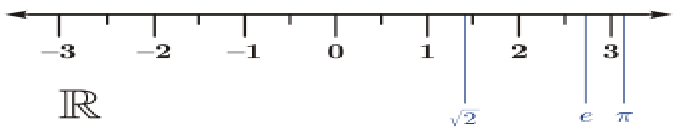
Text 3. Real numbers (Part I)

$\mathbb{R}$ is f symbol of the set of real numbers. In mathematics, a real number is a value that represents a quantity along a continuous line.

The real numbers include all the rational numbers, such as the integer −5 and the fraction 4/3, and all the irrational numbers such as √2 (1.41421356…, the square root of two, an irrational algebraic number) and π (3.14159265…, a transcendental number).

Real numbers can be thought of as points on an infinitely long line called the number line or real line, where the points corresponding to integers are equally spaced. Any real number can be determined by a possibly infinite decimal representation such as that of 8.632, where each consecutive digit is measured in units one tenth the size of the previous one.

The real line can be thought of as a part of the complex plane, and complex numbers include real numbers.



Real numbers can be thought of as points on an infinitely long number line.
These descriptions of the real numbers are not sufficiently rigorous by the modern standards of pure mathematics. The discovery of a suitably rigorous definition of the real numbers – indeed, the realization that a better definition was needed – was one of the most important developments of 19th century mathematics. The currently standard axiomatic definition is that real numbers form the unique Archimedean complete totally ordered field (R;+;·; <), up to an isomorphism, whereas popular constructive definitions of real numbers include declaring them as equivalence classes of Cauchy sequencesof rational numbers, Dedekind cuts, or certain infinite "decimal representations", together with precise interpretations for the arithmetic operations and the order relation. These definitions are equivalent in the realm of classical mathematics.

The reals are uncountable; that is, while both the set of all natural numbers and the set of all real numbers are infinite sets, there can be no one-to-one function from the real numbers to the natural numbers: the cardinality of the set of all real numbers (denoted $\mathfrak{c}$ and called cardinality of the continuum) is strictly greater than the cardinality of the set of all natural numbers (denoted $\aleph_0$). The statement that there is no subset of the reals with cardinality strictly greater than $\aleph_0$ and strictly smaller than $\mathfrak{c}$ is known as the continuum hypothesis. It is known to be neither provable nor refutable using the axioms of Zermelo–Fraenkel set theory, the standard foundation of modern mathematics, provided ZF set theory is consistent.

History. Simple fractions have been used by the Egyptians around 1000 BC; the Vedic "Sulba Sutras" ("The rules of chords") in, 600 BC, include what may be the first "use" of irrational numbers. The concept of irrationality was implicitly accepted by early Indian mathematicians since Manava (750–690 BC), who were aware that the square roots of certain numbers such as 2 and 61 could not be exactly determined. Around 500 BC, the Greek mathematicians led byPythagoras realized the need for irrational numbers, in particular the irrationality of the square root of 2. The Middle Ages brought the acceptance of zero, negative, integral, and fractional numbers, first by Indian and Chinese mathematicians, and then by Arabi mathematicians, who were also the first to treat irrational numbers as algebraic objects, which was made possible by the development of algebra. Arabic mathematicians merged the concepts of "number" and "magnitude" into a more general idea of real numbers. The Egyptian mathematician Abū Kāmil Shujā ibn Aslam was the first to accept irrational numbers as solutions to quadratic equations or as coefficients in an equation, often in the form of square roots, cube roots and fourth roots. In the 16th century, Simon Stevin created the basis for modern decimal notation, and insisted that there is no difference between rational and irrational numbers in this regard.

In the 17th century, Descartes introduced the term "real" to describe roots of a polynomial, distinguishing them from "imaginary" ones. In the 18th and 19th centuries there was much work on irrational and transcendental numbers. Johann Heinrich Lambert (1761) gave the first flawed proof that π cannot be rational; Adrien-Marie Legendre (1794) completed the proof, and showed that π is not the square root of a rational number. Paolo Ruffini (1799) and Niels Henrik Abel (1842) both constructed proofs of the Abel–Ruffini theorem: that the general quintic or higher equations cannot be solved by a general formula involving only arithmetical operations and roots.

Évariste Galois (1832) developed techniques for determining whether a given equation could be solved by radicals, which gave rise to the field of Galois theory. Joseph Liouville (1840) showed that neither e nor e^2 can be a root of an integer quadratic equation, and then established the existence of transcendental numbers, the proof being subsequently displaced by Georg Cantor (1873). Charles Hermite (1873) first proved that e is transcendental, and Ferdinand von Lindemann (1882), showed that π is transcendental. Lindemann's proof was much simplified by Weierstrass (1885), still further by David Hilbert (1893), and has finally been made elementary by Adolf Hurwitz and Paul Gordan. The development of calculus in the 18th century used the entire set of real numbers without having defined them cleanly. The first rigorous definition was given by Georg Cantor in 1871. In 1874 he showed that the set of all real numbers is uncountably infinite but the set of all algebraic numbers is countably infinite. Contrary to widely held beliefs, his first method was not his famous diagonal argument, which he published in 1891.

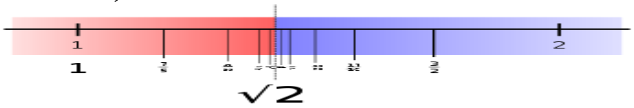
Mathematical terminology

real number - действительное (вещественное) число
rational number/irrational number – рациональное число/иррациональное число
integer - целое число
fraction – дробь; дробное число
square root of – квадратный корень
π (transcendental number) – трансцендентное число
number line (real line) – числовая прямая, [вещественная] цифровая ось
decimal representation – десятичное представление (запись числа в десятичной системе счисления)
complex plane – комплексная плоскость, бесконечная двумерная плоскость, служащая для представления комплексных чисел (complex number); образована перпендикулярными действительной (real axis) и мнимой (imaginary axis) осями, на которых откладываются соответственно действительная и мнимая части комплексного числа
totally ordered field – вполне упорядоченное поле
Isomorphism – изоморфизм (свойство объектов некоторой совокупности иметь однотипную внутреннюю структуру)
equivalence classes – классы эквивалентности
Cauchy sequences – последовательность Коши
Dedekind cuts – дедекиндово сечение
uncountable – неисчисляемый, несчётный (о множестве)
infinite set – бесконечное множество (ant: finite set)
one-to-one function – взаимно однозначная функция
cardinality of the set – мощность (множества), количество элементов множества
continuum – континуум, абсолютно непрерывный объект; сплошная среда
continuum hypothesis – континуум-гипотеза
Zermelo–Fraenkel set theory – теория множеств Цермело-Френкеля с аксиомой выбора (обозначается ZFC), самая распространенная аксиоматическая теория множеств
consistent – непротиворечивый, совместимый, состоятельный (напр. об оценке)
simple fraction – простая дробь
Vedic "Sulba Sutras" – ведийские шу́льба-су́тры – это афоризмы (высказывания) являются единственным источником по индийской математике эпохи Вед, их содержание касается геометрических проектов и задач, относящихся к прямолинейным фигурам, их комбинациям и трансформациям, квадратуре круга, а также алгебраических и арифметических решений данных задач
Pythagoras [piˈthagərəs], [pʌɪˈθagərəs] - Пифагор
negative / positive number – отрицательное / положительное число
integral – целое число

fractional number – дробное число
magnitude – величина; абсолютная величина, значение, модуль
quadratic equations – квадратное уравнение, уравнение второй степени уравнение вида $Ax^2 + Bx + C=0$, где A не равно нулю
coefficient – коэффициент; множитель
equation – уравнение; равенство
cube root – кубический корень, корень третьей степени
fourth root – корень четвёртой степени
decimal notation – десятичная система исчисления
Descartes [ˈdeɪˌkɑrt] – Рене́ Дека́рт (1596 - 1650), французский философ, математик, механик, физик и физиолог, создатель аналитической геометрии и современной алгебраической символики, автор метода радикального сомнения в философии, механицизма в физике.
quintic - 1) уравнение пятой степени; полином 5-ой степени 2) в пятой степени;
uncountably infinite – несчётно-бесконечный
countably infinite – счётно бесконечный
diagonal argument – диагональное доказательство Кантора

Grammar, Lexical, Translation and Speaking Exercises

Task 1. Find the correspondence between English and Russian definitions.

1. Последовательность Коши – последовательность точек метрического пространстватакая, что для любого заданного расстояния существует элемент последовательности, начиная с которого все элементы последовательности находятся друг от друга на расстоянии менее, чем заданное.	a) Galois theory, named after Évariste Galois, provides a connection between field theory and group theory. Using Galois theory, certain problems in field theory can be reduced to group theory, which is in some sense simpler and better understood.
2. Тео́рия Галуа́ – раздел алгебры, позволяющий переформулировать определенные вопросы теории полей на языке теории групп, делая их в некотором смысле более простыми.	b) The continuum hypothesis is a hypothesis about the possible sizes of infinite sets. It states: there is no set whose cardinality is strictly between that of the integers and the real numbers.
3. Конти́нуум-гипо́тезу можно сформулировать следующим образом: любое бесконечное подмножество континуума является либо счётным, либо континуальным. Другими словами, мощность континуума — наименьшая, превосходящая мощность счетного множества, и «промежуточных» мощностей между счетным множеством и континуумом нет.	c) Dedekind cut is a partition of the rational numbers into two non-empty parts A and B, such that all elements of A are less than all elements of B, and A contains no greatest element. Dedekind cuts are one method of construction of the real numbers. Dedekind used his cut to construct the irrational, real numbers. 
4. Дедекиндово сечение – один из способов построения вещественных чисел из рациональных. Множество вещественных чисел определяется как множество дедекиндовых сечений. На них возможно продолжить операции сложения и умножения.	d) Cauchy sequence is a sequence whose elements become arbitrarily close to each other as the sequence progresses. More precisely, given any small positive distance, all but a finite number of elements of the sequence are less than that given distance from each other.

Task 2. Turn direct speech into reported speech.

1. Plato advised, "The principal men of our state must go and learn arithmetic, not as amateurs, but they must carry on the study until they see the nature of numbers with the mind only." 2. Descartes, father of modernism, said, "All nature is a vast geometrical system. Thus all the phenomena of nature are explained and some demonstration of them can be given." 3. In Descartes's words, "You give me extension and motion then I'll construct the universe." 4. The often repeated motto on the entrance to Plato's Academy said, "None ignorant of geometry enter here." 5. J. Kepler affirmed: "The reality of the world consists of its maths relations. Maths laws are true cause of phenomena. " 6. I. Newton said, "I don't know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself now and then by finding a smoother pebble or a prettier shell than usual; whist the great ocean of truth lay all undiscovered before me. If I saw a little farther than others, it is because I stood on the shoulders of giants".

Task 3. Choose the correct variant of translation.

1. We thought that you were going to enter an institute.

a) Мы думали, что вы собираетесь поступить в институт.

b) Мы думали, что вы собирались поступить в институт.

c) Мы думаем, что вы собираетесь идти в институт.

2. Scientists use mathematical formulas to express their findings precisely.

a) Ученые используют математические формулы, чтобы описать свои определения.

b) Ученые используют математические формулы для точного выражения своих находок.

c) Ученые используют математические формулы, чтобы точно выразить полученные данные.

3. Where there is a choice of two expressions, we should always choose the more accurate one.

a) Там, где существует выбор из двух выражений, нам всегда следует выбирать более точное выражение.

b) Там, где есть выбор из двух выражений, мы всегда выберем более точное выражение.

c) Там, где есть выбор из двух выражений, мы бы всегда выбирали более точное выражение.

4. They are likely to have taken a wrong turning in their assumption that all men and women think alike.

a) Они приняли справедливое предположение о том, что не все мужчины и женщины думают одинаково.

b) Вероятно, они ошибочно предположили, что все мужчины и женщины думают одинаково.
c) Они полностью отклонились от своего первоначального допущения, что все мужчины и женщины думамают подобным образом.
5. Very often a proposition is so worded that it requires thought to state the converse proposition correctly.
a) Очень часто утверждение формулируется таким образом, что нужно как следует подумать, чтобы сформулировать обратное утверждение правильно.
b) Зачастую утверждение составляется так, что требуется поразмыслить, чтобы правильно заявить об обратном утверждении.
c) Очень часто утверждение выражается так, что оно требует размышления над правильной формулировкой обратного утверждения.

Task 4. Find Russian equivalents to the English terms.

to reduce a fraction	составная дробь
common fraction; simple fraction; vulgar fraction	десятичная дробь
complex fraction; compound fraction	неправильная дробь
improper fraction	несократимая дробь
irreducible fraction	правильная дробь
proper fraction	простая дробь
decimal fraction	сокращать дробь

Task 5. Translate.

A fraction is a number that can be expressed as a proportion of two whole numbers. For example, 1/2 and 1/3 are both fractions. The students had a grasp of decimals, percentages and fractions. We can enjoy the infinite number of stars in the universe.The use of parentheses will indicate how the result was obtained. Will there be a remainder if you divide 31 by 7? Are subtraction and addition inverse operations?
Are division and multiplication inverse operations? Any operation is called a binary operation when it is applied to only two numbers at a time and gives a single result.

Прежде чем мы решим данное уравнение, мы должны проделать следующие действия. Проиллюстрируйте этот закон каким-нибудь примером. Двоичная система исчисления имеет некоторые преимущества. Каковы недостатки этой системы? Сумма двух натуральных чисел есть тоже натуральное число. Результат умножени называется произведением. Следовательно, мы принимаем свойство замкнутости как аксиому без какого-либо доказательства.

UNIT 4

Text 4. Real numbers (Part II)

Definition. The real number system $(\mathbb{R}; +; \cdot; <)$ can be defined axiomatically up to an isomorphism. There are also many ways to construct "the" real number system, for example, starting from natural numbers, then defining rational numbers algebraically, and finally defining real numbers as equivalence classes of their Cauchy sequences or as Dedekind cuts, which are certain subsets of rational numbers. Another possibility is to start from some rigorous axiomatization of Euclidean geometry (Hilbert, Tarski etc.) and then define the real number system geometrically. From the structuralist point of view all these constructions are on equal footing.

Axiomatic approach. Let $\mathbb{R}$ denote the set of all real numbers. Then: The set $\mathbb{R}$ is a field, meaning that addition and multiplication are defined and have the usual properties. The field $\mathbb{R}$ is ordered, meaning that there is a total order $\geq$ such that, for all real numbers x, y and z: if $x \geq y$ then $x + z \geq y + z$; if $x \geq 0$ and $y \geq 0$ then $xy \geq 0$.

The order is Dedekind-complete; that is, every non-empty subset S of $\mathbb{R}$ with an upper bound in $\mathbb{R}$ has a least upper bound (also called supremum) in $\mathbb{R}$.

The last property is what differentiates the reals from the rationals. For example, the set of rationals with square root less than 2 has a rational upper bound (e.g., 1.5) but no rational least upper bound, because the square root of 2 is not rational.

The real numbers are uniquely specified by the above properties. More precisely, for given any two Dedekind-complete ordered fields $\mathbb{R}_1$ and $\mathbb{R}_2$, there exists a unique field isomorphism from $\mathbb{R}_1$ to $\mathbb{R}_2$, allowing us to think of them as essentially the same mathematical object.

Construction from the rational numbers. The real numbers can be constructed as a completion of the rational numbers in such a way that a sequence defined by a decimal or binary expansion like (3; 3.1; 3.14; 3.141; 3.1415; …) converges to a unique real number, in this case π.

Basic properties. A real number may be either rational or irrational; either algebraic or transcendental; and either positive, negative, or zero. Real numbers are used to measure continuous quantities. They may be expressed by decimal representations that have an infinite sequence of digits to the right of the decimal point; these are often represented in the same form as 324.823122147… The ellipsis (three dots) indicates that there would still be more digits to come. More formally, real numbers have the two basic properties of being an ordered field, and having the least upper bound property. The first says that real numbers comprise a field, with addition and multiplication as well as division by non-zero numbers, which can be totally ordered on a number line in a way compatible with addition and multiplication. The second says that, if a non-empty set of real numbers has an upper bound, then it has a real least upper bound. The second condition distinguishes the real numbers from the rational numbers: for example, the set of rational numbers whose square is less than 2 is a set with an upper bound 1.5 but no (rational) least upper bound: hence the rational numbers do not satisfy the least upper bound property.

Completeness of the real numbers. A main reason for using real numbers is that the reals contain all limits. More precisely, every sequence of real numbers having the property that consecutive terms of the sequence become arbitrarily close to each other necessarily has the property that after some term in the sequence the remaining terms are arbitrarily close to some specific real number. In mathematical terminology, this means that the reals are complete (in the sense of metric spaces or uniform spaces, which is a different sense than the Dedekind completeness). This is formally defined in the following way:

A sequence (x_n) of real numbers is called a *Cauchy sequence* if for any $\varepsilon > 0$ there exists an integer N (possibly depending on ε) such that the distance $|x_n - x_m|$ is less than ε for all n and m that are both greater than N. In other words, a sequence is a Cauchy sequence if its elements x_n eventually come and remain arbitrarily close to each other.

A sequence (x_n) *converges to the limit* x if for any $\varepsilon > 0$ there exists an integer N (possibly depending on ε) such that the distance $|x_n - x|$ is less than ε provided that n is greater than N. In other words, a sequence has limit x if its elements eventually come and remain arbitrarily close to x. Notice that every convergent sequence is a Cauchy sequence.

The converse is also true: *Every Cauchy sequence of real numbers is convergent to a real number. That is, the reals are complete.*

Note that the rationals are not complete. For example, the sequence (1; 1.4; 1.41; 1.414; 1.4142; 1.41421…), where each term adds a digit of the decimal expansion of the positive square root of 2, is Cauchy but it does not converge to a rational number. (In the real numbers, in contrast, it converges to the positive square root of 2.)

The existence of limits of Cauchy sequences is what makes calculus work and is of great practical use. The standard numerical test to determine if a sequence has a limit is to test if it is a Cauchy sequence, as the limit is typically not known in advance.

For example, the standard series of the exponential function:

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to a real number because for every x the sums $\sum_{n=N}^{M} \frac{x^n}{n!}$ can be made arbitrarily small by choosing *N* sufficiently large. This proves that the sequence is Cauchy, so we know that the sequence converges even if the limit is not known in advance.

"The complete ordered field". The real numbers are often described as "the complete ordered field", a phrase that can be interpreted in several ways. First, an order can be lattice-complete. It is easy to see that no ordered field can be lattice-complete, because it can have no largest element (given any element z, $z + 1$ is larger).

Mathematical terminology

addition – сложение
multiplication – умножение
ordered field – упорядоченное поле
total order (linear order, total order, simple order, non-strict ordering) - линейно упорядоченное множество или цепь
non-empty subset – непустое подмножество
upper bound – верхний предел, верхняя граница
least upper bound – точная (наименьшая) верхняя грань (граница), или супрéмум
converge - 1) сходиться; стремиться к (общему) пределу 2) сводить (в-одну точку)
construction of the real numbers-конструктивные способы определения вещественного числа
limit – лимит, предел
exponential function – экспоненциальная функция, показательная функция
lattice-complete – полная решётка, частично упорядоченное множество, в котором всякое непустое подмножество А имеет точную верхнюю и нижнюю грань, называемые обычно объединением и пересечением элементов подмножества А.

Grammar, Lexical, Translation and Speaking Exercises

Task 1. Translate the sentences from Russian into English. State the functions and the forms of the Participles.

1. Mathematicians have developed geometric ideas from the world around us having many physical objects. 2. When naming geometric ideas we usually use letters of the alphabet. 3. The line AB shown below is called a line segment as you might remember. 4. A line segment is a set of points consisting of the two end points and all of the points on the line between them. 5. A geometric figure being formed by a set of points is an abstract concept, it cannot be seen. 6. Having performed the operation of subtraction they found the difference. 7. Drawing a straight line I used a ruler. 8. The program improved by the expert was checked yesterday. 9. The procedure being fulfilled by the researchers needed modern equipment. 10. The translated text dealt with the practical use of geometry. 11. Working in various fields of science Lomonosov also have much of his time to practical application of natural sciences. 12. Testing the new system over and over they found the error at last. 13. All the necessary changes having been made, the experiment showed different result. 14. When asked to compare the two approaches he agreed immediately.

Task 2. Translate the sentences from English into Russian. Denote the function of the Participle in the sentences.

The student writing a new programme works for our research department. Solving these problems we must use a new rule. While/ When solving a problem use a computer. Students are considering the properties of sets. The computers being developed now will be extensively used. Being written on time, the article was published in the journal. The system which is being tested seems very complicated. The proposed method was used in our calculations. The method proposed by the mathematician was used in our calculations. The method just referred to is of great interest. Translated from the language of mathematics into everyday language the relation became easier to understand. As seen from the results the information was carefully collected. When given enough time he will write his paper. Unless properly constructed the device will not be reliable. He was told about some new developments in this field of mathematics. Having answered the instructor's questions the student left. Having been given the problem we began to analyse it.

Task 3. Match the English terms with their Russian equivalents.

1. the process of reasoning	a. пересекаться в одной точке
2. to discover properties of figures	b. существует единственная линия
3. what meaning one attributes	c. какое значение придают
4. to accept without proof	d. процесс рассуждения
5. there is a unique line	e. начинать со слов
6. to lie in the same plane	f. конгруэнтные дуги
7. congruent arcs	g. обнаружить свойства фигур
8. exterior angle	h. принимать без доказательства
9. to intersect at one point	i. внешний угол
10. under this hypothesis	j. провести биссектрису
11. to begin with the words	k. быть обратным данной теореме
12. to draw a bisector	l. по этой гипотезе
13. to be converse to the given theorem	m. лежать на одной плоскости
14. interior angle	n. внутренний угол
15. alternate angle	o. накрест лежащий угол, противолежащий угол

Task 4. Guess what figure possesses the following properties and memorize them (a square, a trapezoid, a kite, a rectangle, a parallelogram, a rhombus).

1. A ... has two parallel pairs of opposite sides. 2. A ... has two pairs of opposite sides parallel, and four right angles. It is also a parallelogram, since it has two pairs of

parallel sides. 3. A ... has two pairs of parallel sides, four right angles, and all four sides are equal. It is also a rectangle and a parallelogram. 4. A ... is defined as a parallelogram with four equal sides. It does not have to have 4 right angles. 5. ... only has one pair of parallel sides. It's a type of quadrilateral that is not a parallelogram. 6. ... has two pairs of adjacent sides that are equal.

Task 5. Translate from English into Russian the description of the following arithmetic operations:

1. **Addition:** The concept of adding stems from such fundamental facts that it does not require a definition and cannot be defined in formal fashion. We can use synonymous expressions, if we so much desire, like saying it is the process of combining. Notation: 8 + 3 = 11; 8 and 3 are the addends, 11 is the sum.
2. **Subtraction:** When one number is subtracted from another the result is called the difference or remainder. The number subtracted is termed the subtrahend, and the number from which the subtrahendis subtracted is called minuend. Notation:15 – 7 = 8; 15 is the subtrahend, 7 is the minuend and 8 is the remainder. Subtraction may be checked by addition: 8 + 7 = 15.
3. **Multiplication:** is the process of taking one number (called the multiplicand) a given number of times (this is the multiplier, which tells us how many times the multiplicand is to be taken). The result is called the product. The numbers multiplied together are called the factors of the products. Notation:12 ×5 = 60 or 12.5 = 60; 12 is the multiplicand, 5 is the multiplier and 60 is the product (here, 12 and 5 are the factors of product).
4. **Division:** is the process of finding one of two factors from the product and the other factor. It is the process of determining how many times one number is contained in another. The number divided by another is called the dividend. The number divided into the dividend is called the divisor, and the answer obtained by division is called the quotient. Notation:48 : 6 = 8; 48 is the dividend, 6 is the divisor and 8 is the quotient.

Division may be checked by multiplication.

Task 6. Learn how to pronounce these symbols in English.

1) ≡	4) →	7) ∃	10) ≥	13) ±
2) ≠	5) <	8) ∀	11) α	14) /
3) ≈	6) >	9) ≤	12) ∞	15)∈

UNIT 5 Text 5. Integer

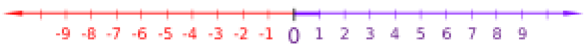
An **integer** (from the Latin *integer* meaning "whole") is a number that can be written without a fractional component. For example, 21, 4, 0, and −2048 are integers, while 9.75, 5½, and √2 are not.

The set of integers consists of zero (0), the natural numbers (1, 2, 3, ...), also called *whole numbers* or *counting numbers*, and their additive inverses (the **negative integers** −1, −2, −3, ...). This is often denoted by a boldface Z ("**Z**") or blackboard bold ℤ standing for the German word *Zahlen* ([ˈtsaːlən], "numbers").

ℤ is a subset of the sets of rational and real numbers and, like the natural numbers, is countably infinite.

The integers form the smallest group and the smallest ring containing the natural numbers. In algebraic number theory, the integers are sometimes called **rational integers** to distinguish them from the more general algebraic integers. In fact, the (rational) integers are the algebraic integers that are also rational numbers.

Algebraic properties



Integers can be thought of as discrete, equally spaced points on an infinitely long number line. In the above, non-negative integers are shown in purple and negative integers in red.

Like the natural numbers, **Z** is closed under the operations of addition and multiplication, that is, the sum and product of any two integers is an integer. However, with the inclusion of the negative natural numbers, and, importantly, 0, **Z** (unlike the natural numbers) is also closed under subtraction. The integers form a unital ring which is the most basic one, in the following sense: for any unital ring, there is a unique ring homomorphism from the integers into this ring. This universal property, namely to be an initial object in the category of rings, characterizes the ring **Z**.

Properties of addition and multiplication on integers

	Addition	Multiplication
Closure:	**a + b is an integer**	***a* × *b* is an integer**
Associativity:	$a + (b + c) = (a + b) + c$	$a \times (b \times c) = (a \times b) \times c$
Commutativity:	$a + b = b + a$	$a \times b = b \times a$
Existence of an identity element:	$a + 0 = a$	$a \times 1 = a$
Existence of inverse elements:	$a + (-a) = 0$	An inverse element usually does not exist at all.
Distributivity:	$a \times (b + c) = (a \times b) + (a \times c)$ and $(a + b) \times c = (a \times c) + (b \times c)$	
No zero divisors:		**If $a \times b = 0$, then $a = 0$ or $b = 0$ (or both)**

Z is not closed under division, since the quotient of two integers (e.g., 1 divided by 2), need not be an integer. Although the natural numbers are closed under exponentiation, the integers are not (since the result can be a fraction when the exponent is negative).The following lists some of the basic properties of addition and multiplication for any integers *a*, *b* and *c*.

In the language of abstract algebra, the first five properties listed above for addition say that **Z** under addition is an abelian group. As a group under addition, **Z** is a cyclic group, since every non-zero integer can be written as a finite sum $1 + 1 + \cdots + 1$ or $(-1) + (-1) + \cdots + (-1)$. In fact, **Z** under addition is the *only* infinite cyclic group, in the sense that any infinite cyclic group is isomorphic to **Z**.

The first four properties listed above for multiplication say that **Z** under multiplication is a commutative monoid. However not every integer has a multiplicative inverse; e.g. there is no integer *x* such that $2x = 1$, because the left hand side is even, while the right hand side is odd. This means that **Z** under multiplication is not a group.

All the rules from the above property table, except for the last, taken together say that **Z** together with addition and multiplication is a commutative ring with unity. It is the prototype of all objects of such algebraic structure. Only those equalities of expressions are true in **Z** for all values of variables, which are true in any unital commutative ring. Note that certain non-zero integers map to zero in certain rings.

At last, the property (*) says that the commutative ring **Z** is an integral domain. In fact, **Z** provides the motivation for defining such a structure.

The lack of multiplicative inverses, which is equivalent to the fact that **Z** is not closed under division, means that **Z** is *not* a field. The smallest field with the usual operations containing the integers is the field of rational numbers. The process of constructing the rationals from the integers can be mimicked to form the field of fractions of any integral domain. And back, starting from an algebraic number field (an extension of rational numbers), its ring of integers can be extracted, which includes **Z** as its subring.

Although ordinary division is not defined on **Z**, the division "with remainder" is defined on them. It is called Euclidean division and possesses the following important property: that is, given two integers *a* and *b* with $b \neq 0$, there exist unique integers *q* and *r* such that $a = q \times b + r$ and $0 \leq r < |b|$, where $|b|$ denotes the absolute value of *b*. The integer *q* is called the *quotient* and *r* is called the *remainder* of the division of *a* by *b*. The Euclidean algorithm for computing greatest common divisors works by a sequence of Euclidean divisions.Again, in the language of abstract algebra, the above says that **Z** is a Euclidean domain. This implies that **Z** is a principal ideal domain and any positive integer can be written as the products of primes in an essentially unique way. This is the fundamental theorem of arithmetic.

Order-theoretic properties

Z is a totally ordered set without upper or lower bound. The ordering of **Z** is given by: ... $-3 < -2 < -1 < 0 < 1 < 2 < 3 < ...$

An integer is *positive* if it is greater than zero and *negative* if it is less than zero. Zero is defined as neither negative nor positive.

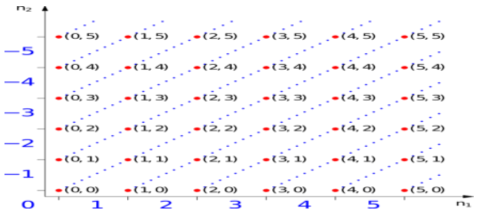
The ordering of integers is compatible with the algebraic operations in the following way: if $a < b$ and $c < d$, then $a + c < b + d$

if $a < b$ and $0 < c$, then $ac < bc$.

It follows that **Z** together with the above ordering is an ordered ring.

The integers are the only nontrivial totally ordered abelian group whose positive elements are well-ordered. This is equivalent to the statement that any Noetherian valuation ring is either a field or a discrete valuation ring.

Construction



Red points represent ordered pairs of natural numbers. Linked red points are equivalence classes representing the blue integers at the end of the line.

In elementary school teaching, integers are often intuitively defined as the disjoint union of the (positive) natural numbers, the singleton set whose only element is zero, and the negations of natural numbers. However, this style of definition leads to many different cases (each arithmetic operation needs to be defined on each combination of types of integer) and makes it difficult to prove that these operations obey the laws of arithmetic. Therefore, in modern set-theoretic mathematics a more abstract construction, which allows one to define the arithmetical operations without any case distinction, is often used instead. The integers can thus be formally constructed as the equivalence classes of ordered pairs of natural numbers (a,b).

The intuition is that (a,b) stands for the result of subtracting b from a.To confirm our expectation that $1 - 2$ and $4 - 5$ denote the same number, we define an equivalence relation ~ on these pairs with the following rule: $(a, b) \sim (c, d)$ precisely when $a + d = b + c$.

Addition and multiplication of integers can be defined in terms of the equivalent operations on the natural numbers; denoting by [(*a*,*b*)] the equivalence class having (*a*,*b*) as a member, one has: $[(a,b)]+[(c,d)]:=[(a+c,b+d)].$

$$[(a,b)]\cdot[(c,d)]:=[(ac+bd,ad+bc)].$$

The negation (or additive inverse) of an integer is obtained by reversing the order of the pair: $-[(a,b)]:=[(b,a)].$

Hence subtraction can be defined as the addition of the additive inverse:

$$[(a,b)]-[(c,d)]:=[(a+d,b+c)].$$

The standard ordering on the integers is given by: $[(a,b)]<[(c,d)]$ iff $a+d<b+c.$

It is easily verified that these definitions are independent of the choice of representatives of the equivalence classes.

Every equivalence class has a unique member that is of the form (*n*,0) or (0,*n*) (or both at once). The natural number *n* is identified with the class [(*n*,0)] (in other words the natural numbers are embedded into the integers by map sending *n* to [(*n*,0)]), and the class [(0,*n*)] is denoted −*n* (this covers all remaining classes, and gives the class [(0,0)] a second time since −0 = 0. Thus, [(*a*,*b*)] is denoted by

$$\begin{cases} a-b, & \text{if } a\ge b \\ -(b-a), & \text{if } a<b. \end{cases}$$

If the natural numbers are identified with the corresponding integers (using the embedding mentioned above), this convention creates no ambiguity.

This notation recovers the familiar representation of the integers as {... −3,−2,−1, 0, 1, 2, 3, ...}. Some examples are:

$$\begin{aligned} 0&=[(0,0)] &&=[(1,1)]=\cdots &&=[(k,k)]\\ 1&=[(1,0)] &&=[(2,1)]=\cdots &&=[(k+1,k)]\\ -1&=[(0,1)] &&=[(1,2)]=\cdots &&=[(k,k+1)]\\ 2&=[(2,0)] &&=[(3,1)]=\cdots &&=[(k+2,k)]\\ -2&=[(0,2)] &&=[(1,3)]=\cdots &&=[(k,k+2)]. \end{aligned}$$

Cardinality. The cardinality of the set of integers is equal to $\aleph_0$ (aleph-null). This is readily demonstrated by the construction of a bijection, that is, a function that is injective and surjective from **Z** to **N**. If **N** = {0, 1, 2, ...} then consider the function:

$$f(x)=\begin{cases} 2|x|, & \text{if } x<0\\ 0, & \text{if } x=0\\ 2x-1, & \text{if } x>0. \end{cases}$$

{... (-4,8) (-3,6) (-2,4) (-1,2) (0,0) (1,1) (2,3) (3,5) ...}

If **N** = {1, 2, 3, ...} then consider the function:

$$g(x)=\begin{cases} 2|x|, & \text{if } x<0\\ 2x+1, & \text{if } x\ge 0. \end{cases}$$

{... (-4,8) (-3,6) (-2,4) (-1,2) (0,1) (1,3) (2,5) (3,7) ...} If the domain is restricted to **Z** then each and every member of **Z** has one and only one corresponding member of **N** and by the definition of cardinal equality the two sets have equal cardinality.

Математическая терминология

fractional component – дробная составляющая
set of integers – множество целых чисел
natural numbers – натуральные числа
additive inverse – аддитивная инверсия, инверсия относительно сложения
boldface – полужирный шрифт, полужирный (о шрифте)
blackboard bold – способ написания жирным шрифтом
countably infinite – счётно бесконечный
algebraic number theory – алгебраическая теория чисел
algebraic integers – целое алгебраическое число
number line – [вещественная] цифровая ось
unital ring – унитальное кольцо (кольцо́ (ассоциативное кольцо) – в общей алгебре – алгебраическая структура, в которой определены операция обратимого сложения и операция умножения, по свойствам похожие на соответствующие операции над числами. Простейшими примерами колец являются числа (целые, вещественные, комплексные), функции, определенные на заданном множестве.
ring homomorphism – гомоморфизм колец
universal property – универсальное свойство
initial object – инициальный объект (в теории категорий начальный объект категории *C* – это её объект *I*, такой что для любого объекта *X* в *C* существует единственный морфизм *I* → *X*.)
exponentiation [ˌɛkspənɛnʃɪˈeɪʃ(ə)n] – возведение в степень
abstract algebra – абстрактная алгебра
abelian group – абелева группа; коммутативная группа
cyclic group – циклическая группа
isomorphic – изоморфный, имеющий идентичную форму; в математике говорят, что между двумя структурами существует изоморфизм, если для каждого компонента одной структуры есть соответствующий компонент в другой структуре, и наоборот
commutative monoid – коммутативный [абелев] моноид
commutative ring – коммутативное кольцо
equality of expressions – равенство выражений
integral domain – область целостности
field of fractions – поле частных, поле отношений
number field – числовое поле; поле чисел
subring – подкольцо (подмножество кольца)
absolute value – абсолютное значение, абсолютная величина, модуль (числа)
remainder – 1) остаток (от деления); 2) разность; 3) остаточный член (ряда)
greatest common divisors – наибольший общий делитель
principal ideal domain – область главных идеалов
fundamental theorem of arithmetic – основная теорема арифметики
totally ordered set – вполне упорядоченное множество
ordered ring – упорядоченное кольцо
Noetherian valuation ring – кольцо нормирования Нетер;

Noether – Эмми Нетер (1882-1935), немецкий математик. С 1933 в США. Труды Нетер по алгебре способствовали созданию нового направления, названного общей алгеброй. Сформулировала (1918) фундаментальную теорему теоретической физики.

discrete valuation ring – кольцо дискретного нормирования

disjoint union – несвязное объединение

singleton set – одноэлементное множество

equivalence classes – классы эквивалентности

ordered pair of natural numbers – упорядоченная пара натуральных чисел

equivalence relation – отношение эквивалентности

to be embedded into – быть вложенным в

embedding (or imbedding) – вложение в математике – это специального вида отображение одного экземпляра некоторой математической структуры во второй экземпляр такого же типа. А именно, вложение некоторого объекта X в Y задаётся инъективным отображением, сохраняющим некоторую структуру. Что означает «сохранение структуры», зависит от типа математической структуры, объектами которой являются X и Y. В терминах теории категорий отображение, «сохраняющее структуру», называют морфизмом.

familiar representation – привычное представление

primitive data type – исходный тип данных

computer languages (programming languages) – языки программирования

cardinality – кардинальное число, мощность множества

aleph-null – алеф-нуль (кардинальное число, характеризующее мощность счетного множества)

bijection – биекция, взаимно-однозначное отображение

injective – инъективный а) реализующий вложение, реализующий инъективное отображение; б) увеличивающий число аргументов (о функции)

surjective – сюръективный

natural numbers – натуральные числа (the positive integers (whole numbers) 1, 2, 3, etc., and sometimes zero as well)

number line – a line on which numbers are marked at intervals, used to illustrate simple numerical operations

exponentiation - the operation of raising one quantity to the power of another

equivalence relation - a relation between elements of a set that is reflexive, symmetric, and transitive. It thus defines exclusive classes whose members bear the relation to each other and not to those in other classes (e.g., "having the same value of a measured property")

Grammar, Lexical, Translation and Speaking Exercises

Task 1. Translate.

1. Рациональное число число, которое может быть представлено в виде отношения a/b, где a и b - целые числа и b - не равно нулю.
2. Целые числа – расширение множества натуральных чисел $\mathbb{N}$, получаемое добавлением к $\mathbb{N}$ нуля иотрицательных чисел вида $-n$. Множество целых чисел обозначается $\mathbb{Z}$. 3. В математике синглетоном называется множество с единственным элементом. Например, множество $\{0\}$ является синглетоном.

Task 2. Fill in the blanks with the necessary words.

1. To (измерить) an angle, compare its side to the corner of this page. 2. The corner represents (прямой угол), whose measurement is 90°. 3. If the angle is smaller than the corner, the angle is (острый угол). 4. If the opening is larger than the corner of the page, the angle is (тупой). Its measure is more than 90°. 5. Locate the point of your (транспортир) which represents the (вершина) and align the vertex with the point. 6. Rotate the protractor keeping the vertex aligned until one (сторона) of the angle is on the 0° – 180° line of the protractor. 7. The angle measure that is (определяется) by the side of the angle that is not on the 0° – 180° line of the protractor. 8. You may have to (продлить) one side of the angle so that it crosses the scale. 9. Use the proper (обозначение), m is the symbol for "measure of".

Task 3. Find the corresponding Russian sentence.

1. From what you already know you may deduce that drawing two rays originating from the same end point forms an angle.

a) Из того, что вы уже знаете, вы можете сделать вывод, что, рисуя два луча, исходящих из одной конечной точки, вы получаете угол.

b) Из того, что вам известно, вы можете сделать вывод, что изображение двух лучей, берущих начало в одной и той же конечной точке, образует угол.

c) Из того, что вы уже узнали, вы, возможно, сделали вывод, что рисунок двух лучей, берущих начало в одной конечной точке, образует угол.

3. The approach to the problem being considered remained traditional.

a) Рассматривался оставшийся подход к традиционной проблеме.

b) Подход к оставшейся проблеме рассматривался традиционно.

c) Подход к рассматриваемой проблеме оставался традиционным.

4. Physical facts expressed in terms of mathematics do not seem unusual nowadays.

a) Выраженные математические факты казались необычными в физических терминах в настоящее время.

b) Физические факты, выраженные в математических терминах, не кажутся необычными сегодня.

c) То, что физические факты в настоящее время выражаются математическими терминами, не кажется сегодня необычным.

5. Having made a number of experiments Faraday discovered electromagnetic induction.

a) Проводя ряд экспериментов, Фарадей открыл электромагнитную индукцию.

b) Проделав ряд экспериментов, Фарадей открыл электромагнитную индукцию.

c) Сделав ряд экспериментов, Фарадей открыл электромагнитную индукцию.

Task 4. The present simple or the past simple. Put the verbs in brackets in the correct forms.

The problem of constructing a regular polygon of nine sides which …………. (require) the trisection of a 60^0 angle ……… (be) the second source of the famous problem.

The Greeks ……… (add) “the trisection problem” to their three famous unsolved problems. It ……… (be) customary to emphasize the futile search of the Greeks for the solution.

The widespread availability of computers …………… (have) in all, probability changed the world for ever.

The microchip technology which …………(make) the PC possible has put chips not only into computers, but also into washing machines and cars.

Fermat almost certainly ………… (write) the marginal note around 1630, when he first ………… (study) Diophantus’s Arithmetica.

I ………… (protest) against the use of infinitive magnitude as something completed, which ……… (be) never permissible in maths, one ………… (have) in mind limits which certain ratio ……….. (approach) as closely as desirable while other ratios may increase indefinitely (Gauss).

In 1676 Robert Hooke ……………(announce) his discovery concerning springs. He ……………..(discover) that when a spring is stretched by an increasing force, the stretch variesdirectly according to the force.

Task 5. Answer the questions below.

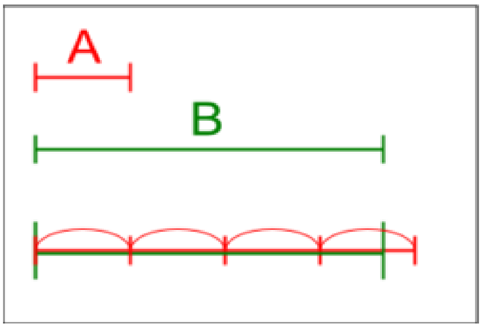
1. What have you seen if you multiply a whole number by 1?
2. Have you changed the fraction when you multiply $\frac{1}{2}$ by $\frac{2}{2}$?
3. What division have you used to change $\frac{6}{8}$ to lower terms?

Task 6. Translate from English into Russian.

1. An angle is the union of two rays which have a common endpoint but which do not lie on the same line.
2. Since an angle is a union of two sets of points, it is itself a set of points. When we say “the angle ABC” we are talking about a set of points - the points lying on the two rays.
3. Two angles occur so often in geometry that they are given special names. An angle of 90° is called a right angle and an angle of 180° is called a straight angle.

UNIT 6

Text 6. Archimedean property

<table>
<tr>
<td>Illustration of the Archimedean property
</td>
<td>In abstract algebra and analysis, the Archimedean property, named after the ancient Greek mathematician Archimedes of Syracuse, is a property held by some algebraic structures, such as ordered or normed groups, and fields. Roughly speaking, it is the property of having no infinitely large or infinitely small elements. It was Otto Stolz who gave the axiom of Archimedes its name because it appears as Axiom V of Archimedes' On the Sphere and Cylinder.</td>
</tr>
</table>

The notion arose from the theory of magnitudes of Ancient Greece; it still plays an important role in modern mathematics such as David Hilbert's axioms for geometry, and the theories of ordered groups, ordered fields, and local fields.

An algebraic structure in which any two non-zero elements are *comparable*, in the sense that neither of them is infinitesimal with respect to the other, is said to be Archimedean. A structure which has a pair of non-zero elements, one of which is infinitesimal with respect to the other, is said to be non-Archimedean. For example, a linearly ordered group that is Archimedean is an Archimedean group.

This can be made precise in various contexts with slightly different ways of formulation. For example, in the context of ordered fields, one has the axiom of Archimedes which formulates this property, where the field of real numbers is Archimedean, but that of rational functions in real coefficients is not.

Definition for linearly ordered groups.

Let x and y be positive elements of a linearly ordered group G. Then x is infinitesimal with respect to y (or equivalently, y is infinite with respect to x) if, for every natural number n, the multiple nx is less than y, that is, the following inequality holds:

$$\underbrace{x+\cdots+x}_{n\ \text{terms}} < y.$$

The group G is Archimedean if there is no pair x, y such that x is infinitesimal with respect to y.

Ordered fields. An ordered field has some additional nice properties.

1) One may assume that the rational numbers are contained in the field.

2) If x is infinitesimal, then $1/x$ is infinite, and vice versa. Therefore to verify that a field is Archimedean it is enough to check only that there are no infinitesimal elements, or to check that there are no infinite elements. 3) If x is infinitesimal

and r is a rational number, then $r\,x$ is also infinitesimal. As a result, given a general element c, the three numbers $c/2$, c, and $2c$ are either all infinitesimal or all non-infinitesimal.

In this setting, an ordered field K is Archimedean precisely when the following statement, called the axiom of Archimedes, holds: ***Let x be any element of K. Then there exists a natural number n such that n > x.***

Alternatively one can use the following characterization: ***For any positive ε in K, there exists a natural number n, such that 1/n < ε.***

Archimedean property of the real numbers. The field of the rational numbers can be assigned one of a number of absolute value functions, including the trivial function $|x| = 1$, when $x \neq 0$, the more usual $|x| = \sqrt{x^2}$, and the p-adic absolute value functions. By Ostrowski's theorem, every non-trivial absolute value on the rational numbers is equivalent to either the usual absolute value or some p-adic absolute value. The rational field is not complete with respect to non-trivial absolute values; with respect to the trivial absolute value, the rational field is a discrete topological space, so complete. The completion with respect to the usual absolute value (from the order) is the field of real numbers. By this construction the field of real numbers is Archimedean both as an ordered field and as a normed field. On the other hand, the completions with respect to the other non-trivial absolute values give the fields of p-adic numbers, where p is a prime integer number (see below); since the p-adic absolute values satisfy the ultrametric property, then the p-adic number fields are non-Archimedean as normed fields (they cannot be made into ordered fields).

In the axiomatic theory of real numbers, the non-existence of nonzero infinitesimal real numbers is implied by the least upper bound property as follows. Denote by$Z$ the set consisting of all positive infinitesimals. This set is bounded above by 1. Now assume for a contradiction that Z is nonempty. Then it has a least upper bound c, which is also positive, so $c/2 < c < 2c$. Since c is an upper bound of Z and $2c$ is strictly larger than c, $2c$ is not a positive infinitesimal. That is, there is some natural number n for which $1/n < 2c$. On the other hand, $c/2$ is a positive infinitesimal, since by the definition of least upper bound there must be an infinitesimal $x$between $c/2$ and c, and if $1/k < c/2 <= x$ then x is not infinitesimal. But $1/(4n) < c/2$, so $c/2$ is not infinitesimal, and this is a contradiction. This means that Z is empty after all: there are no positive, infinitesimal real numbers. One should note that the Archimedean property of real numbers holds also in constructive analysis, even though the least upper bound property may fail in that context.

Every linearly ordered field K contains (an isomorphic copy of) the rationals as an ordered subfield, namely the subfield generated by the multiplicative unit 1 of K, which in turn contains the integers as an ordered subgroup, which contains the natural numbers as an ordered monoid. The embedding of the rationals then gives a way of speaking about the rationals, integers, and natural numbers in K. The following are equivalent characterizations of Archimedean fields in terms of these substructures.

1. The natural numbers are cofinal in K. That is, every element of K is less than some natural number. (This is not the case when there exist infinite elements.) Thus an Archimedean field is one whose natural numbers grow without bound.

2. Zero is the infimum in K of the set $\{1/2, 1/3, 1/4, \ldots\}$. (If K contained a positive infinitesimal it would be a lower bound for the set whence zero would not be the greatest lower bound.)

3. The set of elements of K between the positive and negative rationals is closed. This is because the set consists of all the infinitesimals, which is just the closed set $\{0\}$ when there are no nonzero infinitesimals, and otherwise is open, there being neither a least nor greatest nonzero infinitesimal. In the latter case, (i) every infinitesimal is less than every positive rational, (ii) there is neither a greatest infinitesimal nor a least positive rational, and (iii) there is nothing else in between, a situation that points up both the incompleteness and disconnectedness of any non-Archimedean field.

4. For any x in K the set of integers greater than x has a least element. (If x were a negative infinite quantity every integer would be greater than it.)

5. Every nonempty open interval of K contains a rational. (If x is a positive infinitesimal, the open interval $(x, 2x)$ contains infinitely many infinitesimals but not a single rational.)

6. The rationals are dense in K with respect to both sup and inf. (That is, every element of K is the sup of some set of rationals, and the inf of some other set of rationals.) Thus an Archimedean field is any dense ordered extension of the rationals, in the sense of any ordered field that densely embeds its rational elements.

Mathematical terminology

infinitely large or infinitely small elements – бесконечно большие и бесконечно малые элементы (части)
magnitude – величина; абсолютное значение; модуль
ordered group – упорядоченная группа; **ordered field** – упорядоченное поле
local field – локальное поле
infinitesimal – бесконечно малая величина
linearly ordered group – линейно упорядоченная группа
Archimedean group – архимедова группа
***p*-adic numbers –** р-адические числа
absolute value – абсолютная величина, абсолютное значение
ultrametric property – ультраметрическое свойство

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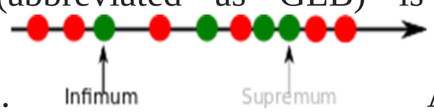
axiomatic theory – аксиоматическая теория
least upper bound property – свойство точной верхней границы
bounded above – ограниченный сверху
proof by contradiction – доказательство от противного
assume for a contradiction – предположим обратное
constructive analysis – конструктивный анализ
monoid – моноид (полугруппа с нейтральным элементом). Моноидом называется множество M, на котором задана бинарная ассоциативная операция, обычно именуемая умножением, и в котором существует такой элемент e, что ex=x=xe для любого x∈M. Элемент e называется единицей и часто обозначается 1. В любом моноиде имеется ровно одна единица.
cofinal – кофинальный, коконцевой; **dense** – плотный

Grammar, Lexical, Translation and Speaking Exercises

Task 1. Give the definition of the Archimedean property.

Task 2. Find the correspondence between English and Russian definitions.

Миноранта или **нижняя грань (граница)** числового множества X – число b, такое что $\forall x \in X \Rightarrow x \geqslant b$.	**The infimum** (abbreviated inf; plural infima) of a subset S of a partially ordered set T is the greatest element of T that is less than or equal to all elements of S.Consequently the term greatest lower bound (abbreviated as GLB) is also commonly used.  A set T of real numbers (red and green balls), a subset S of T (green balls), and the infimum of S. Note that for finite sets the infimum and the minimum are equal.
Пло́тное мно́жество – подмножество пространства, точками которого можно сколь угодно хорошо приблизить любую точку объемлющего пространства. Формально говоря, A плотно в X, если всякая окрестность любой точки x из X содержит элемент из A.	**The supremum** (abbreviated sup; plural suprema) of a subset S of a totally or partially ordered set T is the least element of T that is greater than or equal to all elements of S. Consequently, the supremum is also referred to as the least upper bound (or LUB)  A set A of real numbers (blue balls), a set of upper bounds of A (red diamond and balls), and the smallest such upper bound, that is, the supremum of A (red diamond).
Мажоранта или **верхняя грань (граница)** числового множества X–число a, такое что $\forall x \in X \Rightarrow x \leqslant a$.	A subset A of a topological space X is called dense (in X) if every point x in X either belongs to A or is a limit point of A. Informally, for every point in X, the point is either in A or arbitrarily "close" to a member of A – for instance, every real number is either a rational number or has one arbitrarily close to it.

Task 3. Translate the sentences from English into Russian.

The term **algebraic structure** generally refers to a set (called **carrier set** or **underlying set**) with one or more finitary operations defined on it.

Examples of **algebraic structures** include groups, rings, fields, and lattices. More complex structures can be defined by introducing multiple operations, different underlying sets, or by altering the defining axioms. Examples of more complex algebraic structures include vector spaces, modules, and algebras.

An ordered field necessarily has characteristic 0, all natural numbers, i.e. the elements 0, 1, 1 + 1, 1 + 1 + 1, … are distinct. This implies that an ordered field necessarily contains an infinite number of elements: a finite field cannot be ordered.

Task 4. Practice saying these expressions.

4.1 Fractions: $\frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, \frac{3}{8}, \frac{5}{6}, \frac{2}{7}$

4.2. Equations: $x = \frac{a+b}{c}$, $x+y = \frac{\Delta}{a-b}$, $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$

Task 5. Grammar revision: Change the following sentences using infinitive to express the purpose.

Model: We have to subtract this number from the sum obtained because we want to check the result of addition. – To check the result of addition, we have to subtract this number from the sum obtained.

1) We must know the details because we want to understand the situation. –

2) You must do the following because you want to operate this machine. –

3) He put the figures in a table because he wants to look at the data. –

4) He included the empty set at the beginning because he wants to have a complete table.

5) We made a conjecture and then proved this because we want to have the correct procedure. –

Task 6. Match a line in A with a line in B.

1) We apply the Euclidean algorithm 2) We use the symbol e 3) We use the braces { } 4) To clarify this idea 5) We draw a picture 6) To fix our thoughts 7) To find the negation of some statements, 8) In order to introduce the concept of measure,	a) to denote sets. b) let us use the unit circle. c) to mean “ is an element of ”. d) we return to one-dimensional geometry and line segments. e) we must find a statement that conforms to the rule stated above. f) to express GCD as a linear combination. g) to show the physical realization on this vector sum. h) we present some examples of set.

Task 7. Read these sentences and state the form and the function of the Infinitive. Translate into Russian.

1. To solve the equation was not difficult for her. 2. The speaker at the conference didn't like to be interrupted. 3. The article is difficult to translate. 4. They must have attended his lecture before. 5. He is always the first to come to the University. 6. The method to be applied is rather complicated. 7. He worked hard in order not to be behind the other students. 8. The topic may have been considered at the previous lesson. 9. Our aim is to extend the definition. 10. It isn't easy to speak any foreign language. 11. He must be improving his knowledge of mathematics. 12. The scientist might have been working on this problem for many years.

Task 8. Open the parentheses and give the correct form of the infinitive.

1. I am glad (read) this book now. 2. I hope (award) a scholarship for the coming semester. 3. He is happy (work) at this company for more than five years. 4. He does not like (interrupt) by anybody. 5. Ann was surprised (pass) the exams. 6. The question is too unexpected (answer) at once. 7. I want (solve) these equations. 8. This theorem was the first (prove). 9. She might (forget) to translate the text yesterday. 10. The question must (settle) an hour ago. 11. The article is (write) in time. 12. (Understand) the situation one must (know) the details.

Task 9. Complete the sentences by using infinitives. Supply a preposition after the infinitive if necessary. Use the Model.

1. I'm planning **to fly to** the USA next year. 2. The student promised not … late for the lecture. 3. I need … my homework tonight. 4. I want … computer games after my classes. 5. He intends … a programmer when he graduates from the university. 6. I hope … all of my courses this term. So far my grades have been pretty good. 7. I try … class on time every day. 8. I learned (how) … when I entered the university. 9. I like … a lot of e-mails from my friends. 10. I hate … in front of a large group. 11. My roommate offered … me with my English.

Task 10. Write the correct form (gerund or infinitive) of the verbs given in parentheses. Sometimes more than one answer is possible.

1. He regrets (not study) harder when he was at school. 2. The teacher was very strict and nobody dared (talk) during his lessons. 3. She suggested (go) to the University by taxi. 4. (learn) English involves (speak) as much as you can. 5. (Solve) this equation multiply each term in it by the quantity that precedes it. 6. On (obtain) the data the scientists went on working. 7. The procedure (follow) depends entirely on the student. 8. This equation must (solve) at the previous lesson. 9. Euclid was the first (bring) all the known facts about geometry into one whole system. 10. We don't mind (give) further assistance. 11. The method (apply) is rather complicated. 12. (prove) this theorem means (find) a solution for the whole problem. 13. Students are (study) the laws of mathematics and mechanics.

UNIT 7

Text 7. Series

A series is, informally speaking, the sum of the terms of a sequence. Finite sequences and series have defined first and last terms, whereas infinite sequences and series continue indefinitely.

In mathematics, given an infinite sequence of numbers $\{ a_n \}$, a series is informally the result of adding all those terms together: $a_1 + a_2 + a_3 + \cdots$. These can be written more compactly using the summation symbol $\sum$. An example is the famous series from Zeno's dichotomy and its mathematical representation:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots.$$

The terms of the series are often produced according to a certain rule, such as by a formula, or by an algorithm. As there are an infinite number of terms, this notion is often called an infinite series. Unlike finite summations, infinite series need tools from mathematical analysis, and specifically the notion of limits, to be fully understood and manipulated. In addition to their ubiquity in mathematics, infinite series are also widely used in other quantitative disciplines such as physics, computer science, and finance.

Definition. For any sequence $\{a_n\}$ of rational numbers, real numbers, complex numbers, functions thereof, etc., the associated series is defined as the ordered formal sum

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots.$$

The sequence of partial sums $\{S_k\}$ associated to a series $\sum_{n=0}^{\infty} a_n$ is defined for each k as the sum of the sequence $\{a_n\}$ from a_0 to a_k

$$S_k = \sum_{n=0}^{k} a_n = a_0 + a_1 + \cdots + a_k.$$

By definition the series $\sum_{n=0}^{\infty} a_n$ converges to a limit L if and only if the associated sequence of partial sums $\{S_k\}$ converges to L. This definition is usually written as

$$L = \sum_{n=0}^{\infty} a_n \Leftrightarrow L = \lim_{k\to\infty} S_k.$$

More generally, if $I \xrightarrow{a} G$ is a function from an index set I to a set G, then the series associated to a is the formal sum of the elements $a(x) \in G$ over the index elements $x \in I$ denoted by the $\sum_{x \in I} a(x).$

When the index set is the natural numbers $I = \mathbb{N}$, the function $\mathbb{N} \xrightarrow{a} G$ is a sequence denoted by $a(n) = a_n$. A series indexed on the natural numbers is an ordered formal sum and so we rewrite $\sum_{n\in\mathbb{N}}$ as $\sum_{n=0}^{\infty}$ in order to emphasize the ordering induced by the natural numbers. Thus, we obtain the common notation for a series indexed by the natural numbers:

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots .$$

When the set G is a semigroup, the sequence of partial sums $\{S_k\} \subset G$ associated to a sequence $\{a_n\} \subset G$ is defined for each k as the sum of the terms $a_0, a_1, \cdots, a_k$

$$S_k = \sum_{n=0}^{k} a_n = a_0 + a_1 + \cdots + a_k.$$

When the semigroup G is also a topological space, then the series $\sum_{n=0}^{\infty} a_n$ converges to an element $L \in G$ if and only if the associated sequence of partial sums $\{S_k\}$ converges to L. This definition is usually written as $L = \sum_{n=0}^{\infty} a_n \Leftrightarrow L = \lim_{k\to\infty} S_k.$

Convergent series. A series $\sum a_n$ is said to 'converge' or to 'be convergent' when the sequence S_N of partial sums has a finite limit. If the limit of S_N is infinite or does not exist, the series is said to diverge. When the limit of partial sums exists, it is called the sum of the series $\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=0}^{N} a_n.$

An easy way that an infinite series can converge is if all the a_n are zero for n sufficiently large. Such a series can be identified with a finite sum, so it is only infinite in a trivial sense.

Working out the properties of the series that converge even if infinitely many terms are non-zero is the essence of the study of series. Consider the example

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots .$$

It is possible to "visualize" its convergence on the real number line: we can imagine a line of length 2, with successive segments marked off of lengths 1, ½, ¼, etc. There is always room to mark the next segment, because the amount of line remaining is always the same as the last segment marked: when we have marked off ½, we still have a piece of length ½ unmarked, so we can certainly mark the next ¼. This argument does not prove that the sum is *equal* to 2 (although it is), but it does prove that it is *at most* 2. In other words, the series has an upper bound. Given that the series converges, proving that it is equal to 2 requires only elementary algebra. If the series is denoted S, it can be seen that

$$S/2 = \frac{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots}{2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots .$$

Therefore, $S - S/2 = 1 \Rightarrow S = 2.$

Mathematicians extend the idiom discussed earlier to other, equivalent notions of series. For instance, when we talk about a recurring decimal, as in $x = 0.111\ldots$ we are talking, in fact, just about the series $\sum_{n=1}^{\infty} \frac{1}{10^n}$.

But since these series always converge to real numbers (because of what is called the completeness property of the real numbers), to talk about the series in this way is the same as to talk about the numbers for which they stand. In particular, it should offend no sensibilities if we make no distinction between 0.111… and $^1/_9$. Less clear is the argument that 9 × 0.111… = 0.999… = 1, but it is not untenable when we consider that we can formalize the proof knowing only that limit laws preserve the arithmetic operations.

Examples: A geometric series is one where each successive term is produced by multiplying the previous term by a constant number (called the common ratio in this context). Example:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2^n}.$$

In general, the geometric series $\sum_{n=0}^{\infty} z^n$ converges if and only if $|z| < 1$.

An arithmetico-geometric sequence is a generalization of the geometric series, which has coefficients of the common ratio equal to the terms in an arithmetic series. Example:

$$3 + \frac{5}{2} + \frac{7}{4} + \frac{9}{8} + \frac{11}{16} + \cdots = \sum_{n=0}^{\infty} \frac{(3 + 2n)}{2^n}.$$

The harmonic series is the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}.$

The harmonic series is divergent.

An alternating series is a series where terms alternate signs. Example:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln(2).$$

Calculus and partial summation as an operation on sequences

Partial summation takes as input a sequence, $\{ a_n \}$, and gives as output another sequence, $\{ S_N \}$. It is thus a unary operation on sequences. Further, this function is linear, and thus is a linear operator on the vector space of sequences, denoted Σ. The inverse operator is the finite difference operator, Δ. These behave as discrete analogs of integration and differentiation, only for series (functions of a natural number) instead of functions of a real variable. For example, the sequence {1, 1, 1, ...} has series {1, 2, 3, 4, ...} as its partial summation, which is analogous to the fact that $\int_0^x 1\,dt = x.$

Properties of series. Series are classified not only by whether they converge or diverge, but also by the properties of the terms a_n (absolute or conditional convergence); type of convergence of the series (pointwise, uniform); the class of the term a_n (whether it is a real number, arithmetic progression, trigonometric function); etc.

Non-negative terms. When a_n is a non-negative real number for every n, the sequence S_N of partial sums is non-decreasing. It follows that a series $\sum a_n$ with non-negative terms converges if and only if the sequence S_N of partial sums is bounded.

For example, the series

$\sum_{n\geq 1} \frac{1}{n^2}$ is convergent, because the inequality $\frac{1}{n^2} \leq \frac{1}{n-1} - \frac{1}{n}, \quad n \geq 2,$ and a telescopic sum argument implies that the partial sums are bounded by 2.

Absolute convergence. A series $\sum_{n=0}^{\infty} a_n$ is said to converge absolutely if the series of absolute values $\sum_{n=0}^{\infty} |a_n|$ converges. This is sufficient to guarantee not only that the original series converges to a limit, but also that any reordering of it converges to the same limit.

Conditional convergence. A series of real or complex numbers is said to be conditionally convergent (or semi-convergent) if it is convergent but not absolutely convergent. A famous example is the alternating series

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ which is convergent (and its sum is equal to ln 2), but the series formed by taking the absolute value of each term is the divergent harmonic series.

Mathematical terminology

series – ряд; прогрессия; последовательность
term – член, элемент
terms of a sequence – члены (элементы) последовательности
infinite sequence – бесконечная последовательность
summation – суммирование
Zeno's dichotomy – дихтомия Зенона
formula (**pl. formulae, formulas**) – формула (мн. формулы)
algorithm - алгоритм
formal sum – формальная сумма
converge – 1) сходиться; стремиться к (общему) пределу, 2) сводить (в одну точку)
diverge – 1) расходиться, 2) отклоняться (от линии, направления)
functions thereof – их функций
index set – индексное множество
recurring decimal – периодическая десятичная дробь

completeness property – свойство полноты
geometric series – геометрический ряд, бесконечная геометрическая прогрессия
constant – константа, постоянная (величина)
if and only if – тогда и только тогда, когда
arithmetico-geometric sequence – арифметико-геометрическая прогрессия, последовательность чисел u_n, задаваемая рекуррентным соотношением: $u_1 = a_1, u_{n+1} = qu_n + d$, где q и d - постоянные числа. Частными случаями арифметико-геометрической прогрессии являются арифметическая прогрессия (при $q = 1$) и геометрическая прогрессия (при $d = 0$).
harmonic series – гармонический ряд (ряд, обратные величины членов которого составляют арифметическую прогрессию)
alternating series – знакопеременный ряд
unary operation – унарная операция
linear – линейныйж; **linear operator** – линейный оператор
finite difference – конечная разность, математический термин, широко применяющийся в методах вычисления при интерполировании.
integration – интегрирование
differentiation – дифференцирование, отыскание производной
conditional convergence – условная сходимость

Grammar, Lexical, Translation and Speaking Exercises

Task 1. Answer the questions.

1. What is series?
2. Give the mathematical representation of the famous series from Zeno's dichotomy.
3. How can the associated series be defined for any sequence $\{a_n\}$ of rational numbers, real numbers, complex numbers? Give the formula.
4. Write the common notation for a series indexed by the natural numbers.
5. What is the name of a costant number that serves as a multiplier to get successive term from the previous one in a geometric series like that $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2^n}.$?
6. How do we call a series whose terms are in harmonic progression, as in 1 + 1/2 + 1/3 + ... ?
7. Enumerate the properties of series?

Task 2. Grammar revision. Translate from English into Russian and remember that Passives Voice is very common in technical writing where we are more interested in facts, processes, and events than in people.

Data is transferred from the internal memory to the arithmetic logical unit.
Distributed systems are built using networked computers.
The organization was created to promote the use of computers in education.
A new method for studying geometric figures and curves, both familiar and new were created by Descartes and Fermat.

Task 3. Fill in the gaps using the correct form of the verb in brackets.

All calls …… (register) by the Help Desk staff. Each call …… (evaluate) and then …… (allocate) to the relevant support group. If a visit …… (require), the user…… (contact) by telephone, and an appointment …… (arrange). Most calls …… (deal with) within one working day. In the event of a major problem requiring the removal of a user's PC, a replacement can usually…… (supply).

Task 4. Make the sentences passive. Use "by ..." only if it is necessary to say who does or did the action.

a) Charles Babbage designed a machine which became the basis for building today's computer in the early 1800s.

b) People submerged geometry in a sea of formulas and banished its spirit for more than 150 years.

c) People often appreciate analytical geometry as the logical basis for mechanics and physics.

d) Bill Gates founded Microsoft.

e) People call the part of the processor which controls data transfers between the various input and output devices the central processing unit (CPU).

f) You may use ten digits of the Hindu-Arabic system in various combinations. Thus we will use 1, 2 and 3 to write 123, 132, 213 and so on.

g) Mathematicians refer to a system with which one coordinates numbers and points as a coordinate system or frame of reference.

h) People establish a correspondence between the algebraic and analytic properties of the equation f (x, y) = 0, and geometric properties of the associated curve.

i) In 1946 the University of Pensylvania built the first digital computer.

Task 5. Change the following passive sentences into active.

a) This frame of reference will be used to locate a point in space.

b) Although solid analytic geometry was mentioned by R.Descartes, it was not elaborated thoroughly and exhaustively by him.

c) Most uses of computers in language education can be described as CALL.

d) Since many students are considerably more able as algebraists than as geometers, analytic geometry can be described as the "royal road" in geometry that Euclid thought did not exist.

e) Now new technologies are being developed to make even better machines.

f) Logarithm tables, calculus, and the basis for the modern slide rule were not invented during the twentieth century. g) After World War 2 ended, the transistor was developed by Bell Laboratories. h) The whole subject matter of analytic geometry was well advanced, beyond its elementary stages, by L.Euler.

Task 6. Read the text and answer the questions.

RATIO AND PROPORTION

A ratio is an indicated division. It should be thought of as a fraction. The language used is: ***"the ratio of a to b"*** which means $\boldsymbol{a \div b}$ or $\frac{a}{b}$ and the symbol is $\boldsymbol{a : b}$. In this notation ***a*** is the first term or the antecedent, and ***b*** is the second term or the consequent. It is important to remember that we treat the ratio as a fraction. A proportion is a statement that two ratios are equal. Symbolically we write: **a : b = c : d or** $\frac{a}{b} = \frac{c}{d}$**.** The statement is read ***"a is to b as c is to d"*** and we call ***a*** and ***d*** the extremes, ***b*** and ***c*** the means, and d the fourth proportional. Proportions are treated as equations involving fractions. We may perform all the operations on them that we do on equations, and many of the resulting properties may already have been met in geometry.

Questions: What should a ratio be thought of as?

How is the statement read when we write **a : b = c : d or** $\frac{a}{b} = \frac{c}{d}$?

How are proportions treated?

Task 7. Fine proper definitions to the following mathematical terms: *ellipse, hyperbola, parabola.*

a) A type of cone that has an eccentricity equal to 1. It is an open curve symmetrical about a line.

b) A type of cone that has an eccentricity between 0 and 1 ($0 < e < 1$). It is a closed symmetrical curve like an elongated circle – the higher the eccentricity, the greater the elongation.

c) A type of cone that has an eccentricity (e) greater than 1. It is an open curve with two symmetrical branches.

d) A closed conic section shaped like a flattened circle and formed by an inclined plane that does not cut the base of the cone. Standard equation x2/a2 + y2/b2 = 1, where 2a and 2b are the lengths of the major and minor axes.

e) A conic section formed by a plane that cuts both bases of a cone; it consists of two branches asymptotic to two intersecting fixed lines and has two foci. Standard equation: x2/a2 - y2/b2 = 1 where 2a is the distance between the two intersections with the x-axis and b = a√(e2 - 1), where e is the eccentricity.

f) The graph of a quadratic expression. If y = ax2 + bx + c, where a ≠ 0, then y always has an extreme value when x + -b/2a. This is a minimum if a > 0 and a maximum if a < g) A conic section formed by the intersection of a cone by a plane parallel to its side. Standard equation: y2 = 4ax, where 2a is the distance between focus and directrix.

UNIT 8

Text 8. Functions

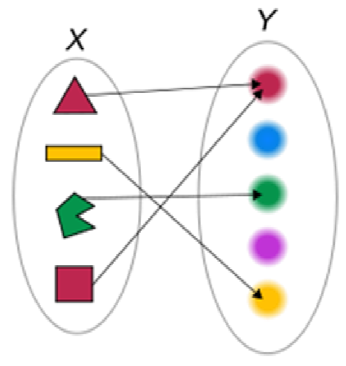
A function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output. An example is the function that relates each real number x to its square x^2. The output of a function f corresponding to an input x is denoted by $f(x)$ (read "f of x"). In this example, if the input is −3, then the output is 9, and we may write $f(-3) = 9$. The input variable(s) are sometimes referred to as the argument(s) of the function.

Functions of various kinds are "the central objects of investigation" in most fields of modern mathematics. There are many ways to describe or represent a function. Some functions may be defined by a formula or algorithm that tells how to compute the output for a given input. Others are given by a picture, called the graph of the function. In science, functions are sometimes defined by a table that gives the outputs for selected inputs. A function could be described implicitly, for example as the inverse to another function or as a solution of a differential equation.

The input and output of a function can be expressed as an ordered pair, ordered so that the first element is the input (or tuple of inputs, if the function takes more than one input), and the second is the output. In the example above, $f(x) = x^2$, we have the ordered pair (−3, 9). If both input and output are real numbers, this ordered pair can be viewed as the Cartesian coordinates of a point on the graph of the function. But no picture can exactly define every point in an infinite set.

In modern mathematics, a function is defined by its set of inputs, called the *domain*; a set containing the set of outputs, and possibly additional elements, as members, called its *codomain*; and the set of all input-output pairs, called its *graph*. (Sometimes the codomain is called the function's "range", but **warning**: the word "range" is sometimes used to mean, instead, specifically the set of outputs. An unambiguous word for the latter meaning is the function's "image". To avoid ambiguity, the words "codomain" and "image" are the preferred language for their concepts.) For example, we could define a function using the rule $f(x) = x^2$ by saying that the domain and codomain are the real numbers, and that the graph consists of all pairs of real numbers (x, x^2). Collections of functions with the same domain and the same codomain are called function spaces, the properties of which are studied in such mathematical disciplines as real analysis, complex analysis, and functional analysis. In analogy with arithmetic, it is possible to define addition, subtraction, multiplication, and division of functions, in those cases where the output is a number. Another important operation defined on functions is function composition, where the output from one function becomes the input to another function.

Introduction and examples

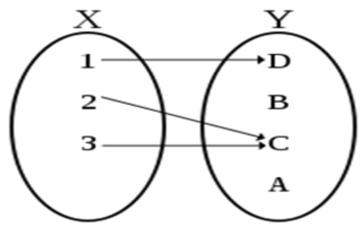
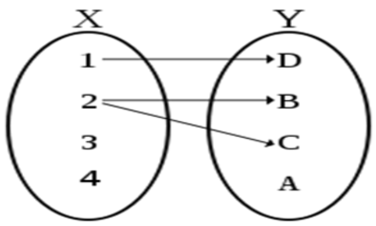
	***A function that associates to any of the four colored shapes its color***. Let *X* be the set consisting of four shapes: a red triangle, a yellow rectangle, a green hexagon, and a red square; and let *Y* be the set consisting of five colors: red, blue, green, pink, and yellow. Linking each shape to its color is a function from *X* to *Y*: each shape is linked to a color (i.e., an element in *Y*), and each shape is "linked", or "mapped", to exactly one color. There is no shape that lacks a color and no shape that has two or more colors. This function will be referred to as the "color-of-the-shape function".

The input to a function is called the argument and the output is called the value. The set of all permitted inputs to a given function is called the domain of the function, while the set of permissible outputs is called the codomain. Thus, the domain of the "color-of-the-shape function" is the set of the four shapes, and the codomain consists of the five colors. The concept of a function does *not* require that every possible output is the value of some argument, e.g. the color blue is not the color of any of the four shapes in *X*.

A second example of a function is the following: the domain is chosen to be the set of natural numbers (1, 2, 3, 4, ...), and the codomain is the set of integers (..., −3, −2, −1, 0, 1, 2, 3, ...). The function associates to any natural number *n* the number 4−*n*. For example, to 1 it associates 3 and to 10 it associates −6.

A third example of a function has the set of polygons as domain and the set of natural numbers as codomain. The function associates a polygon with its number of vertices. For example, a triangle is associated with the number 3, a square with the number 4, and so on. The term range is sometimes used either for the codomain or for the set of all the actual values a function has.

Definition

	The diagram **represents a function** with domain {1, 2, 3}, codomain {A, B, C, D} and set of ordered pairs {(1,D), (2,C), (3,C)}. The image is {C,D}.
	However, this second diagram **doesnot represent a function**. One reason is that 2 is the first element in more than one ordered pair. In particular, (2, B) and (2, C) are both elements of the set of ordered pairs. Another reason, sufficient by itself, is that 3 is not the first element (input) for any ordered pair. A third reason, likewise, is that 4 is not the first element of any ordered pair.

In order to avoid the use of the informally defined concepts of "rules" and "associates", the above intuitive explanation of functions is completed with a formal definition. This definition relies on the notion of the Cartesian product. The Cartesian product of two sets X and Y is the set of all ordered pairs, written (x, y), where x is an element of X and y is an element of Y. The x and the y are called the components of the ordered pair. The Cartesian product of X and Y is denoted by $X \times Y$.

A function f from X to Y is a subset of the Cartesian product $X \times Y$ subject to the following condition: every element of X is the first component of one and only one ordered pair in the subset. In other words, for every x in X there is exactly one element y such that the ordered pair (x, y) is contained in the subset defining the function f. This formal definition is a precise rendition of the idea that to each x is associated an element y of Y, namely the uniquely specified element y with the property just mentioned.

Considering the "color-of-the-shape" function above, the set X is the domain consisting of the four shapes, while Y is the codomain consisting of five colors. There are twenty possible ordered pairs (four shapes times five colors), one of which is ("yellow rectangle", "red").

The "color-of-the-shape" function described above consists of the set of those ordered pairs, (shape, color) where the color is the actual color of the given shape. Thus, the pair ("red triangle", "red") is in the function, but the pair ("yellow rectangle", "red") is not.

Notation. A function f with domain X and codomain Y is commonly denoted by

$$f\colon X \to Y \quad \text{or} \quad X \xrightarrow{f} Y.$$

In this context, the elements of X are called arguments of f. For each argument x, the corresponding unique y in the codomain is called the function value at x or the *image* of x under f. It is written as $f(x)$. One says that f associates y with x or maps x to y. This is abbreviated by $y = f(x).$

A general function is often denoted by f. Special functions have names, for example, the signum function is denoted by sgn. Given a real number x, its image under the signum function is then written as $\operatorname{sgn}(x)$. Here, the argument is denoted by the symbol x, but different symbols may be used in other contexts. For example, in physics, the velocity of some body, depending on the time, is denoted $v(t)$. The parentheses around the argument may be omitted when there is little chance of confusion, thus: $\sin x$; this is known as prefix notation.

In order to denote a specific function, the notation $\mapsto$ (an arrow with a bar at its tail) is used. For example, the above function reads

$$f\colon \mathbb{N} \to \mathbb{Z}$$
$$x \mapsto 4 - x.$$

The first part can be read as:

"f is a function from $\mathbb{N}$ (the set of natural numbers) to $\mathbb{Z}$ (the set of integers)" or "f is a $\mathbb{Z}$-valued function of an $\mathbb{N}$-valued variable".

The second part is read: "x maps to $4-x$."

In other words, this function has the natural numbers as domain, the integers as codomain. Strictly speaking, a function is properly defined only when the domain and codomain are specified. For example, the formula $f(x) = 4 - x$ alone (without specifying the codomain and domain) is not a properly defined function. Moreover, the function

$$g: \mathbb{Z} \to \mathbb{Z}$$
$$x \mapsto 4 - x.$$

(with different domain) is not considered the same function, even though the formulas defining f and g agree, and similarly with a different codomain. Despite that, many authors drop the specification of the domain and codomain, especially if these are clear from the context. So in this example many just write $f(x) = 4 - x$. Sometimes, the maximal possible domain is also understood implicitly: a formula such as $f(x) = \sqrt{x^2 - 5x + 6}$ may mean that the domain of f is the set of real numbers x where the square root is defined (in this case $x \le 2$ or $x \ge 3$).

To define a function, sometimes a dot notation is used in order to emphasize the functional nature of an expression without assigning a special symbol to the variable. For instance, $a(\cdot)^2$ stands for the function $x \mapsto ax^2$, $\int_a^{\cdot} f(u)du$ stands for the integral function $x \mapsto \int_a^x f(u)du$, and so on.

Mathematical terminology

function - функция
relation - отношение
set - множество
inputs – входные (вводные) данные
permissible – допустимый
outputs – выходные данные
input variable – входная величина, входная переменная
argument of the function – аргумент функции
objects of investigation – объект исследования
graph of the function – график функции
inverse - обратная величина; обратный, противоположный
solution – решение
differential equation – дифференциальное уравнение
ordered pair – упорядоченная пара

tuple –1) кортеж, многокомпонентный объект данных; 2) декартово произведение, N-ка, "энка"; 3) запись
Cartesian coordinates – декартовы [прямоугольные] координаты
domain - область определения
codomain – область значений (функции), кообласть
range – слово может обозначать и область значений и выходные данные
unambiguous word – однозначное слово
avoid ambiguity – избегать неоднозначности (неясности, двусмысленности)
image of the function, image domain – область отображения
function space – функциональное пространство
real analysis – анализ действительных чисел
complex analysis – комплексный анализ
functional analysis – функциональный анализ
triangle – треугольник
rectangle – прямоугольник
hexagon – шестиугольник; шестигранник
square – квадрат
linked – связанный
mapped – отображенный; отображаемый
value – значение, величина
integer – целое число
polygon – многоугольник; многогранник; полигон
vertice – вершина
four shapes times five colors – 4 фигуры умноженные на 5 цветов
notation – обозначение; форма записи
signum function - знаковая функция
velocity - скорость
prefix notation – префиксная нотация (бесскобочная запись), одна из возможных бесскобочных форм записи арифметических выражений, функций и их операндов, в которой оператор (имя функции) предшествует всем её операндам, т. е. ставится слева от операндов. В этой нотации алгебраическое выражение (A+B) * C будет выглядеть как * + ABC.
dot notation - точечная запись (нотация)

Grammar, Lexical, Translation and Speaking Exercises

Task 1. Match the English words and word combinations with their Russian equivalents.

1. the undefined term	a) вершина угла
2. to extend indefinitely	b) отличительные черты
3. the vertex of the angle	c) если не указано иное
4. the interior of the angle	d) неопределенный термин
5. distinguishing features	e) внутренняя часть угла
6. the exterior part	f) продлеваться бесконечно
7. unless stated otherwise	g) внешняя часть
8. reflex angle	h) смежные углы
9. perpendicular bisector	i) угол между 180 и 360
10. adjacent angles	j) перпендикулярная биссектриса
11. intersecting lines	k) перпендикулярные прямые
12. parallel lines	l) пересекающиеся прямые
13. perpendicular lines	m) пареллельные прямые
14. acute angle	n) тупой угол
15. right angle	o) острый угол
16. obtuse angle	p) прямой угол
17. straight angle	q) развёрнутый угол

Task 2. Find out whether the statements are True or False according to the information in the text.

Use the introductory phrases:

I think, it is right. Quite so. Absolutely correct. I quite agree to it.	I am afraid, it is wrong. I don't quite agree to it. On the contrary. Far from it.

1. A point has length, width or thickness.
2. A line is limited and extends infinitely in one direction.
3. A line unless stated otherwise is understood to be straight.
4. A line is the shortest distance between two points.
5. A surface has length and width, it doesn't have thickness.
6. Equal angles are angles that have the same number of degrees.
7. Right angles are not congruent.
8. A perpendicular bisector of a line bisects the line and is perpendicular to the line.
9. If two planes intersect, their intersection is a line.
10. A point is a location and it has size. 11. The size of the angle depends on the lengths of the rays forming it.

11. Acute angle – an angle which measure is less than 90°.
12. Right angle – an angle which measure equals 90°.
13. A box in the vertex denotes a right angle.
14. Obtuse angle – an angle which measure is greater than 90° and less than 180°.
15. Straight angle – an angle which measure equals 180°.
16. Reflex angle – an angle which measure is greater than 180° and less than 360°
17. Equal angles are angles that have the same number of degrees.
18. A ray that bisects an angle divides it into 2 equal parts.
19. The line is called the angle bisector.
20. Congruent angles have the same measure.
21. Perpendiculars are lines that form right angles.
22. All right angles are congruent.
23. The sides of a straight angle lie on a straight line.
24. All straight angles are congruent.
25. A perpendicular bisector of a line bisects the line and is perpendicular to the line.

Task 3. Grammar revision. Choose the correct form of the Participle.

1. (to name) geometric ideas we usually use letters of the alphabet.
2. We insisted on the (to follow) notation of the geometric object.
3. (to divide) both the numerator and the denominator by x you will get the following expression.
4. When (to speak) with my science adviser I got better understanding of the latest development in my special field.
5. The properties of the material (to use) in the experiment now are given in the latest article.
6. The advantages of the new system (to prove) by many tests are very important.
7. Two angles (to have) the same vertex and a common side are refered to as adjacent angles.
8. The concepts (to introduce) at the seminar should be considered in detail.
9. The (to obtain) difference must be checked carefully.
10. The (to expect) result must prove that this law holds for similar cases.

Task 4. Fill in the blanks with the necessary words.

1. To (измерить) an angle, compare its side to the corner of this page.
2. The corner represents (прямой угол), whose measurement is 90°.
3. If the angle is smaller than the corner, the angle is (острый угол).
4. If the opening is larger than the corner of the page, the angle is (тупой). Its measure is more than 90°.

5. Locate the point of your (транспортир) which represents the (вершина) and align the vertex with the point.
6. Rotate the protractor keeping the vertex aligned until one (сторона) of the angle is on the 0° – 180° line of the protractor.
7. The angle measure that is (определяется) by the side of the angle that is not on the 0° – 180° line of the protractor.
8. You may have to (продлить) one side of the angle so that it crosses the scale.
9. Use the proper (обозначение), m is the symbol for “measure of”.

Task 5. Match the left and the right parts of the following statements.

1. A group of two angles is known	a) two angles whose measures add up to 180°.
2. Adjacent angles are	b) two nonadjacent angles formed by two intersecting lines.
3. Vertical angles are	
4. Complementary angles are	c) is the complement of the other.
5. One angle	d) two angles whose measures add up to 90°.
6. Supplementary angles are	e) as a pair of angles.
7. If an angle is cut into two adjacent angles	f) two angles that have the same vertex and a common side. g. are congruent.
8. If the exterior sides of a pair of adjacent angles are perpendicular	h) then the sum of the measures of the adjacent angles equals the measure of the original angle.
9. If two angles are congruent and supplementary	i) then the angles are complementary.
10. Vertical angles are	j) then each angle is a right angle.

Task 6. Grammar revision; the Continuous or Perfect Continuous Tenses.
1. I (to look for) a photographs my brother sent to me.
2. They (to have) a meeting now.
3. The phone always (to ring) when I (to have) a bath.
4. Friends always (to talk) to me when I (to try) to concentrate.
5. He (to watch) television when the door bell (to ring).
6. He (to build up) his business all his life.
7. They (to stay) with us for a couple of weeks.
8. By 1992 he (to live) there for ten years.
9. The video industry (to develop) rapidly.
10. He (to work) nights next week.
11. She (to spend) this summer in Europe.
12. Why are you so late? I (to wait) you for hours.
13. The boys must be tired. They (to play) football in the garden all afternoon.

14. The old town theatre is currently (to rebuild). 15. I usually (to go) to work by car, but I (to go) on the bus this week while my car (to repair).

Task 7. Match the definitions of the circles with their names.

1. Tangent circles 2. Concentric circles 3. A circumscribed circle 4. Externally tangent circles 5. An inscribed circle 6. Eccentric circles 7. Inscribed polygon 8. A circumscribed polygon 9. Internally tangent circles	a. are circles that have different centers. b. are both circles which are on the opposite sides of the tangent line. c. is a polygon that is inside a circle so that each of its vertices lies on the circle. d. is a circle to which all the sides of a polygon are tangents. e. is a polygon that is outside the circle in such a way that all of its sides are tangent to the circle. f. is a circle passing through each vertex of a polygon. g. both circles which are on the same side of the tangent line. h. are two or more circles in a plane with the same center, but the lengths of their radii vary. The annulus is the region between concentric circles. i. are two circles that intersect only at one point.

Task 8. Revise the Degrees of Comparison. Give the best English equivalents for the words in parentheses.

1. A circle is (самая простая) of all curved lines.
2. Every point at a distance (больше) than radius (говорят) to be outside the circle.
3. A secant segment is a line segment with an endpoint in the exterior of a circle, and the other endpoint on the circle, (самой далекой) from the external point.
4. Tom comes top in all the exams – he must be (самый умный) student in the group.
5. (Чем меньше) students think, (тем больше) they talk.
6. How are you today? – I’m very (хорошо), thanks.
7. Is this proof (более правильно)?
8. Peter speaks English (наиболее бегло) of all the students in this group.
9. (Чем больше) I learn, (тем больше) I forget and (тем меньше) I know.
10. (Чем скорее) the problem is solved, (тем лучше).
11. This contribution of the ancient Greeks is (намного больше, чем) the formulas of the Egyptians.

UNIT 9

Text 9. Continuous function

A continuous function is, roughly speaking, a function for which small changes in the input result in small changes in the output. Otherwise, a function is said to be a *discontinuous* function. A continuous function with a continuous inverse function is called a homeomorphism.

Continuity of functions is one of the core concepts of topology. This text focuses on the special case where the inputs and outputs of functions are real numbers.

As an example, consider the function *h*(*t*), which describes the height of a growing flower at time *t*. This function is continuous. By contrast, if *M*(*t*) denotes the amount of money in a bank account at time *t*, then the function jumps whenever money is deposited or withdrawn, so the function *M*(*t*) is discontinuous.

History. A form of this epsilon-delta definition of continuity was first given by Bernard Bolzano in 1817. Augustin-LouisCauchy defined continuity of $y = f(x)$ as follows: an infinitely small increment α of the independent variable *x* always produces an infinitely small change $f(x+\alpha) - f(x)$ of the dependent variable *y* . Cauchy defined infinitely small quantities in terms of variable quantities, and his definition of continuity closely parallels the infinitesimal definition used today. The formal definition and the distinction between pointwise continuity and uniform continuity were first given by Bolzano in the 1830s but the work wasn't published until the 1930s. Eduard Heine provided the first published definition of uniform continuity in 1872, but based these ideas on lectures given by Peter Gustav Lejeune Dirichletin 1854.

Real-valued continuous functions. Definition.

A function from the set of real numbers to the real numbers can be represented by a graph in the Cartesian plane; such a function is continuous if, roughly speaking, the graph is a single unbroken curve with no "holes" or "jumps".

There are several ways to make this definition mathematically rigorous. These definitions are equivalent to one another, so the most convenient definition can be used to determine whether a given function is continuous or not. In the definitions below, $f : I \to \mathbf{R}.$ is a function defined on a subset *I* of the set R of real numbers. This subset *I* is referred to as the domain of *f*. Some possible choices include *I* = R, the whole set of real numbers, an open interval

$I = (a,b) = \{x \in \mathbf{R} \mid a < x < b\},$ or a closed interval

$I = [a,b] = \{x \in \mathbf{R} \mid a \le x \le b\}.$ Here, *a* and *b* are real numbers.

Definition in terms of limits of functions. The function *f* is *continuous at some point* *c* of its domain if the limit of *f*(*x*) as *x* approaches *c* through the domain of *f* exists and is equal to *f*(*c*). This is written as $\lim_{x \to c} f(x) = f(c).$

In detail this means three conditions: first, *f* has to be defined at *c*. Second, the limit on the left hand side of that equation has to exist. Third, the value of this limit must equal *f*(*c*). The function *f* is said to be *continuous* if it is continuous at every point of its domain. If the point *c* in the domain of *f* is not a limit point of the domain, then this condition is vacuously true, since *x* cannot approach *c* through values not equal to *c*. Thus, for example, every function whose domain is the set of all integers is continuous.

Definition in terms of limits of sequences. One can instead require that for any sequence $(x_n)_{n\in\mathbb{N}}$ of points in the domain which converges to *c*, the corresponding sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to *f*(*c*). In mathematical notation, $\forall (x_n)_{n\in\mathbb{N}} \subset I : \lim_{n\to\infty} x_n = c \Rightarrow \lim_{n\to\infty} f(x_n) = f(c)\,.$

Weierstrass definition (epsilon-delta) of continuous functions.

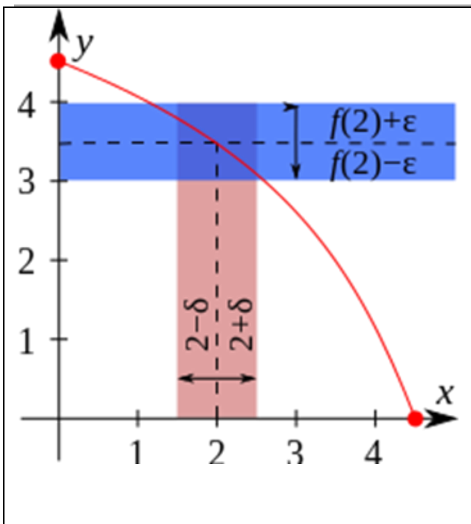


Illustration of the ε-δ-definition: for ε=0.5, c=2, the value δ=0.5 satisfies the condition of the definition. Explicitly including the definition of the limit of a function, we obtain a self-contained definition: Given a function *f* as above and an element *c* of the domain *I*, *f* is said to be continuous at the point *c* if the following holds: For any number $\varepsilon > 0$, however small, there exists some number $\delta > 0$ such that for all *x* in the domain of *f* with $c - \delta < x < c + \delta$, the value of *f*(*x*) satisfies $f(c) - \varepsilon < f(x) < f(c) + \varepsilon.$

Alternatively written, continuity of $f : I \to R$ at $c \in I$ means that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in I$,: $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon.$

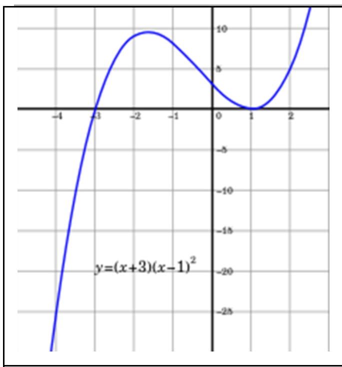
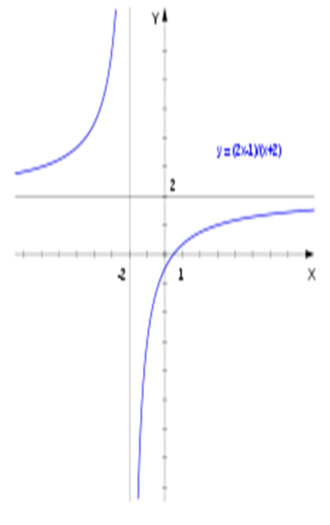
More intuitively, we can say that if we want to get all the *f*(*x*) values to stay in some small neighborhood around *f*(*c*), we simply need to choose a small enough neighborhood for the *x* values around *c*, and we can do that no matter how small the*f*(*x*) neighborhood is; *f* is then continuous at *c*.

In modern terms, this is generalized by the definition of continuity of a function with respect to a basis for the topology, here the metric topology.

Definition using the hyperreals. Cauchy defined continuity of a function in the following intuitive terms: an infinitesimal change in the independent variable corresponds to an infinitesimal change of the dependent . Non-standard analysis is a way of making this mathematically rigorous. The real line is augmented by the addition of infinite and infinitesimal numbers to form the hyperreal numbers. In nonstandard analysis, continuity can be defined as follows.

A real-valued function *f* is continuous at *x* if its natural extension to the hyperreals has the property that for all infinitesimal *dx*, *f*(*x*+*dx*) − *f*(*x*) is infinitesimal.

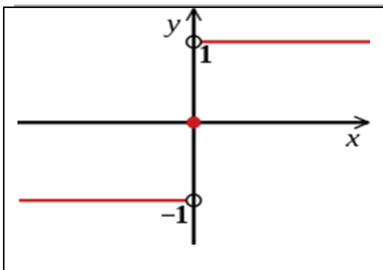
In other words, an infinitesimal increment of the independent variable always produces to an infinitesimal change of the dependent variable, giving a modern expression to Augustin-Louis Cauchy's definition of continuity. **Examples:**

	The graph of a cubic function has no jumps or holes. The function is continuous. All polynomial functions,such as $f(x) = x^3 + x^2 - 5x + 3$, are continuous. This is a consequence of the fact that, given two continuous functions $f, g\colon I \to \mathbf{R}$ defined on the same domain *I*, then the sum *f* + *g*, and the product *fg* of the two functions are continuous (on the same domain *I*). Moreover, the function $\frac{f}{g}\colon \{x \in I \mid g(x) \neq 0\} \to \mathbf{R}, x \mapsto \frac{f(x)}{g(x)}$ is continuous.
	The graph of a rational function. The function is not defined for *x*=−2. The vertical and horizontal lines are asymptotes. The function $f(x) = \frac{2x-1}{x+2}$ is defined for all real numbers $x \neq -2$ and is continuous at every such point. The question of continuity at $x = -2$ does not arise, since $x = -2$ is not in the domain of *f*. There is no continuous function *F*: R → R that agrees with *f*(*x*) for all $x \neq -2$. The sinc function $g(x) = (\sin x)/x$, defined for all $x \neq 0$ is continuous at these points. However, this function *can* be extended to a continuous function on all real numbers, namely $G(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases}$ since the limit of *g*(*x*), when *x* approaches 0, is 1. Therefore, the point x=0 is called a removable singularity of g.

Given two continuous functions

$f\colon I \to J (\subset \mathbf{R}), g\colon J \to \mathbf{R},$ the composition $g \circ f\colon I \to \mathbf{R}, x \mapsto g(f(x))$ is continuous.

An example of a discontinuous function is the function *f* defined by *f*(*x*) = 1 if *x* > 0, *f*(*x*) = 0 if *x* ≤ 0. Pick for instance ε = ½. There is no δ-neighborhood around *x* = 0 that will force all the *f*(*x*) values to be within ε of *f*(0). Intuitively we can think of this type of discontinuity as a sudden jump in function values. Similarly, the signum or sign function

	Plot of the signum function. The hollow dots indicate that sgn(*x*) is 1 for all *x*>0 and −1 for all *x*<0. $\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$ is discontinuous at *x* = 0 but continuous everywhere else.

Yet another example: the function

$$f(x) = \begin{cases} \sin\left(\frac{1}{x^2}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

is continuous everywhere apart from $x = 0$.

$$D(x) = \begin{cases} 0 \text{ if } x \text{ is irrational } (\in \mathbb{R} \setminus \mathbb{Q}) \\ 1 \text{ if } x \text{ is rational } (\in \mathbb{Q}) \end{cases}$$

Properties. Intermediate value theorem.

The intermediate value theorem is an existence theorem, based on the real number property of completeness, and states:

If the real-valued function f is continuous on the closed interval $[a, b]$ and k is some number between $f(a)$ and $f(b)$, then there is some number c in $[a, b]$ such that $f(c) = k$.

For example, if a child grows from 1 m to 1.5 m between the ages of two and six years, then, at some time between two and six years of age, the child's height must have been 1.25 m.

As a consequence, if f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, then, at some point c in $[a, b]$, $f(c)$ must equal zero.

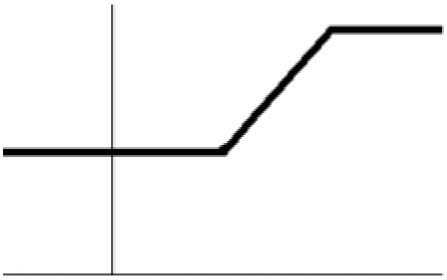
Extreme value theorem. The extreme value theorem states that if a function f is defined on a closed interval $[a,b]$ (or any closed and bounded set) and is continuous there, then the function attains its maximum, i.e. there exists $c \in [a,b]$ with $f(c) \geq f(x)$ for all $x \in [a,b]$. The same is true of the minimum of f. These statements are not, in general, true if the function is defined on an open interval (a,b) (or any set that is not both closed and bounded), as, for example, the continuous function $f(x) = 1/x$, defined on the open interval (0,1), does not attain a maximum, being unbounded above.

Relation to differentiability and integrability. Every differentiable function $f: (a, b) \to \mathbf{R}$ is continuous, as can be shown. The converse does not hold: for example, the absolute value function $f(x) = |x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$ is everywhere continuous. However, it is not differentiable at $x = 0$ (but is so everywhere else). Weierstrass's function is everywhere continuous but nowhere differentiable.

The derivative $f'(x)$ of a differentiable function $f(x)$ need not be continuous. If $f'(x)$ is continuous, $f(x)$ is said to be continuously differentiable. The set of such functions is denoted $C^1((a, b))$. More generally, the set of functions $f: \Omega \to \mathbf{R}$ (from an open interval (or open subset of R) Ω to the reals) such that f is n times differentiable and such that the n-th derivative of f is continuous is denoted $C^n(\Omega)$. See differentiability class. In the field of computer graphics, these three levels are sometimes called G^0 (continuity of position), G^1 (continuity of tangency), and G^2(continuity of curvature).

Every continuous function $f : [a, b] \to \mathbf{R}$ is integrable (for example in the sense of the Riemann integral). The converse does not hold, as the (integrable, but discontinuous) sign function shows.

Pointwise and uniform limit



A sequence of continuous functions $f_n(x)$ whose (pointwise) limit function $f(x)$ is discontinuous. The convergence is not uniform.

Given a sequence $f_1, f_2, \ldots : I \to \mathbf{R}$ of functions such that the limit

$f(x) := \lim_{n\to\infty} f_n(x)$ exists for all x in I, the resulting function $f(x)$ is referred to as the pointwise limit of the sequence of functions $(f_n)_{n\in N}$. The pointwise limit function need not be continuous, even if all functions f_n are continuous, as the animation at the right shows. However, f is continuous when the sequence converges uniformly, by the uniform convergence theorem. This theorem can be used to show that the exponential functions, logarithms, square root function, trigonometric functions are continuous.

Mathematical terminology

continuous function - непрерывная функция
discontinuous function – дискретная, прерывная функция
continuous inverse function – непрерывная обратная функция
homeomorphism – гомеоморфизм, топологическое отображение
height – высота, вершина, верх
epsilon-delta definition – эпсилон-дельта определение
Bernard Bolzano – Бернард Больца́но (1781-1848), чешский математик, философ, автор первой строгой теории вещественных чисел и один из основоположников теории множеств.
Augustin-Louis Cauchy - Огюсте́н Луи́ Коши́ (1789-1857), великий французский математик и механик, член Парижской академии наук, Лондонского королевского общества, Петербургской академии наук. Разработал фундамент математического анализа, внёс огромный вклад в анализ, алгебру, математическую физику. Его имя внесено в список величайших учёных Франции, помещённый на первом этаже Эйфелевой башни.
uniform continuity - равномерная непревывность
curve - кривая; изгибать; изгиб; график; дуга; закругление; искривление
equivalent - эквивалент, эквивалентный

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open interval - открытый интервал; **closed interval** - замкнутый интервал
limit point - предельная точка
vacuously true – бессодержательно истинный; **vacuous set** – пустое множество
neighborhood – окрестность (точки)
metric topology - метрическая топология
infinitesimal - бесконечно малая (величина)
Non-standard analysis - нестандартный анализ
hyperreal number - гипервещественное число
Augustin-Louis Cauchy's definition of continuity – определение непрерывности по Коши
cubic function - кубическая функция
polynomial functions – полиномиальная функция, полином
rational function – рациональная функция
asymptote - асимптота
sinc function [ˈsɪŋk] *sinus cardinalis* (cardinal sine function) - «кардина́льный си́нус», математическая функция
signum (sign function) - сигнум (функция), знаковая функция
intermediate value theorem - теорема о промежуточном значении
existence theorem – теорема существования
property of completeness – свойство полноты (напр. системы функций)
closed interval – замкнутый интервал
extreme value theorem – теорема об экстремальном значении, экстремумах функции
differentiable function – дифференцируемая функция, гладкая функция
Weierstrass's function – Функция Вейерштрасса, непрерывная функция, нигде не имеющая производной
derivative - производная функция
open subset – открытое подмножество
differentiability – дифференцируемость (свойство функции, означающее возможность вычисления производной по какому-л. аргументу в какой-л. точке; в случае с функцией полезности означает, что поверхности безразличных множеств не имеют изломов)
integrable – интегрируемый, суммируемый; **absolutely integrable** -абсолютно интегрируемый
Riemann integral – интегра́л Ри́мана, определённый интеграл
pointwise limit – точечный предел
uniform convergence theorem – равномерная сходимость последовательности функций (отображений) – свойство последовательности $f_n : X \to Y$, где X — произвольное множество, $Y = (Y, d)$ - метрическое пространство, $n = 1, 2, \ldots$ сходится к функции (отображению) $f : X \to Y$, означающее, что для любого $\varepsilon > 0$ существует такой номер N_ε, что для всех номеров $n > N_\varepsilon$ и всех точек $x \in X$ выполняется неравенство $|f_n(x) - f(x)| < \varepsilon$. Обычно обозначается $f_n \rightrightarrows f$. Это условие равносильно тому, что $\lim_{n\to\infty} \sup_{x \in X} |f_n(x) - f(x)| = 0$.
exponential function – экспоненциальная функция, показательная функция
logarithm – логарифм; **common logarithm** – десятичный логарифм, **natural logarithm** – натуральный логарифм; **square root function** – функция квадратный корень
trigonometric function – тригонометрическая функция

Grammar, Lexical, Translation and Speaking Exercises

Task 1. Translate the definitions into Russian and find the suitable term from the opposite column.

1) an interval on the real line including its end points, as [0, 1], the set of reals between and including 0 and 1; 2) any of the set of numbers formed by the addition of infinite numbers and infinitesimal numbers to the set of real numbers; 3) a straight line that is closely approached by a plane curve so that the perpendicular distance between them decreases to zero as the distance from the origin increases to infinity; 4) the exponent indicating the power to which a fixed number, the base, must be raised to obtain a given number or variable. It is used esp to simplify multiplication and division: if ax = M, then the logarithm of M to the base a (written logaM) is x Often shortened to: log; 5) a function whose value is a constant raised to the power of the argument, esp. the function where the constant is e.	a) closed interval b) hyperreal number c) asymptote d) logarithm e) exponential function

Task 2. Complete each of the sentences below by choosing one of the pronouns in brackets.

1. ... arrived in good time and the meeting started promptly at 3.30 (anybody/ nobody/ everybody)
2. ... in the village went to the party but ... enjoyed it very much. (everybody/ no one/ some one), (anybody/ somebody/ nobody)
3. ... heard anything. (everyone/ nobody/ somebody)
4. “Who shall I give this one to? – You can give it to ... . It doesn’t matter.” (everyone/ nobody/ anybody)
5. That’s a very easy job. ... can do it. (everyone/ nobody/ somebody).

6. Would you like ... to drink? (anything/ something/ nothing)
7. I thought I'd seen you ... . (anywhere/ somewhere/ nowhere)
8. There was ... to hide. (anywhere/ somewhere/ nowhere)
9. You still haven't told me ... . (anything/ something/ nothing)
10. Does ... agree with me? (anybody/ somebody/ nobody)
11. I want to introduce you to ... (no one/ someone/ any one)
12. The box was completely empty. There was ... in it. (nothing/ anything)
13. "Excuse me, you've dropped ... . Yes, look. It's passport." (something/ anything/ everything)
14. It's all finished. I am afraid there's ... left. (nothing/ anything/ something)
15. I heard a noise, but I didn't see ... . (any one/ no one)
16. It's too late. We can't do ... to help. (anything/ nothing)
17. I agree with most of what he said, but I don't agree with ... . (something/ everything/ anything)
18. ... offered to help. They probably didn't have the time. (anybody/ nobody/ everybody)
19. If ... asks, you can tell them I'll be back soon. (somebody/ anybody/ everybody)
20. There was ... in the box, it was completely empty. (nothing/ anything/ something)

Task 3. Ask special questions to which the sentences below are the answers.
1. A statement satisfying certain conditions is true. (what)
2. Like terms being arranged in the following way will be enclosed in the parenthesis.(where)
3. Reference is made to the commonly accepted system. (what ... to)
4. The force keeping all material bodies including people on the Earth is called gravitation. (what kind)
5. Having used the classification suggested by my science adviser I found it very convenient. (when)
6. Having been given little information they couldn't continue the research. (why) 7. Having followed the procedure they obtained the required results. (how)
8. Any fraction represents the quotient of its numerator divided by its denominator. (what)
9. Having obtained a proper interpretation of this fact they realized the importance of the problem. (when)
10. The created method has no advantages over the old one. (what)
11. Differential equation is an equation containing differentials or derivatives of a function of one independent variable. (what)

Task 4. Find the corresponding Russian sentence.

1. Geometry is a branch of mathematics concerned with questions of shape, size, relative position of figures, and the properties of space.

a. Геометрия – это область математики, которая рассматривала форму, размер, относительное расположение фигур и свойства пространства.

b. Геометрия – это область математики, рассматривающая вопросы формы, размера, относительного расположения фигур и свойства пространства.

c. Геометрия – это раздел математики, в котором рассматривали форму, размер, относительное расположение фигур и свойства пространства.

2. From what you already know you may deduce that drawing two rays originating from the same end point forms an angle.

a. Из того, что вы уже знаете, вы можете сделать вывод, что, рисуя два луча, исходящих из одной конечной точки, вы получаете угол.

b. Из того, что вам известно, вы можете сделать вывод, что изображе- ние двух лучей, берущих начало в одной и той же конечной точке, образует угол.

c. Из того, что вы уже узнали, вы, возможно, сделали вывод, что рисунок двух лучей, берущих начало в одной конечной точке, образует угол.

3. The approach to the problem being considered remained traditional.

a. Рассматривался оставшийся подход к традиционной проблеме.

b. Подход к оставшейся проблеме рассматривался традиционно.

c. Подход к рассматриваемой проблеме оставался традиционным.

4. Physical facts expressed in terms of mathematics do not seem unusual nowadays.

a. Выраженные математические факты казались необычными в физических терминах в настоящее время.

b. Физические факты, выраженные в математических терминах, не кажутся необычными сегодня.

c. То, что физические факты в настоящее время выражаются математическими терминами, не кажется сегодня необычным.

5. Having made a number of experiments Faraday discovered electromagnetic induction.

a. Проводя ряд экспериментов, Фарадей открыл электромагнитную индукцию.

b. Проделав ряд экспериментов, Фарадей открыл электромагнитную индукцию.

c. Сделав ряд экспериментов, Фарадей открыл электромагнитную индукцию.

Task 5. Translate into English.

1. Первая линия, с которой мы знакомимся, изучая математику, – это прямая линия.
2. Дать строгое определение этого понятия совсем непросто.
3. В работах Евклида (Euclid) линия определялась как длина без толщины.
4. Угол – самая простая геометрическая фигура после точки, прямой, луча и отрезка.
5. Если в плоскости из точки O провести два различных луча OA и OB, то они разделят плоскость на две части, каждая из которых называется углом с вершиной O и сторонами OA и OB.
6. Луч, делящий угол пополам и берущий начало в вершине угла, называется его биссектрисой.
7. Биссектриса развернутого угла делит его на два смежных угла, называемых прямыми углами.
8. Большое значение для теории и практики имеет определение величины или меры угла.
9. Основное свойство меры угла должно заключаться в том, чтобы равные углы имели одинаковую меру.
10. Градусная мера используется в элементарной геометрии для измерения углов.
11. Каждый, наверное, знаком с транспортиром – измерителем углов на чертежах.
12. Углы меньше прямого называются острыми, а углы больше прямого, но меньше развернутого, называются тупыми.
13. Первая книга Евклида начинается с 23 «определений», среди них такие: точка есть то, что не имеет частей; линия есть длина без ширины; линия ограничена точками; прямая есть линия, одинаково расположенная относительно своих точек; наконец, две прямые, лежащие в одной плоскости, называются параллельными, если они, сколь угодно продолженные, не встречаются.
14. Изложение геометрии в «Началах» Евклида считалось образцом, которому стремились следовать ученые и за пределами математики.
15. Эту теорему используют, чтобы показать, что экспоненциальные функции, логарифмические, тригонометрические функции непрерывны.
16. В качестве основания логарифма обычно используют число 10 или число e; соответственно говорят о десятичном (decimal logarithm) либо о натуральном логарифме (natural logarithm).

UNIT 10

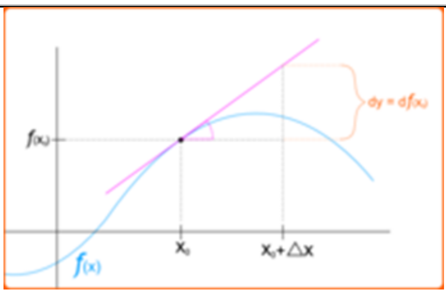
Text 10. Differential of a function

In calculus, the differential represents the principal part of the change in a function $y = f(x)$ with respect to changes in the independent variable. The differential dy is defined by $dy = f'(x)\,dx,$ where $f'(x)$ is the derivative of f with respect to x, and dx is an additional real variable (so that dy is a function of x and dx). The notation is such that the equation $dy = \frac{dy}{dx}\,dx$ holds, where the derivative is represented in the Leibniz notation dy/dx, and this is consistent with regarding the derivative as the quotient of the differentials. One also writes $df(x) = f'(x)\,dx.$
The precise meaning of the variables dy and dx depends on the context of the application and the required level of mathematical rigor. The domain of these variables may take on a particular geometrical significance if the differential is regarded as a particular differential form, or analytical significance if the differential is regarded as a linear approximation to the increment of a function. Traditionally, the variables dx and dy are considered to be very small (infinitesimal), and this interpretation is made rigorous in non-standard analysis.
History and usage. The differential was first introduced via an intuitive or heuristic definition by Gottfried Wilhelm Leibniz, who thought of the differential dy as an infinitely small (or infinitesimal) change in the value y of the function, corresponding to an infinitely small change dx in the function's argument x. For that reason, the instantaneous rate of change of y with respect to x, which is the value of the derivative of the function, is denoted by the fraction $\frac{dy}{dx}$ what is called the Leibniz notation for derivatives. The quotient dy/dx is not infinitely small; rather it is a real number.

The use of infinitesimals in this form was widely criticized, for instance by the famous pamphlet The Analyst by Bishop Berkeley. Augustin-Louis Cauchy (1823) defined the differential without appeal to the atomism of Leibniz's infinitesimals. Instead, Cauchy, following d'Alembert, inverted the logical order of Leibniz and his successors: the derivative itself became the fundamental object, defined as a limit of difference quotients, and the differentials were then defined in terms of it. That is, one was free to *define* the differential dy by an expression $dy = f'(x)\,dx$ in which dy and dx are simply new variables taking finite real values, not fixed infinitesimals as they had been for Leibniz.

Cauchy's approach was a significant logical improvement over the infinitesimal approach of Leibniz because, instead of invoking the metaphysical notion of infinitesimals, the quantities *dy* and *dx* could now be manipulated in exactly the same manner as any other real quantities in a meaningful way. Cauchy's overall conceptual approach to differentials remains the standard one in modern analytical treatments, although the final word on rigor, a fully modern notion of the limit, was ultimately due to Karl Weierstrass.

Following twentieth-century developments in mathematical analysis and differential geometry, it became clear that the notion of the differential of a function could be extended in a variety of ways. In real analysis, it is more desirable to deal directly with the differential as the principal part of the increment of a function. This leads directly to the notion that the differential of a function at a point is a linear functional of an increment Δ*x*. This approach allows the differential to be developed for a variety of more sophisticated spaces, ultimately giving rise to such notions as the Gâteaux derivative. Likewise, in differential geometry, the differential of a function at a point is a linear function of a tangent vector , which exhibits it as a kind of one-form: the exterior derivative of the function. In non-standard calculus, differentials are regarded as infinitesimals, which can themselves be put on a rigorous footing.



Definition. The differential of a function *f*(x) at a point x0. The differential is defined in modern treatments of differential calculus as follows.The differential of a function *f*(*x*) of a single real variable *x* is the function *df* of two independent real variables *x* and *Δx* given by $df(x,\Delta x) \stackrel{\text{def}}{=} f'(x)\,\Delta x.$

One or both of the arguments may be suppressed, i.e., one may see *df*(*x*) or simply *df*. If *y* = *f*(*x*), the differential may also be written as *dy*. Since *dx*(*x*, Δ*x*) = Δ*x* it is conventional to write *dx* = Δ*x*, so that the following equality holds: $df(x) = f'(x)\,dx$.

This notion of differential is broadly applicable when a linear approximation to a function is sought, in which the value of the increment Δ*x* is small enough. More precisely, if *f* is a differentiable function at *x*, then the difference in *y*-values

$\Delta y \stackrel{\text{def}}{=} f(x+\Delta x) - f(x)$ satisfies $\Delta y = f'(x)\,\Delta x + \varepsilon = df(x) + \varepsilon$

where the error ε in the approximation satisfies ε/Δ*x* → 0 as Δ*x* → 0. In other words, one has the approximate identity $\Delta y \approx dy$ in which the error can be made as small as desired relative to Δ*x* by constraining *Δx* to be sufficiently small; that is to say,

$\frac{\Delta y - dy}{\Delta x} \to 0$ as $\Delta x \to 0$. For this reason, the differential of a function is known as the principal (linear) part in the increment of a function: the differential is a linear function of the increment Δx, and although the error ε may be nonlinear, it tends to zero rapidly as Δx tends to zero.

Differentials in several variables. For functions of more than one independent variable, $y = f(x_1, \ldots, x_n)$, the partial differential of y with respect to any one of the variables x_1 is the principal part of the change in y resulting from a change dx_1 in that one variable. The partial differential is therefore $\frac{\partial y}{\partial x_1} dx_1$ involving the partial derivative of y with respect to x_1. The sum of the partial differentials with respect to all of the independent variables is the total differential

$$dy = \frac{\partial y}{\partial x_1} dx_1 + \cdots + \frac{\partial y}{\partial x_n} dx_n,$$

which is the principal part of the change in y resulting from changes in the independent variables x_i.

More precisely, in the context of multivariable calculus, if f is a differentiable function, then by the definition of the differentiability, the increment

$$\Delta y \stackrel{\text{def}}{=} f(x_1 + \Delta x_1, \ldots, x_n + \Delta x_n) - f(x_1, \ldots, x_n)$$
$$= \frac{\partial y}{\partial x_1} \Delta x_1 + \cdots + \frac{\partial y}{\partial x_n} \Delta x_n + \varepsilon_1 \Delta x_1 + \cdots + \varepsilon_n \Delta x_n$$

where the error terms ε_i tend to zero as the increments Δx_i jointly tend to zero. The total differential is then rigorously defined as

$$dy = \frac{\partial y}{\partial x_1} \Delta x_1 + \cdots + \frac{\partial y}{\partial x_n} \Delta x_n.$$

Since, with this definition, $dx_i(\Delta x_1, \ldots, \Delta x_n) = \Delta x_i$, one has $dy = \frac{\partial y}{\partial x_1} dx_1 + \cdots + \frac{\partial y}{\partial x_n} dx_n.$

As in the case of one variable, the approximate identity holds $dy \approx \Delta y$ in which the total error can be made as small as desired relative to $\sqrt{\Delta x_1^2 + \cdots + \Delta x_n^2}$ by confining attention to sufficiently small increments.

Properties. A number of properties of the differential follow in a straightforward manner from the corresponding properties of the derivative, partial derivative, and total derivative. These include:

Linearity: For constants a and b and differentiable functions f and g,

$$d(af + bg) = a\,df + b\,dg.$$

Product rule: For two differentiable functions f and g, $d(fg) = f\,dg + g\,df.$

An operation d with these two properties is known in abstract algebra as a derivation. They imply the Power rule $d(f^n) = nf^{n-1}df$

In addition, various forms of the chain rule hold, in increasing level of generality:

If $y = f(u)$ is a differentiable function of the variable u and $u = g(x)$ is a differentiable function of x, then $dy = f'(u)\,du = f'(g(x))g'(x)\,dx.$

If $y = f(x_1, ..., x_n)$ and all of the variables $x_1, ..., x_n$ depend on another variable t, then by the chain rule for partial derivatives, one has

$$\begin{aligned} dy &= \frac{dy}{dt}dt \\ &= \frac{\partial y}{\partial x_1}dx_1 + \cdots + \frac{\partial y}{\partial x_n}dx_n \\ &= \frac{\partial y}{\partial x_1}\frac{dx_1}{dt}\,dt + \cdots + \frac{\partial y}{\partial x_n}\frac{dx_n}{dt}\,dt. \end{aligned}$$

Heuristically, the chain rule for several variables can itself be understood by dividing through both sides of this equation by the infinitely small quantity dt.

More general analogous expressions hold, in which the intermediate variables x_i depend on more than one variable.

Mathematical terminology

differential – дифференциал, дифференциальный
derivative – производная
variable – переменная
Leibniz notation – система обозначений Лейбница
differential form – дифференциальная форма
linear approximation – линейная аппроксимация, линейное приближение
Gottfried Wilhelm Leibniz (1646 – 1716) – немецкий философ, логик, математик, механик, физик, юрист, историк, дипломат, изобретатель и языковед. Основатель и первый президент Берлинской Академии наук, иностранный член Французской Академии наук.
Важнейшие научные достижения: Лейбниц, независимо от Ньютона , создал математический анализ - дифференциальное и интегральное исчисления, основанные на бесконечно малых; Лейбниц создал комбинаторику как науку; только он во всей истории математики одинаково свободно работал как с непрерывным, так и с дискретным; он заложил основы математической логики; описал двоичную систему счисления с цифрами 0 и 1, на которой основана современная компьютерная техника; в механике ввёл понятие «живой силы» (прообраз современного понятия кинетической энергии) и сформулировал закон сохранения энергии; в психологии выдвинул понятие бессознательно «малых перцепций» и развил учение о бессознательной психической жизни.
appeal to – ссылаться

atomism - атомизм, атомистическая теория (учение о дискретном строении материи); атомистика (редк.) – учение о прерывистом, дискретном строении материи, т. е. о реальности как о совокупности множества независимых частиц.

Jean-Baptiste le Rond d'Alembert (1717 – 1783) – Жан Леро́н Д'Аламбе́р (д'Аламбер, Даламбер) французский учёный-энциклопедист. Широко известен как математик и механик

Karl Theodor Wilhelm Weierstrass (1815 – 1897) – Карл Те́одор Вильге́льм Ве́йерштрасс, немецкий математик, «отец современного анализа»

linear functional – линейный функционал

Gâteaux derivative – производная Гато расширяет концепцию производной на локально выпуклые топологические векторные пространства. Название дано в честь французского математика Рене Гато

differential geometry – дифференциальная геометрия

tangent vector – тангенциальный вектор

exterior derivative – внешняя производная

linear approximation – линейная аппроксимация, линейное приближение

differentiable function – дифференцируемая функция, гладкая функция

linear part – линейная часть

partial derivative – частная производная

differentiability – дифференцируемость (свойство функции, означающее возможность вычисления производной по какому-л. аргументу в какой-л. точке; в случае с функцией полезности означает, что поверхности безразличных множеств не имеют изломов)

linearity – линейность (свойство линейной функции; также свойство или состояние переменной, находящейся в линейной зависимости от другой переменной)

product rule – теорема умножения; **chain rule –** цепное правило

Grammar, Lexical, Translation and Speaking Exercises

Task 1. Combine the columns.

1) a theorem that may be used in the differentiation of the function of a function. It states that du/dx = (du/dy)(dy/dx), where y is a function of x and u a function of y; 2) a derivative of a function of two or more variables with respect to one variable, the other(s) being treated as constant; 3) the application of differential calculus to geometrical problems; the study of objects that remain unchanged by transformations that preserve derivatives.	a) chain rule b) partial derivative c) differential geometry

Task 2. Read, memorize and translate into Russian.

1. Euler's theorem: The relationship V - E + F = 2 for any simple closed polyhedron, where V is the number of vertices, E the number of edges, and F the number of faces. (A simple closed polyhedron is one that is topologically equivalent to a sphere) The

expression V - E + F = 2 is called the Euler characteristic, and its value serves to indicate the topological genus.

2. Euler's formula: The formula: $e^{ix} = \cos x + i \sin x$

It was introduced by Euler in 1748, and is used as a method of expressing complex numbers. The special case in which x = π leads to the formula $e^{i\pi}$= -1.

Task 3. Match the left and the right parts of the sentences.

1. The interior angles of a triangle	a. are all different in measure
2. A triangle is a polygon with	b. is degenerate
3. In a scalene triangle	c. two sides are equal in length
4. A right-angled triangle has	d. always add up to 360°
5. The three angles of a scalene triangle	e. the cosine rule and sine rule
6. A triangle with an interior angle of 180°	f. all sides have the same length
7. The exterior angles of a triangle	g. always add up to 180°
8. In an equilateral triangle	h. three vertices and three edges
9. In an isosceles triangle	i. all sides are unequal
10. Angles and sides in triangles are related by	j. one of its interior angles measuring 90°

Task 4. Let's revise Perfect Tenses. Complete the sentences using the following words: *already before ever for just by since so still yet never*

1. Have you … dreamt of going to London? 2. I haven't worked out how to set the timer on the video …. 3. My dad's lived in the same house … he was born. 4. The film's only been on … a couple of minutes. 5. Kate has passed three exams out of five … far. 6. He will have translated the text … 3 o'clock tomorrow. 7. He's only … got home. 8. It's eleven o'clock and he … hasn't come home. Where could he be? 9. I've … met Ann …. What's she like? 10. He has … finished doing his homework.

Task 5. Transform the sentences from Perfect Active into Perfect Passive.

1. She has just typed her report for the conference.
2. The teacher told us that she had checked all the tests.
3. The student will have written his degree work by May.
4. They have learnt a lot of new English words.
5. He hasn't found the answer yet.
6. I've just received my exam results.
7. By the end of the conference, the participants had discussed a number of important questions concerning the problem.
8. They will have read two books on topology by the end of the month.

9. We had planned the meeting months in advance, but we still had problems.
10. I had discussed the plan of my work with my science adviser before the end of the class.

Task 6. Choose the correct variant of translation.

1. It's difficult to study a foreign language.

a) Это трудный иностранный язык для изучения.
b) Трудно изучать иностранный язык.
c) Изучать иностранный язык было трудно.

2. He hopes to pass his examination in mathematical analysis.

a. Он надеется сдать экзамен по математическому анализу.
b. Он надеялся на сдачу экзамена по математическому анализу.
c. Он будет надеяться на сдачу экзамена по математическому анализу.

3. She was writing the dictation very carefully in order not to make mistakes.

a. Она написала диктант очень осторожно и не сделала ошибок.
b. Она писала диктант внимательно и в правильном порядке, не делая ошибок.
c. Она писала диктант очень внимательно, чтобы не сделать ошибок.

4. I'm sorry not to have seen this film in English at the lesson.

a. Мне жаль, что на уроке я не посмотрела этот фильм на английском языке.
b. Я сожалею о том, что не посмотрю этот английский фильм на уроке.
c. Я не сожалею о том, что не посмотрел этот фильм на уроке английского.

5. He read the rule several times to understand it better.

a. Он читает правило несколько раз, чтобы понять его лучше.
b. Он прочитал правило несколько раз, чтобы лучше понять его.
c. Он читал правило несколько раз и понимал его лучше.

6. This is just the person to speak to on this problem.

a. Вот человек, о котором говорится в этой проблеме.
b. Это как раз тот человек, с которым можно поговорить на эту тему.
c. Только с этим человеком говорят об этой проблеме.

Task 7. Match the columns.

1) Abbrevation and symbol for the imaginary part of a complex number. 2) A square matrix in which all the entries not in the main diagonal are zero. 3) A variable or function which either takes the value true or false. 4) A set whose elements are themselves sets may be called a family. In certain other circumstances, for example where less formal language is appropriate, the word 'family' may be used as an alternative to 'set'.	a) im b) boolean c) diagonal matrix d) family

UNIT 11

Text 11. Specifying a function

Specifying a function. A function can be defined by any mathematical condition relating each argument (input value) to the corresponding output value. If the domain is finite, a function *f* may be defined by simply tabulating all the arguments *x* and their corresponding function values *f*(*x*). More commonly, a function is defined by a formula, or (more generally) an algorithm – a recipe that tells how to compute the value of *f*(*x*) given any *x* in the domain.

There are many other ways of defining functions. Examples include piecewise definitions, induction or recursion, algebraic or analytic closure, limits, analytic continuation, infinite series, and as solutions to integral and differential equations. The lambda calculus provides a powerful and flexible syntax for defining and combining functions of several variables. In advanced mathematics, some functions exist because of an axiom, such as the Axiom of Choice.

Graph. The *graph* of a function is its set of ordered pairs *F*. This is an abstraction of the idea of a graph as a picture showing the function plotted on a pair of coordinate axes; for example, (3, 9), the point above 3 on the horizontal axis and to the right of 9 on the vertical axis, lies on the graph of $y=x^2$.

Formulas and algorithms. Different formulas or algorithms may describe the same function. For instance $f(x) = (x+1)(x-1)$ is exactly the same function as $f(x) = x^2 - 1$. Furthermore, a function need not be described by a formula, expression, or algorithm, nor need it deal with numbers at all: the domain and codomain of a function may be arbitrary sets. One example of a function that acts on non-numeric inputs takes English words as inputs and returns the first letter of the input word as output.

As an example, the factorial function is defined on the nonnegative integers and produces a nonnegative integer. It is defined by the following inductive algorithm: 0! is defined to be 1, and *n*! is defined to be $n(n-1)!$ for all positive integers *n*. The factorial function is denoted with the exclamation mark (serving as the symbol of the function) after the variable (postfix notation).

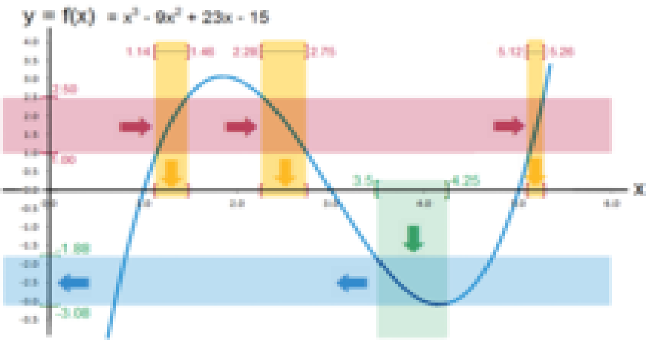
Computability.

Functions that send integers to integers, or finite strings to finite strings, can sometimes be defined by an algorithm, which gives a precise description of a set of steps for computing the output of the function from its input. Functions definable by an algorithm are called *computable functions*. For example, the Euclidean algorithm gives a precise process to compute the greatest common divisor of two positive integers. Many of the functions studied in the context of number theory are computable.

Fundamental results of computability theory show that there are functions that can be precisely defined but are not computable. Moreover, in the sense of cardinality, almost all functions from the integers to integers are not computable. The number of computable functions from integers to integers is countable, because the number of possible algorithms is. The number of all functions from integers to integers is higher: the same as the cardinality of the real numbers. Thus most functions from integers to integers are not computable. Specific examples of uncomputable functions are known, including the busy beaver function and functions related to the halting problem and other undecidable problems.

Basic properties. There are a number of general basic properties and notions. In this section, *f* is a function with domain *X* and codomain *Y*.

Image and preimage



The graph of the function $f(x) = x^3 - 9x^2 + 23x - 15$. The interval $A = [3.5, 4.25]$ is a subset of the domain, thus it is shown as part of the *x*-axis (green). The image of *A* is (approximately) the interval $[-3.08, -1.88]$. It is obtained by projecting to the *y*-axis (along the blue arrows) the intersection of the graph with the light green area consisting of all points whose *x*-coordinate is between 3.5 and 4.25. the part of the (vertical) *y*-axis shown in blue. The preimage of $B = [1, 2.5]$ consists of three intervals. They are obtained by projecting the intersection of the light red area with the graph to the *x*-axis.

If *A* is any subset of the domain *X*, then *f*(*A*) is the subset of the codomain *Y* consisting of all images of elements of A. We say the *f*(*A*) is the *image* of A under f. The *image* of *f* is given by *f*(*X*). On the other hand, the *inverse image* or *preimage*, *complete inverse image* of a subset *B* of the codomain *Y* under a function *f* is the subset of the domain *X* defined by $f^{-1}(B) = \{x \in X : f(x) \in B\}$.

So, for example, the preimage of {4, 9} under the squaring function is the set $\{-3,-2,2,3\}$. By definition of a function, the image of an element *x* of the domain is always a single element *y* of the codomain. Conversely, though, the preimage of a singleton set (a set with exactly one element) may in general contain any number of elements. For example, if $f(x) = 7$ (the constant function taking value 7), then the preimage of {5} is the empty set but the preimage of {7} is the entire domain. It is customary to write $f^{-1}(b)$ instead of $f^{-1}(\{b\})$, i.e. $f^{-1}(b) = \{x \in X : f(x) = b\}$.

Use of $f(A)$ to denote the image of a subset $A \subseteq X$ is consistent so long as no subset of the domain is also an element of the domain. In some fields (e.g., in set theory, where ordinals are also sets of ordinals) it is convenient or even necessary to distinguish the two concepts; the customary notation is $f[A]$ for the set $\{ f(x): x \in A \}$.

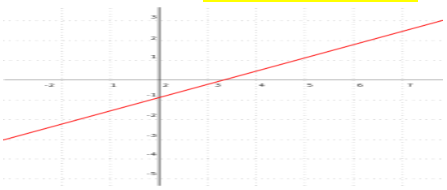
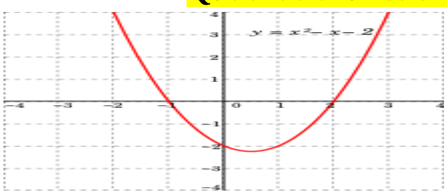
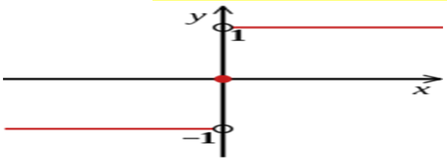
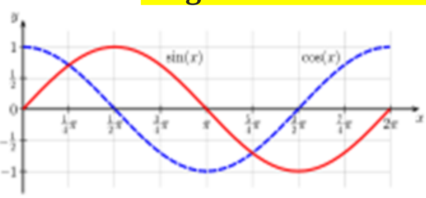
Real-valued functions. A real-valued function f is one whose codomain is the set of real numbers or a subset thereof. If, in addition, the domain is also a subset of the reals, f is a real valued function of a real variable. The study of such functions is called real analysis.

Real-valued functions enjoy so-called pointwise operations. That is, given two functions $f, g: X \to Y$, where Y is a subset of the reals (and X is an arbitrary set), their (pointwise) sum $f+g$ and product $f \cdot g$ are functions with the same domain and codomain. They are defined by the formulas:

$$(f+g)(x) = f(x) + g(x),$$
$$(f \cdot g)(x) = f(x) \cdot g(x).$$

In a similar vein, complex analysis studies functions whose domain and codomain are both the set of complex numbers. In most situations, the domain and codomain are understood from context, and only the relationship between the input and output is given, but if $f(x) = \sqrt{x}$, then in real variables the domain is limited to non-negative numbers.

The following table contains a few particularly important types of real-valued functions:

Linear function	**Quadratic function**
 A linear function: $f(x) = ax + b$.	 A quadratic function: $f(x) = ax^2 + bx + c$.
Discontinuous function	**Trigonometric functions**
 The signum function is not continuous, since it "jumps" at 0. Roughly speaking, a continuous function is one whose graph can be drawn without lifting the pen.	 The sine and cosine functions. f(x) = sin(x) (red), f(x) = cos(x)

Mathematical terminology

formula – 1) (mathematical formula; мн.ч - formulae) - [математическая] формула в математике - формализованная запись некоторой функциональной зависимости string formula 2) уравнение Syn: equation 3) аналитическое выражение 4) формулировка

algorithm – 1) алгоритм математическая функция или конечный чёткий набор описаний логической последовательности действий (правил, инструкций), 2) метод, правило

to compute the value of *f*(*x*) – вычислить значение функции *f(x)*

piecewise definability – кусочная определимость

induction - математическая индукция – метод математического доказательства, используется чтобы доказать истинность некоторого утверждения для всех натуральных чисел. Для этого сначала проверяется истинность утверждения с номером 1 – база (базис) индукции, а затем доказывается, что, если верно утверждение с номером n, то верно и следующее утверждение с номером n + 1 – шаг индукции, или индукционный переход. Доказательство по индукции наглядно может быть представлено в виде так называемого принципа домино. Пусть какое угодно число косточек домино выставлено в ряд таким образом, что каждая косточка, падая, обязательно опрокидывает следующую за ней косточку (в этом заключается индукционный переход). Тогда, если мы толкнём первую косточку (это база индукции), то все косточки в ряду упадут.

recursion – реку́рсия в определении, описании, изображении какого-либо объекта или процесса внутри самого этого объекта или процесса, то есть ситуация, когда объект является частью самого себя. В математике рекурсия имеет отношение к методу определения функций и числовых рядов: рекурсивно заданная функция определяет своё значение через обращение к себе самой с другими аргументами.

algebraic or analytic closure - алгебраическое или аналитическое замыкание

limit - предел

analytic continuation – аналитическое продолжение

infinite series – бесконечные ряды

integral and differential equations – интегральные и дифференциальные уравнения

lambda calculus – лямбда-исчисление математическая система для определения функций, вычисления значений выражений (lambda expression) и доказательства равенства выражений

flexible syntax – гибкий (адаптивный) синтаксис

function of several variables – функция с несколькими переменными

Axiom of Choice – аксиома выбора

postfix notation – постфиксная нотация, постфиксная запись известна также под названием "обратная польская запись"; метод бесскобочной записи математических выражений, при котором операция записывается после операндов, например, (2+3) * (4+5) в постфиксной нотации будет выглядеть как 2 3 + 4 5 + *. Такая запись используется в языке Forth

computable function – вычислимая функция функция вычислима, если можно найти алгоритм, позволяющий вычислить её выходное значение для любого действительного входного; известно, что существует много функций, для которых это сделать не удаётся

Euclidean algorithm – евклидов алгоритм

greatest common divisor – наибольший общий делитель

number theory – number theory теория чисел математическая дисциплина, изучающая свойства чисел. Применяется, в частности, в криптографии

computability theory – теория вычислимости, также известная как теория рекурсивных функций, - это раздел современной математики, лежащий на стыке математической логики, теории алгоритмов и информатики, возникший в результате изучения понятий вычислимости и невычислимости. Изначально теория была посвящена вычислимым и невычислимым функциям и сравнению различных моделей вычислений. Сейчас поле исследования теории вычислимости расширилось — появляются новые определения понятия вычислимости и идёт слияние с математической логикой, где вместо вычислимости и невычислимости идёт речь о доказуемости и недоказуемости (выводимости и невыводимости) утверждений в рамках каких-либо теорий.
cardinality – мощность множества, число элементов множества, кардинальное число
countable – исчисляемый, счётный (о множестве)
busy beaver function – невычислимая функция (A busy beaver function quantifies th upper limits on a given measure, and is a noncomputable function.)
halting problem – проблема останова в теории вычислений - проблема определения, остановится ли (завершится ли) данная программа при вычислении данного набора входных данных. Эта проблема относится к числу алгоритмически неразрешимых задач.
undecidable problem – неразрешимая задача
inverse image or preimage – образ в инверсии или прообраз
singleton set – одноэлементное множество
in a similar vein – в том же духе, подобным образом
in a critical vein – в критическом духе
Linear function – линейная функция (функция вида y = kx + b; основное свойство такой функции заключается в том, что ее приращение пропорционально приращению аргумента)
Trigonometric functions - тригонометрическая функция
Discontinuous function – разрывная функция
Quadratic function – квадратическая функция

Grammar, Lexical, Translation and Speaking Exercises

Task 1. Irregular plural nouns of Latin and Greek origin. A lot of Latin and Greek original nouns have the plural forms ending in –i (e.g. calculus – calculi) and –ae (e.g. abscissa – abscissae). Put the words in the correct plural form using the Model: *focus – foci (–us → –i); hyperbola – hyperbolae (–a → –ae).*
Focus, formula, corona, genius, locus, hyperbola, lacuna, radius, nebula, modulus, nucleus, rhombus.

Task 2. Fill in the gaps using the words above.
a) Two equations are called equivalent if they have the same …………
b) Up until quite recently, when functions were mentioned in the mathematical literature they were usually considered to be …………
c) In the figure, we can sketch the ………… determined by an equation of the form.
d) The simplest …………, that of common hydrogen, has a single proton.
e) The area of an ellipse equals π/4 times the product of the long and the short diametres or π times the product of the long and the short …… .

Note: Besides that, some nouns always have similar forms, example: an apparatus – apparatus a headquarters – headquarters; a means – means; news – news; a series – series; a species – species.

Task 3. Match the columns.

1) The total distance travelled divided by the total time taken. 2) The total displacement (as a vector) divided by the total time taken. 3) One of the planes containing two of the coordinate axes in a 3-dimensional Cartesian coordinate system. For example, one of the axial planes is the yz-plane, or (y, z)-plane, containing the y-axis and the z-axis, and it has equation x = 0. 4) A statement whose truth is either to be taken as self-evident or to be assumed. Certain areas of mathematics involve choosing a set of axioms and discovering what results can be derived from them, providing proofs for the theorems that are obtained.	a) axiom b) axial plane c) average velocity d) average speed

Task 4. Find mistakes in the following sentences. Mind the use of Perfect Tenses in the Active and Passive Voice.

1. They finished their experiment by 5 o'clock yesterday. 2. The production of such computers has reduced by the end of the previous year. 3. I can't do the exercise. I had forgotten my text-book at home. 4. The article just translates by all the students. 5. By the time Kate returned from her studies, her brother goes to his friends. 6. His graduation paper will present by 3 o'clock tomorrow. 7. He is doing this work by tomorrow. 8. The solution for the problem is found by the end of the meeting yesterday. 9. The students already pass their credits. 10. She is written her course-paper by next month. 11. The advantages of this program already spoke of by the scientists at the conference. 12. The algorithm carefully hadn't worked out at the recent seminar yet.

Task 5. Ask special questions using question words given in parentheses.

The development of geometry 1. The earliest recorded beginnings of geometry can be traced to early predecessors. (to whom) 2. They discovered obtuse triangles in the ancient Indus Valley and ancient Babylonia from around 3000 BC. (where; when) 3. Early geometry was a collection of empirically discovered principles concerning lengths, angles, areas, and volumes. (what collection) 4. In geometry a spatial point is a primitive notion upon which other concepts may be defined. (where) 5. Points have neither volume, area, length, nor any other higher dimensional analogue. (what (question to the subject)) 6. In branches of mathematics dealing with a set theory, an element is often referred to as a point. (where; how) 7. A point could also be defined as a sphere which has a diameter of zero. (how)

Task 6. Translate these sentences from Russian into English. 1. Треугольник – это плоская фигура, ограниченная тремя линиями и содержащая три угла. 2. Треугольники бывают равносторонние, разносторонние, равнобедренные. 3. Треугольник с вершинами A, B, C обозначается ΔABC. 4. У равнобедренного треугольника два угла имеют одинаковую величину. 5. Название треугольника происходит от латинского слова «триангулум» – треугольный. 6. Существует семь видов треугольников в зависимости от формы и градусной меры углов. 7. Равносторонний треугольник – это тот, у которого три стороны равны. 8. Другие математики определяют равнобедренный треугольник как треугольник, по крайней мере, с двумя равными сторонами. 9. Площадь треугольника может быть вычислена при помощи формулы. 10. Треугольник, у которого все внутренние углы меньше 90°, является остроугольным.

Task 7. Make up the sentences with the words in their general non-mathematical meaning: *formulae, area, line, cube, nought, infinity, interval, series,*

E.g.: The limit (informal) – a person or thing that is intolerably exasperating, i.e. несносный человек; что-либо невыносимое

It's the limit! — Это уже слишком!

That's the giddy limit! — Да как можно такое терпеть!

Oh, Harry, you are the limit. — Ох, Гарри, как ты меня достал!

You're the limit! Can't you make up your mind? — Ты просто невыносим! Ты что, никак не решишь, что тебе надо?

That child is the limit! — С этим ребенком сладу нет.

She does seem to be about the limit — Ну она и дает!

Isn't he the limit? — Во дает!

Well, aren't you the bloody limit! — Как ты себя ведешь?

I think you kids are the absolute limit. — С такими детьми, как вы, нужно адское терпение

He's the frozen limit — Он совершенно невыносим.

This is the limit! — Это переходит всякие границы!

UNIT 12

Text 12. Fermat's Last Theorem

In number theory, Fermat's Last Theorem (sometimes called Fermat's conjecture, especially in older texts) states that no three positive integers *a*, *b*, and *c* can satisfy the equation $a^n + b^n = c^n$ for any integer value of *n* greater than two. This theorem was first conjectured by Pierre de Fermat in 1637 in the margin of a copy of *Arithmetica* where he claimed he had a proof that was too large to fit in the margin. The first successful proof was released in 1994 by Andrew Wiles, and formally published in 1995, after 358 years of effort by mathematicians. The unsolved problem stimulated the development of algebraic number theory in the 19th century and the proof of the modularity theorem in the 20th century. It is among the most notable theorems in the history of mathematics and prior to its proof it was in the *Guinness Book of World Records* for "most difficult mathematical problems".

Fermat's Last Theorem stood as an unsolved riddle in mathematics for over three and a half centuries. The theorem itself is a deceptively simple statement that Fermat stated he had proved around 1637. His claim was discovered some 30 years later, after his death, written in the margin of a book, but with no proof provided.

The claim eventually became one of the most notable unsolved problems of mathematics. Attempts to prove it prompted substantial development in number theory, and over time Fermat's Last Theorem gained prominence as an unsolved problem in popular mathematics. It is based on the Pythagorean theorem, which states that $a^2 + b^2 = c^2$, where *a* and *b* are the lengths of the legs of a right triangle and *c* is the length of the hypotenuse.

The Pythagorean equation has an infinite number of positive integer solutions for *a*, *b*, and *c*; these solutions are known as Pythagorean triples. Fermat stated that the more general equation $a^n + b^n = c^n$ had no solutions in positive integers, if *n* is an integer greater than 2. Although he claimed to have a general proof of his conjecture, Fermat left no details of his proof apart from the special case $n = 4$.

Subsequent developments and solution. With the special case $n = 4$ proven, the problem was to prove the theorem for exponents *n* that are prime numbers (this limitation is considered trivial to prove). Over the next two centuries (1637–1839), the conjecture was proven for only the primes 3, 5, and 7, although Sophie Germain innovated and proved an approach that was relevant to an entire class of primes. In the mid-19th century, Ernst Kummer extended this and proved the theorem for all regular primes, leaving irregular primes to be analyzed individually. Building on Kummer's work and using sophisticated computer studies, other mathematicians were able to extend the proof to cover all prime exponents up to four million, but a proof for all exponents was inaccessible (meaning that mathematicians generally considered a proof to be either impossible, or at best exceedingly difficult, or not achievable with current knowledge).

The proof of Fermat's Last Theorem in full, for all *n*, was finally accomplished, however, after 357 years, by Andrew Wiles in 1994, an achievement for which he was honoured and received numerous awards. The solution came in a roundabout manner, from a completely different area of mathematics.

Around 1955 Japanese mathematicians Goro Shimura and Yutaka Taniyama suspected a link might exist between elliptic curves and modular forms, two completely different areas of mathematics. Known at the time as the Taniyama–Shimura-Weil conjecture, and (eventually) as the modularity theorem, it stood on its own, with no apparent connection to Fermat's Last Theorem. It was widely seen as significant and important in its own right, but was (like Fermat's equation) widely considered to be completely inaccessible to proof.

In 1984, Gerhard Frey noticed an apparent link between the modularity theorem and Fermat's Last Theorem. This potential link was confirmed two years later by Ken Ribet (see: *Ribet's Theorem* and *Frey curve*). On hearing this, English mathematician Andrew Wiles, who had a childhood fascination with Fermat's Last Theorem, decided to try to prove the modularity theorem as a way to prove Fermat's Last Theorem. In 1993, after six years working secretly on the problem, Wiles succeeded in proving enough of the modularity theorem to prove Fermat's Last Theorem. Wiles' paper was massive in size and scope. A flaw was discovered in one part of his original paper during peer review and required a further year and collaboration with a past student, Richard Taylor, to resolve.

As a result, the final proof in 1995 was accompanied by a second, smaller, joint paper to that effect. Wiles's achievement was reported widely in the popular press, and was popularized in books and television programs. The remaining parts of the modularity theorem were subsequently proven by other mathematicians, building on Wiles's work, between 1996 and 2001.

Pythagorean triples. A Pythagorean triple – named for the ancient Greek Pythagoras – is a set of three integers (*a*, *b*, *c*) that satisfy a special case of Fermat's equation ($n = 2$) $a^2 + b^2 = c^2$.

Examples of Pythagorean triples include (3, 4, 5) and (5, 12, 13). There are infinitely many such triples, and methods for generating such triples have been studied in many cultures, beginning with the Babylonians and later ancient Greek, Chinese, and Indian mathematicians. The traditional interest in Pythagorean triples connects with the Pythagorean theorem; in its converse form, it states that a triangle with sides of lengths *a*, *b*, and *c* has a right angle between the *a* and *b*legs when the numbers are a Pythagorean triple. Right angles have various practical applications, such as surveying, carpentry, masonry, and construction. Fermat's Last Theorem is an extension of this problem to higher powers, stating that no solution exists when the exponent 2 is replaced by any larger integer.

Diophantine equations Fermat's equation, $x^n + y^n = z^n$ with positive integer solutions, is an example of a Diophantine equation, named for the 3rd-century Alexandrian mathematician, Diophantus, who studied them and developed methods for the solution of some kinds of Diophantine equations. A typical Diophantine problem is to find two integers x and y such that their sum, and the sum of their squares, equal two given numbers A and B, respectively: $A = x + y,\ B = x^2 + y^2.$

Diophantus's major work is the *Arithmetica*, of which only a portion has survived. Fermat's conjecture of his Last Theorem was inspired while reading a new edition of the *Arithmetica*, that was translated into Latin and published in 1621 by Claude Bachet.

Diophantine equations have been studied for thousands of years. For example, the solutions to the quadratic Diophantine equation $x^2 + y^2 = z^2$ are given by the Pythagorean triples, originally solved by the Babylonians (1800 BC). Solutions to linear Diophantine equations, such as $26x + 65y = 13$, may be found using the Euclidean algorithm (5th century BC). Many Diophantine equations have a form similar to the equation of Fermat's Last Theorem from the point of view of algebra, in that they have no *cross terms* mixing two letters, without sharing its particular properties. For example, it is known that there are infinitely many positive integers x, y, and z such that $x^n + y^n = z^m$ where n and m are relatively prime natural numbers.

Fermat's conjecture. Problem II.8 of the *Arithmetica* asks how a given square number is split into two other squares; in other words, for a given rational number k, find rational numbers u and v such that $k^2 = u^2 + v^2$. Diophantus shows how to solve this sum-of-squares problem for $k = 4$ (the solutions being $u = 16/5$ and $v = 12/5$).

Around 1637, Fermat wrote his **Last Theorem** in the margin of his copy of the *Arithmetica* next to Diophantus' sum-of-squares problem:

> *It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers. I have discovered a truly marvellous proof of this, which this margin is too narrow to contain*.

After Fermat's death in 1665, his son Clément-Samuel Fermat produced a new edition of the book (1670) augmented with his father's comments. The margin note became known as *Fermat's Last Theorem*, as it was the last of Fermat's asserted theorems to remain unproven.

It is not known whether Fermat had actually found a valid proof for all exponents n, but it appears unlikely. Only one related proof by him has survived, namely for the case $n = 4$, as described in the section *Proofs for specific exponents*. While Fermat posed the cases of $n = 4$ and of $n = 3$ as challenges to his mathematical correspondents, such as Marin Mersenne, Blaise Pascal, and John Wallis, he never posed the general case. Moreover, in the last thirty years of his life, Fermat never again wrote of his "truly marvellous proof" of the general case, and never published it. Van der Poorten suggests that while the absence of a proof is insignificant, the lack of challenges means Fermat realised he did not have a proof; he quotes Weil as saying Fermat must have briefly deluded himself with an irretrievable idea.

The techniques Fermat might have used in such a "marvellous proof" are unknown. Taylor and Wiles's proof relies on 20th century techniques. Fermat's proof would have had to have been elementary by comparison, given the mathematical knowledge of his time.

While Harvey Friedman's grand conjecture implies that any provable theorem (including Fermat's last theorem) can be proved using only 'elementary function arithmetic', such a proof need only be 'elementary' in a technical sense but could involve millions of steps, and thus be far too long to have been Fermat's proof.

Proofs for specific exponents. Only one relevant proof by Fermat has survived, in which he uses the technique of infinite descent to show that the area of a right triangle with integer sides can never equal the square of an integer. His proof is equivalent to demonstrating that the equation $x^4 - y^4 = z^2$ has no primitive solutions in integers (no pairwise coprime solutions). In turn, this proves Fermat's Last Theorem for the case $n = 4$, since the equation $a^4 + b^4 = c^4$ can be written as $c^4 - b^4 = (a^2)^2$.

Alternative proofs of the case $n = 4$ were developed later by Frénicle de Bessy (1676), Leonhard Euler (1738), Kausler (1802), Peter Barlow (1811), Adrien-Marie Legendre (1830), Terquem (1846), Bertrand (1851), Victor Lebesgue (1859, 1862), Theophile Pepin (1883),Tafelmacher (1893), David Hilbert (1897), Bendz (1901), Gambioli (1901), Leopold Kronecker (1901), Bang (1905), Sommer (1907), Bottari(1908), Karel Rychlík (1910), Nutzhorn (1912), Robert Carmichael (1913), Hancock (1931), and Vrănceanu (1966). For various proofs for $n=4$ by infinite descent, see Grant and Perella (1999), Barbara (2007), and Dolan (2011).

After Fermat proved the special case $n = 4$, the general proof for all n required only that the theorem be established for all odd prime exponents. In other words, it was necessary to prove only that the equation $a^n + b^n = c^n$ has no integer solutions (a, b, c) when n is an odd prime number. This follows because a solution (a, b, c) for a given n is equivalent to a solution for all the factors of n. For illustration, let n be factored into d and e, $n = de$. The general equation $a^n + b^n = c^n$ implies that (a^d, b^d, c^d) is a solution for the exponent e: $(a^d)^e + (b^d)^e = (c^d)^e$.

Thus, to prove that Fermat's equation has no solutions for $n > 2$, it would suffice to prove that it has no solutions for at least one prime factor of every n. Each integer $n > 2$ is divisible by 4 or an odd prime number (or both). Therefore, Fermat's Last Theorem could be proved for all n if it could be proved for $n = 4$ and for all odd primes p.

In the two centuries following its conjecture (1637–1839), Fermat's Last Theorem was proven for three odd prime exponents $p = 3$, 5 and 7. The case $p = 3$ was first stated by Abu-Mahmud Khojandi (10th century), but his attempted proof of the theorem was incorrect. In 1770, Leonhard Euler gave a proof of $p = 3$, but his proof by infinite descent contained a major gap. However, since Euler himself had proven the lemma necessary to complete the proof in other work, he is generally credited with the first proof.

Independent proofs were published by Kausler (1802), Legendre (1823, 1830), Calzolari (1855), Gabriel Lamé (1865), Peter Guthrie Tait (1872), Günther (1878), Gambioli (1901), Krey (1909), Rychlík (1910), Stockhaus (1910), Carmichael (1915), Johannes van der Corput (1915), Axel Thue (1917), and Duarte (1944). The case $p = 5$ was proven independently by Legendre and Peter Gustav Lejeune Dirichlet around 1825. Alternative proofs were developed by Carl Friedrich Gauss (1875, posthumous), Lebesgue (1843), Lamé (1847), Gambioli (1901), Werebrusow (1905), Rychlík (1910), van der Corput (1915), and Guy Terjanian (1987). The case $p = 7$ was provenby Lamé in 1839. His rather complicated proof was simplified in 1840 by Lebesgue, and still simpler proofs were published by Angelo Genocchi in 1864, 1874 and 1876. Alternative proofs were developed by Théophile Pépin (1876) and Edmond Maillet (1897).

Fermat's Last Theorem has also been proven for the exponents $n = 6$, 10, and 14. Proofs for $n = 6$ have been published by Kausler, Thue, Tafelmacher, Lind, Kapferer, Swift, and Breusch. Similarly, Dirichlet and Terjanian each proved the case $n = 14$, while Kapferer and Breusch each proved the case $n = 10$. Strictly speaking, these proofs are unnecessary, since these cases follow from the proofs for $n = 3$, 5, and 7, respectively. Nevertheless, the reasoning of these even-exponent proofs differs from their odd-exponent counterparts. Dirichlet's proof for $n = 14$ was published in 1832, before Lamé's 1839 proof for $n = 7$.

All proofs for specific exponents used Fermat's technique of infinite descent, either in its original form, or in the form of descent on elliptic curves or abelian varieties. The details and auxiliary arguments, however, were often *ad hoc* and tied to the individual exponent under consideration. Since they became ever more complicated as *p* increased, it seemed unlikely that the general case of Fermat's Last Theorem could be proved by building upon the proofs for individual exponents. Although some general results on Fermat's Last Theorem were published in the early 19th century by Niels Henrik Abel and Peter Barlow, the first significant work on the general theorem was done by Sophie Germain.

Computational studies. In the latter half of the 20th century, computational methods were used to extend Kummer's approach to the irregular primes. In 1954, Harry Vandiver used a SWAC computer to prove Fermat's Last Theorem for all primes up to 2521. By 1978, Samuel Wagstaff had extended this to all primes less than 125,000. By 1993, Fermat's Last Theorem had been proven for all primes less than four million.

However despite these efforts and their results, no proof existed of Fermat's Last Theorem. Proofs of individual exponents by their nature could never prove the *general* case: even, if all exponents were verified up to an extremely large number X, a higher exponent beyond X might still exist for which the claim was not true.

Wiles's general proof. Ribet's proof of the epsilon conjecture in 1986 accomplished the first of the two goals proposed by Frey. Upon hearing of Ribet's success, Andrew Wiles, an English mathematician with a childhood fascination with Fermat's Last Theorem, and a prior study area of elliptical equations, decided to commit himself to accomplishing the second half: proving a special case of the modularity theorem (then known as the Taniyama–Shimura conjecture) for semistable elliptic curves.

Wiles worked on that task for six years in near-total secrecy, covering up his efforts by releasing prior work in small segments as separate papers and confiding only in his wife. His initial study suggested proof by induction, and he based his initial work and first significant breakthrough on Galois theory before switching to an attempt to extend Horizontal Iwasawa theory for the inductive argument around 1990–91 when it seemed that there was no existing approach adequate to the problem. However, by the summer of 1991, Iwasawa theory also seemed to not be reaching the central issues in the problem. In response, he approached colleagues to seek out any hints of cutting edge research and new techniques, and discovered an Euler system recently developed by Victor Kolyvagin and Matthias Flach that seemed "tailor made" for the inductive part of his proof. Wiles studied and extended this approach, which worked. Since his work relied extensively on this approach, which was new to mathematics and to Wiles, in January 1993 he asked his Princeton colleague, Nick Katz, to check his reasoning for subtle errors. Their conclusion at the time was that the techniques used by Wiles seemed to be working correctly.

By mid-May 1993 Wiles felt able to tell his wife he thought he had solved the proof of Fermat's Last Theorem, and by June he felt sufficiently confident to present his results in three lectures delivered on 21–23 June 1993 at the Isaac Newton Institute for Mathematical Sciences. Specifically, Wiles presented his proof of the Taniyama–Shimura conjecture for semistable elliptic curves; together with Ribet's proof of the epsilon conjecture, this implied Fermat's Last Theorem. However, it became apparent during peer review that a critical point in the proof was incorrect. It contained an error in a bound on the order of a particular group. The error was caught by several mathematicians refereeing Wiles's manuscript including Katz (in his role as reviewer), who alerted Wiles on 23 August 1993.

The error would not have rendered his work worthless – each part of Wiles' work was highly significant and innovative by itself, as were the many developments and techniques he had created in the course of his work, and only one part was affected. However without this part proven, there was no actual proof of Fermat's Last Theorem. Wiles spent almost a year trying to repair his proof, initially by himself and then in collaboration with Richard Taylor, without success.

On 19 September 1994, on the verge of giving up, Wiles had a flash of insight that the proof could be saved by returning to his original Horizontal Iwasawa theory approach, which he had abandoned in favour of the Kolyvagin–Flach approach, this time strengthening it with expertise gained in Kolyvagin–Flach's approach. On 24 October 1994, Wiles submitted two manuscripts, "Modular elliptic curves and Fermat's Last Theorem" and "Ring theoretic properties of certain Hecke algebras", the second of which was co-authored with Taylor and proved that certain conditions were met that were needed to justify the corrected step in the main paper. The two papers were vetted and published as the entirety of the May 1995 issue of the *Annals of Mathematics*. These papers established the modularity theorem for semistable elliptic curves, the last step in proving Fermat's Last Theorem, 358 years after it was conjectured.

Subsequent developments. The full Taniyama–Shimura–Weil conjecture was finally proved by Diamond (1996), Conrad, Diamond & Taylor (1999), and Breuil et al. (2001) who, building on Wiles' work, incrementally chipped away at the remaining cases until the full result was proved. The now fully proved conjecture became known as the modularity theorem.

Several other theorems in number theory similar to Fermat's Last Theorem also follow from the same reasoning, using the modularity theorem. For example: no cube can be written as a sum of two coprime n-th powers, $n \geq 3$. (The case $n = 3$ was already known by Euler.)

Exponents other than positive integers

Reciprocal Integers (Inverse Fermat Equation)

The equation $a^{1/m} + b^{1/m} = c^{1/m}$ can be considered the "inverse" Fermat equation. All solutions of this equation were computed by Lenstra in 1992. In the case in which the *m*th roots are required to be real and positive, all solutions are given by

$a = rs^m, \quad b = rt^m, \quad c = r(s+t)^m$ for positive integers *r, s, t* with *s* and *t* coprime.

Rational exponents. For the Diophantine equation $a^{n/m} + b^{n/m} = c^{n/m}$ with *n* not equal to 1, in 2004, for $n > 2$, Bennett, Glass, and Szekely proved that if *n* and *m* are coprime, then there are integer solutions if and only if 6 divides *m*, and $a^{1/m}$, $b^{1/m}$, and $c^{1/m}$ are different complex 6th roots of the same real number.

Negative exponents. ***n*** **= –1.** All primitive (pairwise coprime) integer solutions to $a^{-1} + b^{-1} = c^{-1}$ can be written as $a = mn + m^2, \quad b = mn + n^2, \quad c = mn$ for positive, coprime integers *m, n*.

n **= –2.** The case $n = -2$ also has an infinitude of solutions, and these have a geometric interpretation in terms of right triangles with integer sides and an integer altitude to the hypotenuse. All primitive solutions to $a^{-2} + b^{-2} = d^{-2}$ are given by $a = (v^2 - u^2)(v^2 + u^2), \quad b = 2uv(v^2 + u^2), \quad d = 2uv(v^2 - u^2),$ for coprime integers *u, v* with $v > u$. The geometric interpretation is that *a* and *b* are the integer legs of a right triangle and *d* is the integer altitude to the hypotenuse. Then the hypotenuse itself is the integer $c = (v^2 + u^2)^2$, so (*a, b, c*) is a Pythagorean triple.

Integer ***n*** **< –2.** There are no solutions in integers for $a^n + b^n = c^n$ for integers $n < -2$. If there were, the equation could be multiplied through by $a^{|n|}b^{|n|}c^{|n|}$ to obtain $(bc)^{|n|} + (ac)^{|n|} = (ab)^{|n|}$, which is impossible by Fermat's Last Theorem. Values other than positive integers Fermat's last theorem can easily be extended to positive rationals: $\left(\frac{a}{x}\right)^n + \left(\frac{b}{y}\right)^n = \left(\frac{c}{z}\right)^n$ can have no solutions, because any solution could be rearranged as: $(ayz)^n + (bxz)^n = (cxy)^n$, to which Fermat's Last Theorem applies.

Monetary prizes. In 1816 and again in 1850, the French Academy of Sciences offered a prize for a general proof of Fermat's Last Theorem. In 1857, the Academy awarded 3000 francs and a gold medal to Kummer for his research on ideal numbers, although he had not submitted an entry for the prize. Another prize was offered in 1883 by the Academy of Brussels.

In 1908, the German industrialist and amateur mathematician Paul Wolfskehl bequeathed 100,000 gold marks, a very large sum at that time, to the Göttingen Academy of Sciences to be offered as a prize for a complete proof of Fermat's Last Theorem. On 27 June 1908, the Academy published nine rules for awarding the prize. Among other things, these rules required that the proof be published in a peer-reviewed journal; the prize would not be awarded until two years after the publication; and that no prize would be given after 13 September 2007, roughly a century after the competition was begun. Wiles collected the Wolfskehl prize money, then worth $50,000, on 27 June 1997.

Prior to Wiles' proof, thousands of incorrect proofs were submitted to the Wolfskehl committee, amounting to roughly 10 feet (3 meters) of correspondence. In the first year alone (1907–1908), 621 attempted proofs were submitted, although by the 1970s, the rate of submission had decreased to roughly 3–4 attempted proofs per month. According to F. Schlichting, a Wolfskehl reviewer, most of the proofs were based on elementary methods taught in schools, and often submitted by "people with a technical education but a failed career". In the words of mathematical historian Howard Eves, "Fermat's Last Theorem has the peculiar distinction of being the mathematical problem for which the greatest number of incorrect proofs have been published."

Mathematical terminology

Fermat's Last Theorem – Последняя теорема Ферма (или Вели́кая теоре́ма Ферма́)
number theory – теория чисел (математическая дисциплина, изучающая свойства чисел; применяется, в частности, в криптографии
positive integer – положительное целое число
conjecture – гипотеза, догадка, предположение
conjectured – гипотетический
Pierre de Fermat – Пьер де Ферма́ (1601 – 1665) – французский математик, один из создателей аналитической геометрии, математического анализа, теории вероятностей и теории чисел. По профессии юрист, с 1631 года — советник парламента в Тулузе. Блестящий полиглот. Наиболее известен формулировкой Великой теоремы Ферма.
successful proof – успешное доказательство
Andrew Wiles–Сэр Эндрю Джон Уайлс (родился 11 апреля 1953, Кембридж, Великобритания рыцарь-командор Ордена Британской Империи с 2000) – английский и американский математик, профессор математики Принстонского университета, заведующий его кафедрой математики, член научного совета Института математики Клэя
algebraic number theory – алгебраическая теория чисел
modularity theorem – теорема о модулярности
Guinness Book of World Records – Кни́га реко́рдов Ги́ннесса, ежегодный сборник мировых рекордов, достижений человека, животных и природных величин. Впервые опубликована в 1955 году по заказу ирландской пивоваренной компании «Гиннесс».
unsolved problem – нерешенный вопрос
Pythagorean theorem – теорема Пифагора
right triangle – прямоугольный треугольник
hypotenuse [haɪˈpɒtɪˌnjuːz] – гипотенуза
equation – уравнение

Pythagorean triple – Пифагорова тройка, в математике **пифагоровой тройкой** называется упорядоченный конечный набор из трёх натуральных чисел (x, y, z), удовлетворяющих следующему однородному квадратному уравнению: $x^2 + y^2 = z^2$. При этом числа, образующие пифагорову тройку, называются **пифагоровыми числами**.
exponent – показатель степени, показатель, экспонента
prime number – простое число
Sophie Germain – Софи́ Жерме́н (1776–1831) – французский математик, философ и механик. Внесла весомый вклад в дифференциальную геометрию, теорию чисел и механику. Самостоятельно училась в библиотеке отца-ювелира и с детства увлекалась математическими сочинениями, особенно известной историей математика Монтюкла, хотя родители препятствовали её занятиям как не подходящим для женщины.
regular prime – регулярное простое число
elliptic curve – эллиптическая кривая
modular form – модулярная форма
Frey curve – кривая Фрея, т.е. эллиптическая кривая $y^2 = x(x - a^\ell)(x + b^\ell)$, ассоциируемая с решением уравнения Ферма $a^\ell + b^\ell = c^\ell$.
Ribet's Theorem – теорема Рибета, ранее называлась эпсилон-гипотеза (epsilon conjecture or ε-conjecture)
succeed in proving – преуспеть в доказательстве (чего-то), доказать
peer review – экспертная оценка, проводить экспертную оценку; независимая (внешняя) экспертиза (оценка)
Richard Taylor – Ричард Лоуренс Тейлор (1962) – английский математик, занимающийся проблемами теории чисел.
surveying – общий анализ; выполнение общего анализа
Diophantine equation [ˌdaɪəʊ'fæntaɪn] – диофантово уравнение
Euclidean algorithm – евклидов алгоритм
relatively prime – взамно простой
grand conjecture – великая догадка (предположение)
elementary function – элементарная функция
infinite descent – бесконечный спуск
proof by infinite descent – метод бесконечного спуска, это метод доказательства от противного, основанный на том, что множество натуральных чисел вполне упорядочено.
pairwise coprime – попарно взаимно простые числа
odd prime number – нечетное простое число
proof by induction – доказательство посредством индуктивного метода
Galois theory – **тео́рия Галуа́,** раздел алгебры, позволяющий переформулировать определенные вопросы теории полей на языке теории групп, делая их в некотором смысле более простыми
Euler system – эйлерова система
coprimes – взаимно простые числа
side – грань треугольника
altitude to the hypotenuse – высота, проведенная к гипотенузе

Grammar, Lexical, Translation and Speaking Exercises

Task 1. Match the columns.

1) the branch of mathematics that deals with the properties and relationships of numbers; 2) a triangle one angle of which is a right angle; 3) the side in a right triangle that is opposite the right angle; 4) At right angles to one another. Can be of two lines, two planes, a line and a plane, or a line and a surface; 5) A million million (10^{12}). In Britain (10^{18}).	a) hypotenuse b) perpendicular c) trillion d)number theory e) right triangle (right-angled triangle)

Task 2. Translate from Russian into English the text: Великая теорема Ферма.

Для любого натурального числа $n > 2$ уравнение $a^n + b^n = c^n$ не имеет натуральных решений *a, b, c*. Ферма широко известен благодаря так называемой великой (или последней) теореме Ферма. Теорема была сформулирована им в 1637 году, на полях книги «Арифметика» Диофанта с припиской, что найденное им остроумное доказательство этой теоремы слишком длинно, чтобы привести его на полях. Вероятнее всего, его доказательство не было верным, так как позднее он опубликовал доказательство только для случая n = 4. Доказательство, найденное в 1994 году Эндрю Уайлсом, содержит 129 страниц и опубликовано в журнале «Annals of Mathematics» в 1995 году. Простота формулировки этой теоремы привлекла много математиков-любителей, так называемых ферматистов. Даже и после решения Уайлса во все академии наук идут письма с «доказательствами» великой теоремы Ферма.

Task 3. Change the sentences using the Participle Forms, follow the model.

Model: *I have got a book which deals with computers.*

I've got a book dealing with computers.

1) I know the man who teaches you English. 2) Give me the journal which lies on the table. 3) I must see the scientists who work in this lab. 4) The letters which name the angles are A, B, C.

Model: *The material which is used in the article is true.*

The material used in the article is true.

1) The most prevalent calculator in the United States is the slide rule, which is based on the principle of logarithms.

2) One of the original calculators was undoubtedly a version of the Japanese abacus, which is still in use today.

3) Most calculators are based on the fundamental mathematical principle which is called the binary number system.

4) The calculators which were traced back to the Tigris Euphrates Valley 5000 years ago are original.

Task 4. Choose the correct item.

1. She had the feeling of ... .

a) being deceived b) deceiving c) having deceived

2. It's a waste of time ... over trifles.

a) having argued b) having been argued c) arguing

3. My watch doesn't keep good time. It needs ... .

a) having been repaired b) being repaired c) repairing

4. He mentioned ... it in the paper.

a) being read b) reading c) having read

5. Is it worth while your ... to convince him of being wrong?

a) being tried b) trying c) having tried

6. He insisted on ... with a certain respect.

a) having been treated b) treating c) being treated

7. Father didn't approve of my ... the offer.

a) having rejected b) having been rejected c) rejecting

8. Many apologies for not ... to your letter.

a) having replied b) replying c) being replied

9. She remembers ... him the message.

a) having been given b) giving c) being given

10. I'm really looking forward to ... all your news.

a) being heard b) having heard c) hearing

Task 5. Join the two sentences to make one sentence, beginning with a gerund. Model. She's a teacher. It's hard work. Being a teacher is hard work / Teaching is hard work. 1. Capital letters are used to name geometrical objects. It is very convenient. 2. You are to classify these quadrilaterals. It requires the knowledge of some properties. 3. We are going to locate this point on the y axis. It will give us the first point on the line. 4. The student intends to divide a circle into a certain number of congruent parts. It will help him to obtain a regular polygon. 5. The base and the altitude of a rectangle are to be multiplied. It will give the product of its dimensions or the area of the rectangle. 6. Don't argue! It's no use. In a crossed quadrilateral, the interior angles on either side of the crossing add up to 720°. 7. Don't deny this fact! It is useless. A square is a quadrilateral, a parallelogram, a rectangle and a rhombus. 8. You are going to divide a heptagon (a 7-sided polygon) into five triangles. Is it any good?

Task 6. Choose the right preposition. Make sensible sentences.

1. Are you interested	a) disturbing you.
2. She is very good	b) looking after the children.
3. He insisted	c) learning foreign languages.
4. I apologize	d) having more time for doing things he wants to.
5. The teacher is fed up	e) understanding this – its too difficult.
6. She succeeded	f) answering our stupid questions.
7. My friend is keen	g) studying.
8. Professor is looking forward	h) considering his solution of the problem.
9. This student is not capable	i) doing sums.
10. His sister is tired on of to at in with for	j) getting good education.

Task 7. Complete the sentences using a gerund as an attribute.

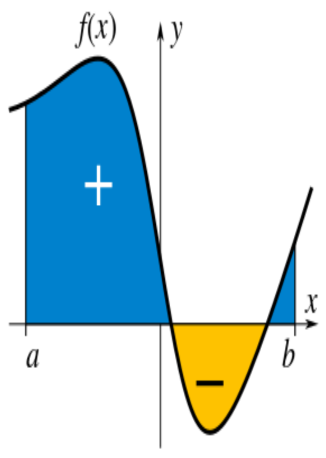
1. I didn't very much like the idea of … . 2. What is the purpose of … ? 3. She had no difficulty (in) … . 4. You have made great progress in … . 5. He was late, and he was afraid of … . 6. Can you imagine the pleasure of … . 7. He always produces the impression of … . 8. I am afraid you do not realize the importance of … . Ex. 5. Complete the second sentence so that it has a similar meaning to the first one. Use the word in bold and other words to complete each sentence. 1. I'll be happy when I can have a rest after exams. forward to I'm looking ... a rest after exams. 2. Learning new geometric theorems is something I like doing. interested in I'm always ... new geometric theorems 3. If I study a lot at night, it keeps me awake. prevents from ... a lot at night ... sleeping. 4. I often operate the computer at university. am used to I ... the computer at university. 5. He didn't want to take the books back to the library. feel like He didn't... the books back to the library. 6. He hates it if he has to do a lot of boring exercises. can't stand He ... a lot of boring exercises. 7. 'I'm sorry. I've broken the speed limit', said Sue. apologized for Sue ... the speed limit. 8. Let us write a new program. suggest I ... a new program.

Task 8. Find and correct the mistakes in the sentences. Some of them are right sentences.

1. I'm looking forward to go on holiday. 2. To cheat in examination is not allowed.
3. It was kind of you inviting me joining you. 4. It's a waste of time watching TV.
5. She said she was too busy to do this. 6. Do you think that drawing a polygon is easier than drawing a circle? 7. Please stop to make that noise, it's driving me mad.

UNIT 13

Text 13. Integral



A definite integral of a function can be represented as the signed area of the region bounded by its graph.

The integral is an important concept in mathematics. Integration is one of the two main operations in calculus, with its inverse, differentiation, being the other. Given a function f of a real variable x and an interval $[a, b]$ of the real line, the definite integral $\int_a^b f(x)\,dx$ is defined informally as the signed area of the region in the xy-plane that is bounded by the graph of f, the x-axis and the vertical lines $x = a$ and $x = b$. The area above the x-axis adds to the total and that below the x-axis subtracts from the total.

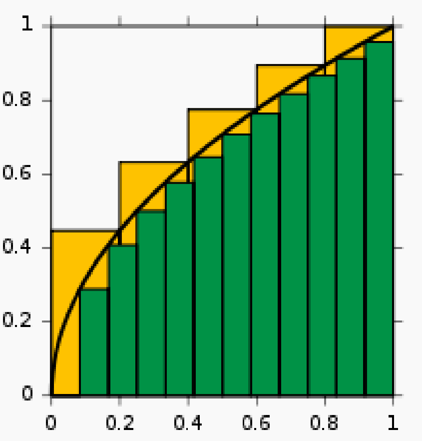
The term *integral* may also refer to the related notion of the antiderivative, a function F whose derivative is the given function f. In this case, it is called an *indefinite integral* and is written: $F(x) = \int f(x)\,dx.$

However, the integrals discussed in this article are those termed *definite integrals*. The principles of integration were formulated independently by Isaac Newton and Gottfried Leibniz in the late 17th century. Through the fundamental theorem of calculus, which they independently developed, integration is connected with differentiation: if f is a continuous real-valued function defined on a closed interval $[a, b]$, then, once an antiderivative F of f is known, the definite integral of over that interval is given by $\int_a^b f(x)\,dx = F(b) - F(a).$

Integrals and derivatives became the basic tools of calculus, with numerous applications in science and engineering. The founders of calculus thought of the integral as an infinite sum of rectangles of infinitesimal width. A rigorous mathematical definition of the integral was given by Bernhard Riemann. It is based on a limiting procedure which approximates the area of a curvilinear region by breaking the region into thin vertical slabs. Beginning in the nineteenth century, more sophisticated notions of integrals began to appear, where the type of the function as well as the domain over which the integration is performed has been generalised. A line integral is defined for functions of two or three variables, and the interval of integration $[a, b]$ is replaced by a certain curve connecting two points on the plane or in the space. In a surface integral, the curve is replaced by a piece of a surface in the three-dimensional space.

Integrals of differential forms play a fundamental role in modern differential geometry. These generalizations of integrals first arose from the needs of physics, and they play an important role in the formulation of many physical laws, notably those of electrodynamics. There are many modern concepts of integration, among these, the most common is based on the abstract mathematical theory known as Lebesgue integration, developed by Henri Lebesgue.

Introduction. Integrals appear in many practical situations. If a swimming pool is rectangular with a flat bottom, then from its length, width, and depth we can easily determine the volume of water it can contain (to fill it), the area of its surface (to cover it), and the length of its edge (to rope it). But if it is oval with a rounded bottom, all of these quantities call for integrals. Practical approximations may suffice for such trivial examples, but precision engineering (of any discipline) requires exact and rigorous values for these elements.



Approximations to integral of √xfrom 0 to 1, with 5 ■ (yellow) right endpoint partitions and 12 ■ (green) left endpoint partitions.

To start off, consider the curve $y = f(x)$ between $x = 0$ and $x = 1$ with $f(x) = \sqrt{x}$. We ask: *What is the area under the function f, in the interval from 0 to 1?* and call this (yet unknown) area the **integral** of *f*. The notation for this integral will be $\int_0^1 \sqrt{x}\,dx.$

As a first approximation, look at the unit square given by the sides $x = 0$ to $x = 1$ and $y = f(0) = 0$ and $y = f(1) = 1$. Its area is exactly 1. As it is, the true value of the integral must be somewhat less. Decreasing the width of the approximation rectangles shall give a better result; so cross the interval in five steps, using the approximation points 0, 1/5, 2/5, and so on to 1. Fit a box for each step using the right end height of each curve piece, thus √1/5, √2/5, and so on to √1 = 1. Summing the areas of these rectangles, we get a better approximation for the sought integral, namely

$$\sqrt{\tfrac{1}{5}}\left(\tfrac{1}{5}-0\right)+\sqrt{\tfrac{2}{5}}\left(\tfrac{2}{5}-\tfrac{1}{5}\right)+\cdots+\sqrt{\tfrac{5}{5}}\left(\tfrac{5}{5}-\tfrac{4}{5}\right)\approx 0.7497.$$

We are taking a sum of finitely many function values of *f*, multiplied with the differences of two subsequent approximation points. We can easily see that the approximation is still too large. Using more steps produces a closer approximation, but will never be exact: replacing the 5 subintervals by twelve in the same way, but with the left end height of each piece, we will get an approximate value for the area of 0.6203, which is too small. The key idea is the transition from adding *finitely many* differences of approximation points multiplied by their respective function values to using infinitely many fine, or *infinitesimal* steps.

As for the *actual calculation of integrals*, the fundamental theorem of calculus, due to Newton and Leibniz, is the fundamental link between the operations of differentiating and integrating. Applied to the square root curve, $f(x) = x^{1/2}$, it says to look at the antiderivative $F(x) = 2/3x^{3/2}$, and simply take $F(1) - F(0)$, where 0 and 1 are the boundaries of the interval [0, 1]. So the *exact* value of the area under the curve is computed formally as

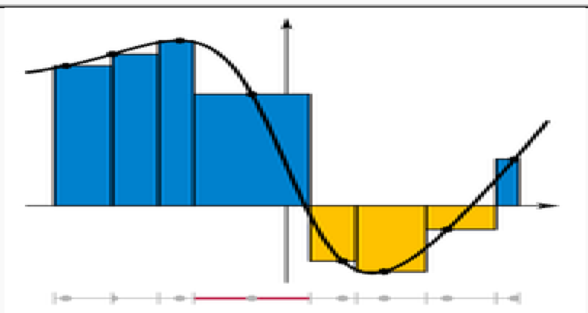
$$\int_0^1 \sqrt{x}\,dx = \int_0^1 x^{1/2}\,dx = F(1) - F(0) = \frac{2}{3}.$$

(This is a case of a general rule, that for $f(x) = x^q$, with $q \neq -1$, the related function, the so-called antiderivative is $F(x) = x^{q+1}/(q + 1)$.)

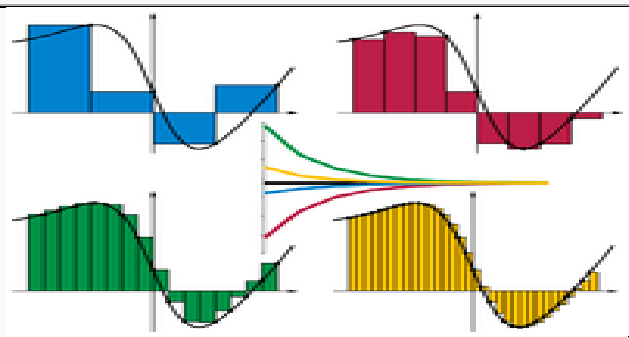
The notation $\int f(x)\,dx$ conceives the integral as a weighted sum, denoted by the elongated *s*, of function values, $f(x)$, multiplied by infinitesimal step widths, the so-called *differentials*, denoted by dx. The multiplication sign is usually omitted.

Historically, after the failure of early efforts to rigorously interpret infinitesimals, Riemann formally defined integrals as a limit of weighted sums, so that the dxsuggested the limit of a difference (namely, the interval width). Shortcomings of Riemann's dependence on intervals and continuity motivated newer definitions, especially the Lebesgue integral, which is founded on an ability to extend the idea of "measure" in much more flexible ways. Thus the notation $\int_A f(x)\,d\mu$ refers to a weighted sum in which the function values are partitioned, with μ measuring the weight to be assigned to each value. Here A denotes the region of integration.

Formal definitions



Integral example with irregular partitions (largest marked in red)



Riemann sums converging

There are many ways of formally defining an integral, not all of which are equivalent. The differences exist mostly to deal with differing special cases which may not be integrable under other definitions, but also occasionally for pedagogical reasons. The most commonly used definitions of integral are Riemann integrals and Lebesgue integrals.

Riemann integral. The Riemann integral is defined in terms of Riemann sums of functions with respect to *tagged partitions* of an interval. Let $[a, b]$ be a closed interval of the real line; then a *tagged partition* of $[a, b]$ is a finite sequence

$$a = x_0 \le t_1 \le x_1 \le t_2 \le x_2 \le \cdots \le x_{n-1} \le t_n \le x_n = b.$$

This partitions the interval $[a, b]$ into n sub-intervals $[x_{i-1}, x_i]$ indexed by i, each of which is "tagged" with a distinguished point $t_i \in [x_{i-1}, x_i]$. A *Riemann sum* of a function f with respect to such a tagged partition is defined as

$$\sum_{i=1}^{n} f(t_i)\Delta_i;$$

thus each term of the sum is the area of a rectangle with height equal to the function value at the distinguished point of the given sub-interval, and width the same as the sub-interval width. Let $\Delta_i = x_i - x_{i-1}$ be the width of sub-interval i; then the *mesh* of such a tagged partition is the width of the largest sub-interval formed by the partition, $\max_{i=1\ldots n} \Delta_i$. The *Riemann integral* of a function f over the interval $[a, b]$ is equal to S if:

For all $\varepsilon > 0$ there exists $\delta > 0$ such that, for any tagged partition $[a, b]$ with mesh less than δ, we have $\left|S - \sum_{i=1}^{n} f(t_i)\Delta_i\right| < \varepsilon.$

When the chosen tags give the maximum (respectively, minimum) value of each interval, the Riemann sum becomes an upper (respectively, lower) Darboux sum, suggesting the close connection between the Riemann integral and the Darboux integral.

Properties. Linearity. The collection of Riemann integrable functions on a closed interval $[a, b]$ forms a vector space under the operations of pointwise addition and multiplication by a scalar, and the operation of integration $f \mapsto \int_a^b f(x)\,dx$ is a linear functional on this vector space. Thus, firstly, the collection of integrable functions is closed under taking linear combinations; and, secondly, the integral of a linear combination is the linear combination of the integrals,

$$\int_a^b (\alpha f + \beta g)(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$$

Similarly, the set of real-valued Lebesgue integrable functions on a given measure space E with measure μ is closed under taking linear combinations and hence form a vector space, and the Lebesgue integral $f \mapsto \int_E f\,d\mu$ is a linear functional on this vector space, so that $\int_E (\alpha f + \beta g)\,d\mu = \alpha \int_E f\,d\mu + \beta \int_E g\,d\mu.$

Inequalities. A number of general inequalities hold for Riemann-integrable functions defined on a closed and bounded interval $[a, b]$ and can be generalized to other notions of integral (Lebesgue and Daniell).

1) *Upper and lower bounds.* An integrable function f on $[a, b]$, is necessarily bounded on that interval. Thus there are real numbers m and M so that $m \le f(x) \le M$ for all x in $[a, b]$. Since the lower and upper sums of f over $[a, b]$ are therefore bounded by, respectively, $m(b - a)$ and $M(b - a)$, it follows that

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

2) *Inequalities between functions.* If $f(x) \le g(x)$ for each x in $[a, b]$ then each of the upper and lower sums of f is bounded above by the upper and lower sums, respectively, of g. Thus $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$

This is a generalization of the above inequalities, as $M(b - a)$ is the integral of the constant function with value M over $[a, b]$.

In addition, if the inequality between functions is strict, then the inequality between integrals is also strict. That is, if $f(x) < g(x)$ for each x in $[a, b]$, then

$$\int_a^b f(x)\,dx < \int_a^b g(x)\,dx.$$

3) *Subintervals.* If $[c, d]$ is a subinterval of $[a, b]$ and $f(x)$ is non-negative for all x, then $\int_c^d f(x)\,dx \le \int_a^b f(x)\,dx.$

4) *Products and absolute values of functions.* If f and g are two functions then we may consider their pointwise products and powers, and absolute values:

$$(fg)(x) = f(x)g(x),\ f^2(x) = (f(x))^2,\ |f|(x) = |f(x)|.$$

If f is Riemann-integrable on $[a, b]$ then the same is true for $|f|$, and

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Moreover, if f and g are both Riemann-integrable then fg is also Riemann-integrable, and $\left(\int_a^b (fg)(x)\,dx\right)^2 \le \left(\int_a^b f(x)^2\,dx\right)\left(\int_a^b g(x)^2\,dx\right).$

This inequality, known as the Cauchy–Schwarz inequality, plays a prominent role in Hilbert space theory, where the left hand side is interpreted as the inner product of two square-integrable functions f and g on the interval $[a, b]$.

Conventions. In this section f is a real-valued Riemann-integrable function. The integral $\int_a^b f(x)\,dx$ over an interval $[a, b]$ is defined if $a < b$. This means that the upper and lower sums of the function f are evaluated on a partition $a = x_0 \le x_1 \le \ldots \le x_n = b$ whose values x_i are increasing. Geometrically, this signifies that integration takes place "left to right", evaluating f within intervals $[x_i, x_{i+1}]$ where an interval with a higher index lies to the right of one with a lower index. The values a and b, the end-points of the interval, are called the limits of integration of f. Integrals can also be defined if $a > b$: **Reversing limits of integration.** If $a > b$ then define

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$ This, with $a = b$, implies: *Integrals over intervals of length zero*. If a is a real number then $\int_a^a f(x)\,dx = 0.$

The first convention is necessary in consideration of taking integrals over subintervals of [a, b]; the second says that an integral taken over a degenerate interval, or a point, should be zero. One reason for the first convention is that the integrability of f on an interval [a, b] implies that f is integrable on any subinterval [c, d], but in particular integrals have the property that:

Additivity of integration on intervals. If c is any element of [a, b], then

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$ With the first convention the resulting relation

$$\begin{aligned}\int_a^c f(x)\,dx &= \int_a^b f(x)\,dx - \int_c^b f(x)\,dx \\ &= \int_a^b f(x)\,dx + \int_b^c f(x)\,dx\end{aligned}$$

is then well-defined for any cyclic permutation of a, b, and c.

Instead of viewing the above as conventions, one can also adopt the point of view that integration is performed of differential forms on *oriented* manifolds only. If $M$is such an oriented m-dimensional manifold, and M' is the same manifold with opposed orientation and ω is an m-form, then one has: $\int_M \omega = -\int_{M'} \omega\,.$ These conventions correspond to interpreting the integrand as a differential form, integrated over a chain. In measure theory, by contrast, one interprets the integrand as a function f with respect to a measure μ and integrates over a subset A, without any notion of orientation; one writes $\int_A f\,d\mu = \int_{[a,b]} f\,d\mu$ to indicate integration over a subset A.

The fundamental theorem of calculus is the statement that differentiation and integration are inverse operations: if a continuous function is first integrated and then differentiated, the original function is retrieved. An important consequence, sometimes called the *second fundamental theorem of calculus*, allows one to compute integrals by using an antiderivative of the function to be integrated.

Fundamental theorem of calculus. Let f be a continuous real-valued function defined on a closed interval [a, b]. Let F be the function defined, for all x in [a, b], by $F(x) = \int_a^x f(t)\,dt.$ Then, F is continuous on [a, b], differentiable on the open interval (a, b), and $F'(x) = f(x)$ for all x in (a, b).

Second fundamental theorem of calculus. Let f be a real-valued function defined on a closed interval *[a, b]* that admits an antiderivative F on [a, b].

That is, f and F are functions such that for all x in [a, b], $f(x) = F'(x).$

If f is integrable on $[a, b]$ then $$\int_a^b f(x)\,dx = F(b) - F(a).$$

Mathematical terminology

antiderivative – неопределённый интеграл, первообразная функция
indefinite integral – неопределенный интеграл
derivative – производная, производная величина
fundamental theorem of calculus - основная теорема матанализа
closed interval – замкнутый интервал
infinite sum of rectangles – бесконечная сумма прямоугольников
infinitesimal width – бесконечно малая ширина
curvilinear – криволинейный
curve – кривая, график
surface integral – интеграл по поверхности
differential form – дифференциальная форма
differential geometry – дифференциальная геометрия
physical law – физическая закономерность
electrodynamics – электродинамика
Lebesgue integral - интеграл Лебе́га, это обобщение интеграла Римана на более широкий класс функций
precision engineering – точное машиностроение
differentiating – дифференцирование
limit of weighted sums – предел взвешенных сумм
Riemann sum – сумма Римана
Darboux sums – суммы Дарбу
Darboux integral – интеграл Дарбу
vector space – векторное пространство
pointwise addition – поточечное сложение
linear functional – линейный функционал
linear combination – линейная комбинация (функций или векторов)
real-valued Lebesgue-integrable (Riemann-integrable) function – действительная функция, интегрируемая по Лебегу (Риману)
measure space – пространство с мерой
closed and bounded interval – ограниченный и замкнутый интервал
Cauchy–Schwarz inequality – неравенство Коши-Шварца
Hilbert space – гильбертово пространство
inner product – скалярное произведение
square-integrable functions – квадратично интегрируемые функции
conventions - условные обозначения
end-points of the interval – конечные точки интервала
limits of integration – пределы интегрирования
oriented manifolds – ориентированное множество
measure theory – теория меры
continuous function – непрерывная функция

Grammar, Lexical, Translation and Speaking Exercises

Task 1. Match the columns.

1) an interval on the real line including its end points, as [0, 1], the set of reals between and including 0 and 1; 2) another function, obtained by multiplying the image of the two functions at each value in the domain. If *f* and *g* are both functions with domain *X* and codomain *Y*, and elements of *Y* can be multiplied (for instance, *Y* could be some set of numbers), then the pointwise product of *f* and *g*is another function from *X* to *Y* which maps $x \in X$ to $f(x)g(x)$; 3) a line segment of a circle passing through the centre of the circle; 4) the amount remaining after one quantity is subtracted from another (e.g., in 5 — 3 = 2, 2 is the difference); 5) any one of the ten numerals: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9; 6) a space consisting of vectors, together with the associative and commutative operation of addition of vectors, and the associative and distributive operation of multiplication of vectors by scalars; 7) measurements of a figure (e.g., the length, width, and height of a 3-D object); 8) a series of points not connected in a line graph; 9) the length of the line segment joining two points.	a) pointwise product of two functions; b) closed interval; c) vector space; d) diameter; e) difference; f) digit; g) dimensions; h) discrete element; i) distance

Task 2. Read the formulae and translate them into Russian.

(a, b)	open interval a and b
[ab]	closed interval a and b
(ab]	half – open interval a b, open on the left and closed on the right
X=(−∞;+∞)	capital x equals the open interval minus infinite plus infinite
$X \to x_o$	x approaches x nought; or x tends to x nought
$\lim_{x \to x1} f(x) = L$	the limit of f x as x tends to x one is capital L

$\lim_{x \to xo} f(x) \neq f(xo)$	the limit of f of x tends to x nought is not equal to f of x nought
$\lim_{n \to \infty} an = 0$	the limit of a sub n is zero as n tends to infinity

Task 3. Translate from English into Russian and retell the text.

FERMAT'S LAST THEOREM

Pierre de Fermat was born in Toulouse in 1601 and died in 1665. Today we think of Fermat as a number theorist, infact as perhaps the most famous number theorist who ever lived. The history of Pythagorean triples goes back to 1600 B.C, but it was not until the seventeenth century A.D that mathematicians seriously attacked, in general terms, the problem of finding positive integer solutions to the equation $x^n + y^n = z^n$. Many mathematicians conjectured that there are no positive integer solutions to this equation if n isgreater than 2. Fermat's now famous conjecture was inscribed in the marginof his copy of the Latin translation of Diophantus's Arithmetica. The note read: "To divide a cube intotwo cubes, a fourth power or in general any power whatever into two powers of the same denomination above the second is impossible and I have assuredly found an admirable proof of this, but the margin is too narrow to contain it".

Despite Fermat's confident proclamation the conjecture, referred to as "Fermat's last theorem" remains unproven. Fermat gave elsewhere a proof for the case n = 4. it was not until the next century that L.Euler supplied a proof for the case n = 3, and still another century passed before A.Legendre and L.Dirichlet arrived at independent proofs of the case n = 5. Not long after, in 1838, G.Lame established the theorem for n = 7. In 1843, the German mathematician E.Kummer submitted a proof of Fermat's theorem to Dirichlet. Dirichlet found an error in the argument and Kummer returned to the problem. After developing the algebraic "theory of ideals", Kummer produced a proof for "most small n". Subsequent progress in the problem utilized Kummer's ideals and many more special cases were proved. It is now known that Fermat's conjecture is true for all n < 4.003 and many special values of n, but no general proof has been found.

Fermat's conjecture generated such interest among mathematicians that in 1908 the German mathematician P.Wolfskehl bequeathed DM 100.000 to the Academy of Science at Gottingen as a prize for the first complete proof of the theorem. This prize induced thousands of amateurs to prepare solutions, with the result that Fermat's theorem is reputed to be the maths problem for which the greatest number of incorrect proofs was published. However, these faulty arguments did not tarnish the reputation of the genius – P.Fermat. Richard Lawrence Taylor (born 19 May 1962) is a British mathematician working in the field of number theory. A former research student of Andrew Wiles, he returned to Princeton to help his advisor complete the proof of Fermat's Last Theorem. Taylor received a $3 million 2014 Breakthrough Prize in Mathematics "For numerous breakthrough results in the theory of automorphic forms, including the Taniyama-Weil conjecture, the local Langlands conjecture for general linear groups, and the Sato-Tate conjecture." He also received the 2007 Shaw Prize in Mathematical Sciences for his work on the Langlands program with Robert Langlands.

Task 4. Answer the following questions.

How old was Pierre Fermat when he died?
Which problem did mathematicians face in the 17 century A.D?
What did many mathematiciansconjecture at that time?
Who first gave a proof to Fermat's theorem?
What proof did he give?
Did any mathematicians prove Fermat's theorem after him? Who were they?

Task 5. Are the statements True (T) or False (F)? Correct the false sentences.

The German mathematician E.Kummer was the first to find an error in the argument.
With the algebraic "theory of ideals" in hand, Kummer produced a proof for "most small n" and many special cases.
A general proof has been found for all value of n.
The German mathematician P.Wolfskehl won DM 100.000 in 1908 for the first complete proof of the theorem.

Task 6. Memorize the fractions pronunciation.

Fractional numbers – Дробные числительные
Common Fractions (Простые дроби)
½ - a (one) half; $^1/_3$ - a (one) third; $^2/_3$ – two thirds;
¼ - a (one) quarter, a (one) fourth; ¾ - three quarters, three fourths;
$^1/_5$ – a (one) fifth; $^2/_5$ – two fifths; $^1/_6$ – one sixth; $^5/_6$ – five sixths;
1 ½ - one and a half; 2 ¼ - two and a (one) quarter
Decimal Fractions – Десятичные дроби
0.1 - nought point one; point one (Br. Eng.); zero point one (Am. Eng.)
0.01 - nought point nought one; point nought one
2.35 – two point three five
32.305 – three two (or thirty-two) point three nought five

Task 7. Choose the correct variant of translation.

1. The numerator and the denominator of this fraction are sure to be divisible by two.
a) Числитель и знаменатель дроби делится на два.
b) Мы уверяем вас, что числитель и знаменатель можно сократить на два.
c) Без сомнения, числитель и знаменатель этой дроби делится на два.

2. Scientists were determined to develop further that idea.

a) Приняли решение, что ученые будут содействовать развитию этой идеи.

b) Ученые вычислили, то необходимо в будущем развивать эту идею.

c) Ученых заставили развивать в дальнейшем эту идею.

d) Ученые решили, что будут развивать эту идею в дальнейшем.

3. Where there is a choice of two expressions, we should always choose the more accurate one.

a) Там, где существует выбор из двух выражений, нам всегда следует выбирать более точное выражение.

b) Там, где есть выбор из двух выражений, мы всегда выберем более точное выражение.

c) Там, где есть выбор из двух выражений, мы бы всегда выбирали более точное выражение.

4. Thus the circumference of a circle may be defined as the limit of the perimeter of an inscribed regular n-gon as n increases.

a) Окружность определяется как периметр вписанного многогранника, где количество сторон обозначается как n.

б) Длина окружности круга может быть определена как предел от периметра правильного вписанного n-угольника, где n – возрастает.

c) Окружность, описанная вокруг любого многоугольника, где n – количество сторон, определяется по формуле $C=2\pi r$.

5. Very often a proposition is so worded that it requires thought to state the converse proposition correctly.

a) Очень часто утверждение формулируется таким образом, что нужно как следует подумать, чтобы сформулировать обратное утверждение правильно.

b) Зачастую утверждение составляется так, что требуется поразмыслить, чтобы правильно заявить об обратном утверждении.

c) Очень часто утверждение выражается так, что оно требует размышления над правильной формулировкой обратного утверждения.

Task 8. Translate the following sentences into English.

1. В данном случае обе теоремы – как прямая, так и обратная – оказываются справедливыми. 2. Пять аксиом Евклида – это предложения, вводящие отношения равенства или неравенства величин. 3. Учебник Евклида по геометрии «Начала» читали, читают и будут читать многие (люди). 4. Предложение, которое следует непосредственно из аксиомы, называется следствием. 5. Следующие две теоремы обратны друг другу. 6. Одно и то же предложение может быть или не быть истинным относительно другого

множества допущений. 7. В любой теореме есть две части: гипотеза и вывод. 8. Вас просят записать кратко предположения, которые вы сделали. 9. Аксиома – это истинное, исходное положение теории. 10. Постулат – это утверждение, принимаемое в какой-либо научной теории как истинное, хотя и не доказуемое ее средствами, и поэтому он играет в ней роль аксиомы.

Task 9. Read the text and find the answers to the following questions.

1. What is logical deduction? 2. Do we proceed from the general to the particular or from the particular to the general in induction? 3. Which method of thinking is more useful: deductive or inductive?

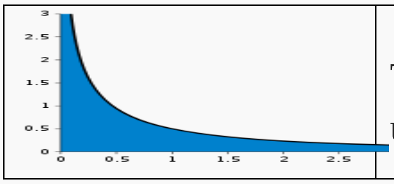
Deduction and Induction

The scientists have proved a chain of theorems and have come to recognize the entire structure of undefined terms, definitions, assumptions, and theorems as constituting an abstract logical system. In such a system we say that each proposition is derived from its predecessor by the process of logical deduction. This process of logical deduction is scientific reasoning. This scientific reasoning must not be confused with the mode of thinking employed by the scientist when he is feeling his way toward a new discovery. At such times the scientist, curious about the sum of the angles of a triangle, proceeds to measure the angles of a great many triangles very carefully. In every instance he notices that the sum of the three angles is very close to 180°; so he puts forward a guess that this will be true of every triangle he might draw. This method of deriving a general principle from a limited number of special instances is called induction.

The method of induction always leaves the possibility that further measurement and experimentation may necessitate some modification of the general principle. The method of deduction is not subject to upsets of this sort. When the mathematician is groping for (ищет) new mathematical ideas, he uses induction. On the other hand, when he wishes to link his ideas together into a logical system, he uses deduction. The laboratory scientist also uses deduction when he wishes to order and classify the results of his observations and his inspired guesses and to arrange them all in a logical system. While building this logical system he must have a pattern (модель) to guide him, an ideal of what a logical system ought to be. The simplest exposition (изложение) of this ideal is to be found in the abstract logical system of demonstrative geometry. It is clear that both deductive and inductive thinking are very useful to the scientist.

UNIT 14

Text 14. Improper integrals

	The improper integral $\int_0^\infty \frac{dx}{(x+1)\sqrt{x}} = \pi$ has unbounded intervals for both domain and range.

A "proper" Riemann integral assumes the integrand is defined and finite on a closed and bounded interval, bracketed by the limits of integration. An improper integral occurs when one or more of these conditions is not satisfied. In some cases such integrals may be defined by considering the limit of a sequence of proper Riemann integrals on progressively larger intervals.

If the interval is unbounded, for instance at its upper end, then the improper integral is the limit as that endpoint goes to infinity. $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$

If the integrand is only defined or finite on a half-open interval, for instance $(a, b]$, then again a limit may provide a finite result. $\int_a^b f(x)\,dx = \lim_{\epsilon\to 0} \int_{a+\epsilon}^b f(x)\,dx$

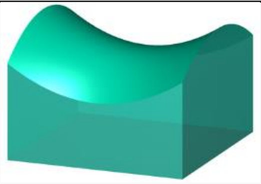
That is, the improper integral is the limit of proper integrals as one endpoint of the interval of integration approaches either a specified real number, or ∞, or −∞. In more complicated cases, limits are required at both endpoints, or at interior points.

Consider, for example, the function $1/((x + 1)\sqrt{x})$ integrated from 0 to ∞ (shown right). At the lower bound, as x goes to 0 the function goes to ∞, and the upper bound is itself ∞, though the function goes to 0. Thus this is a doubly improper integral. Integrated, say, from 1 to 3, an ordinary Riemann sum suffices to produce a result of π/6. To integrate from 1 to ∞, a Riemann sum is not possible. However, any finite upper bound, say t (with $t > 1$), gives a well-defined result, $2 \arctan(\sqrt{t}) - \pi/2$. This has a finite limit as t goes to infinity, namely π/2. Similarly, the integral from 1/3 to 1 allows a Riemann sum as well, coincidentally again producing π/6. Replacing 1/3 by an arbitrary positive value s (with $s < 1$) is equally safe, giving $\pi/2 - 2 \arctan(\sqrt{s})$. This, too, has a finite limit as s goes to zero, namely π/2. Combining the limits of the two fragments, the result of this improper integral is

$$\begin{aligned}\int_0^\infty \frac{dx}{(x+1)\sqrt{x}} &= \lim_{s\to 0}\int_s^1 \frac{dx}{(x+1)\sqrt{x}} + \lim_{t\to\infty}\int_1^t \frac{dx}{(x+1)\sqrt{x}} \\ &= \lim_{s\to 0}\left(\frac{\pi}{2} - 2\arctan\sqrt{s}\right) + \lim_{t\to\infty}\left(2\arctan\sqrt{t} - \frac{\pi}{2}\right) \\ &= \frac{\pi}{2} + \left(\pi - \frac{\pi}{2}\right) \\ &= \frac{\pi}{2} + \frac{\pi}{2} \\ &= \pi .\end{aligned}$$

This process does not guarantee success; a limit might fail to exist, or might be unbounded. For example, over the bounded interval from 0 to 1 the integral of $1/x$ does not converge; and over the unbounded interval from 1 to ∞ the integral of $1/\sqrt{x}$ does not converge.

Multiple integration

	Double integral as volume under a surface. Integrals can be taken over regions other than intervals. In general, an integral over a set E of a function f is written: $\int_E f(x)\,dx.$

Here x need not be a real number, but can be another suitable quantity, for instance, a vector in $\mathbf{R}^3$. Fubini's theorem shows that such integrals can be rewritten as an *iterated integral*. In other words, the integral can be calculated by integrating one coordinate at a time.

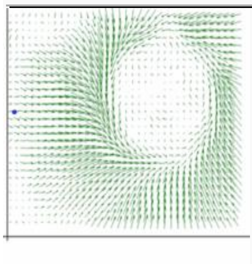
Just as the definite integral of a positive function of one variable represents the area of the region between the graph of the function and the x-axis, the *double integral* of a positive function of two variables represents the volume of the region between the surface defined by the function and the plane which contains its domain. (The same volume can be obtained via the *triple integral* – the integral of a function in three variables – of the constant function $f(x, y, z) = 1$ over the above-mentioned region between the surface and the plane.) If the number of variables is higher, then the integral represents a hypervolume, a volume of a solid of more than three dimensions that cannot be graphed. For example, the volume of the cuboid of sides 4 × 6 × 5 may be obtained in two ways:

1) By the double integral $\iint_D 5\ dx\ dy$ of the function $f(x, y) = 5$ calculated in the region D in the xy-plane which is the base of the cuboid. For example, if a rectangular base of such a cuboid is given via the xy inequalities $3 \le x \le 7$, $4 \le y \le 10$, our above double integral now reads $\int_4^{10}\left[\int_3^7 5\ dx\right] dy.$

From here, integration is conducted with respect to either x or y first; in this example, integration is first done with respect to x as the interval corresponding to $x$is the inner integral. Once the first integration is completed via the $F(b) - F(a)$ method or otherwise, the result is again integrated with respect to the other variable. The result will equate to the volume under the surface.

2) By the triple integral $\iiint_{\text{cuboid}} 1\, dx\, dy\, dz$ of the constant function 1 calculated on the cuboid itself.

Line integrals



A line integral sums together elements along a curve.

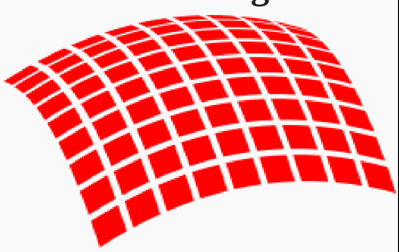
The concept of an integral can be extended to more general domains of integration, such as curved lines and surfaces. Such integrals are known as line integrals and surface integrals respectively. These have important applications in physics, as when dealing with vector fields.

A *line integral* (sometimes called a *path integral*) is an integral where the function to be integrated is evaluated along a curve. Various different line integrals are in use. In the case of a closed curve it is also called a *contour integral*.

The function to be integrated may be a scalar field or a vector field. The value of the line integral is the sum of values of the field at all points on the curve, weighted by some scalar function on the curve (commonly arc length or, for a vector field, the scalar product of the vector field with a differential vector in the curve). This weighting distinguishes the line integral from simpler integrals defined on intervals. Many simple formulas in physics have natural continuous analogs in terms of line integrals; for example, the fact that work is equal to force, **F**, multiplied by displacement, **s**, may be expressed (in terms of vector quantities) as: $W = \mathbf{F} \cdot \mathbf{s}$.

For an object moving along a path *C* in a vector field **F** such as an electric field or gravitational field, the total work done by the field on the object is obtained by summing up the differential work done in moving from **s** to **s** + *d***s**. This gives the line integral $W = \int_C \mathbf{F} \cdot d\mathbf{s}.$

Surface integrals



A *surface integral* is a definite integral taken over a surface (which may be a curved set in space); it can be thought of as the double integral analog of the line integral. The function to be integrated may be a scalar field or a vector field. The value of the surface integral is the sum of the field at all points on the surface. This can be achieved by splitting the surface into surface elements, which provide the partitioning for Riemann sums.

For an example of applications of surface integrals, consider a vector field **v** on a surface *S*; that is, for each point *x* in *S*,**v**(*x*) is a vector. Imagine that we have a fluid flowing through S, such that $\mathbf{v}(x)$ determines the velocity of the fluid at x. The flux is defined as the quantity of fluid flowing through S in unit amount of time. To find the flux, we need to take the dot product of $\mathbf{v}$ with the unit surface normal to S at each point, which will give us a scalar field, which we integrate over the surface: $\int_s \mathbf{v} \cdot d\mathbf{S}$. The fluid flux in this example may be from a physical fluid such as water or air, or from electrical or magnetic flux. Thus surface integrals have applications in physics, particularly with the classical theory of electromagnetism.

Integrals of differential forms. A differential form is a mathematical concept in the fields of multivariable calculus, differential topology and tensors. The modern notation for the differential form, as well as the idea of the differential forms as being the wedge products of exterior derivatives forming an exterior algebra, was introduced by Élie Cartan. We initially work in an open set in $\mathbf{R}^n$. A 0-form is defined to be a smooth function f. When we integrate a function f over an m-dimensional subspace S of $\mathbf{R}^n$, we write it as $\int_S f\, dx^1 \ldots dx^m$.

We can consider dx^1 through dx^n to be formal objects themselves, rather than tags appended to make integrals look like Riemann sums. Alternatively, we can view them as covectors, and thus a measure of "density" (hence integrable in a general sense). We call the dx^1, …, dx^n *basic* 1-*forms*. We define the wedge product, "∧", a bilinear "multiplication" operator on these elements, with the *alternating* property that $dx^a \wedge dx^a = 0$ for all indices a.

Alternation along with linearity and associativity implies $dx^b \wedge dx^a = -dx^a \wedge dx^b$.

This also ensures that the result of the wedge product has an orientation. We define the set of all these products to be *basic* 2-*forms*, and similarly we define the set of products of the form $dx^a \wedge dx^b \wedge dx^c$ to be *basic* 3-*forms*. A general k-form is then a weighted sum of basic k-forms, where the weights are the smooth functions f. Together these form a vector space with basic k-forms as the basis vectors, and 0-forms (smooth functions) as the field of scalars. The wedge product then extends to k-forms in the natural way. Over $\mathbf{R}^n$ at most n covectors can be linearly independent, thus a k-form with $k > n$ will always be zero, by the alternating property. In addition to the wedge product, there is also the exterior derivative operator d. This operator maps k-forms to $(k+1)$-forms. For a k-form $\omega = f\, dx^a$ over $\mathbf{R}^n$, we define the action of d by:

$$d\omega = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx^i \wedge dx^a.$$

with extension to general k-forms occurring linearly.

This more general approach allows for a more natural coordinate-free approach to integration on manifolds. It also allows for a natural generalisation of the fundamental theorem of calculus, called Stokes' theorem, which we may state as $\int_\Omega d\omega = \int_{\partial\Omega} \omega$, where ω is a general k-form, and $\partial\Omega$ denotes the boundary of the region Ω. Thus, in the case that ω is a 0-form and Ω is a closed interval of the real line, this reduces to the fundamental theorem of calculus. In the case that ω is a 1-form and Ω is a two-dimensional region in the plane, the theorem reduces to Green's theorem. Similarly, using 2-forms, and 3-forms and Hodge duality, we can arrive at Stokes' theorem and the divergence theorem. In this way we can see that differential forms provide a powerful unifying view of integration.

Summation. The discrete equivalent of integration is summation. Summations and integrals can be put on the same foundations using the theory of Lebesgue integrals or time scale calculus.

Computation. The most basic technique for computing definite integrals of one real variable is based on the fundamental theorem of calculus. Let $f(x)$ be the function of x to be integrated over a given interval $[a, b]$. Then, find an antiderivative of f; that is, a function F such that $F' = f$ on the interval. Provided the integrand and integral have no singularities on the path of integration, by the fundamental theorem of calculus, $\int_a^b f(x)\,dx = F(b) - F(a)$.

The integral is not actually the antiderivative, but the fundamental theorem provides a way to use antiderivatives to evaluate definite integrals. The most difficult step is usually to find the antiderivative of f. It is rarely possible to glance at a function and write down its antiderivative. More often, it is necessary to use one of the many techniques that have been developed to evaluate integrals. Most of these techniques rewrite one integral as a different one which is hopefully more tractable. Techniques include:

Integration by substitution	Integration by reduction formulae
Integration by parts	Integration using parametric derivatives
Inverse function integration	Integration using Euler's formula
Changing the order of integration	Euler substitution
Integration by trigonometric substitution	Differentiation under the integral sign
Tangent half-angle substitution	Contour integration
Integration by partial fractions	

Alternative methods exist to compute more complex integrals. Many nonelementary integrals can be expanded in a Taylor series and integrated term by term. Occasionally, the resulting infinite series can be summed analytically. There are also many less common ways of calculating definite integrals; for instance, Parseval's identity can be used to transform an integral over a rectangular region into an infinite sum. Occasionally, an integral can be evaluated by a trick.

Symbolic. Many problems in mathematics, physics, and engineering involve integration where an explicit formula for the integral is desired. Extensive tables of integrals have been compiled and published over the years for this purpose. With the spread of computers, many professionals, educators, and students have turned to computer algebra systems that are specifically designed to perform difficult or tedious tasks, including integration. Symbolic integration has been one of the motivations for the development of the first such systems, like Macsyma.

A major mathematical difficulty in symbolic integration is that in many cases, a closed formula for the antiderivative of a rather simple-looking function does not exist. For instance, it is known that the antiderivatives of the functions $\exp(x^2)$, x^x and $(\sin x)/x$ cannot be expressed in the closed form involving only rational and exponential functions, logarithm, trigonometric and inverse trigonometric functions, and the operations of multiplication and composition; in other words, none of the three given functions is integrable in elementary functions, which are the functions which may be built from rational functions, roots of a polynomial, logarithm, and exponential functions. The Risch algorithm provides a general criterion to determine whether the antiderivative of an elementary function is elementary, and, if it is, to compute it. Unfortunately, it turns out that functions with closed expressions of antiderivatives are the exception rather than the rule. Consequently, computerized algebra systems have no hope of being able to find an antiderivative for a randomly constructed elementary function. On the positive side, if the 'building blocks' for antiderivatives are fixed in advance, it may be still be possible to decide whether the antiderivative of a given function can be expressed using these blocks and operations of multiplication and composition, and to find the symbolic answer whenever it exists. The Risch algorithm, implemented in Mathematica and other computer algebra systems, does just that for functions and antiderivatives built from rational functions, radicals, logarithm, and exponential functions.

Some special integrands occur often enough to warrant special study. In particular, it may be useful to have, in the set of antiderivatives, the special functions of physics (like the Legendre functions, the hypergeometric function, the Gamma function, the Incomplete Gamma function and so on). Extending the Risch's algorithm to include such functions is possible but challenging and has been an active research subject.

More recently a new approach has emerged, using *D*-finite function, which are the solutions of linear differential equations with polynomial coefficients. Most of the elementary and special functions are *D*-finite and the integral of a *D*-finite function is also a *D*-finite function. This provide an algorithm to express the antiderivative of a *D*-finite function as the solution of a differential equation. This theory allows also to compute a definite integrals of a *D*-function as the sum of a series given by the first coefficients and an algorithm to compute any coefficient.

Mathematical terminology

improper integral – определенный (несобственный) интеграл
limit of a sequence – предел последовательности
limit of proper integral – предел собственного интеграл
iterated integral – повторный интеграл
Fubini's theorem – теоре́ма Тоне́лли-Фуби́ни в математическом анализе, теории вероятностей и смежных дисциплинах сводит вычисление двойного интеграла к повторным
line integral (path integral) – линейный интеграл (интеграл по траектории)
contour integral – криволинейный интеграл, интеграл по контуру
scalar field – скалярное поле
scalar product, dot product – скалярное произведение, внутреннее произведение (векторов)
vector field – векторное поле
differential vector in the curve – дифференциальный интеграл на кривой
electric field - электрическое поле
gravitational field - гравитационное поле
hypervolume – гиперобъём
cuboid – прямоугольный параллелепипед
iterated integral – повторный интеграл
flux – движение, поток, магнитный поток
surface normal – нормаль к поверхности
classical theory of electromagnetism – классическая теория электромагнетизма
multivariable calculus – анализ функций многих переменных (многомерный анализ (также известный как многомерное или многовариантное исчисление)
differential topology – дифференциальная топология
tensor – тензор; **tensor calculus –** тензорное исчисление
wedge product; exterior product – V-произведение
tensor product – тензорное произведение
exterior derivative – внешняя производная
exterior algebra – внешняя алгебра
Élie Cartan – Эли́ Жозе́ф Карта́н (1869 – 1951), французский математик, основные труды по теории непрерывных групп, дифференциальных уравнений и дифференциальной геометрии.
open set – открытое множество
smooth function – гладкая функция
***m*-dimensional –** m-мерный
exterior derivative – внешняя производная
covector – ковектор
orientation – направление, расположение, направленность
vector space – векторное пространство
exterior derivative – внешняя производная
manifold – множество
Stokes' theorem – теорема Стокса, одна из основных теорем дифференциальной геометрии и математического анализа об интегрировании дифференциальных форм, которая

обобщает несколько теорем анализа. Названа в честь Дж. Г. Стокса.
boundary – предел; граница
Green's theorem – теорема Грина устанавливает связь между криволинейным интегралом по замкнутому контуру C и двойным интегралом по односвязной области D, ограниченной этим контуром. Фактически, эта теорема является частным случаем более общей теоремы Стокса. Теорема названа в честь английского математика Джорджа Грина.
Hodge duality – звезда́ Хо́джа, важный линейный оператор из пространства *p*-векторов в пространство *n*-*p*-форм.
divergence theorem – теорема о дивергенции, теорема Гаусса-Остроградского
summation – суммирование, сложение; подведение итога
time scale – 1) шкала времени 2) масштаб времени
singularity – 1) сингулярность, особенность 2) особая точка функции
integration by substitution – интегрирование подстановкой
integration by parts – интегрирование по частям
inverse function integration – интегрирование обратной функции
Changing the order of integration – изменение порядка интегрирования
Integration by trigonometric substitution – интегрирование посредством тригонометрической подстановки
Tangent half-angle substitution – подстановка с использованием тангенса половинного угла. Универсальная тригонометрическая подстановка, в англоязычной литературе называемая в честь Карла Вейерштрасса подстановкой Вейерштрасса, применяется в интегрировании для нахождения первообразных, определённых и неопределённых интегралов от рациональных функций от тригонометрических функций. Без потери общности можно считать в данном случае такие функции рациональными функциями от синуса и косинуса.
Integration by partial fractions – интегрирование посредством простейших [элементарных] дробей
Integration by reduction formulae - интегрирование посредством сокращения формул
Integration using parametric derivatives – интегрирование посредством использования параметрических производных
Integration using Euler's formula – интегрирование с использованием формулы Эйлера
Euler substitution – подстановка Эйлера
Differentiation under the integral sign – дифференцирование под знаком интеграла
Contour integration – контурное интегрирование
nonelementary integrals – неэлементарные интегралы
Taylor series – ряды Тейлора (an infinite sum giving the value of a function f(z) in the neighborhood of a point ***a*** in terms of the derivatives of the function evaluated at ***a***)
Parseval's identity – равенство Парсеваля
solids of revolution – тела вращения
list (table) of integrals – перечень (таблица) интегралов
computer algebra system (CAS) – система компьютерной алгебры (СКА) - это прикладная программа для символьных вычислений, то есть выполнения преобразований и работы с математическими выражениями в аналитической (символьной) форме.
Macsyma – Максима, система компьютерной алгебры, первая версия которой была разработана с 1968 по 1982 год в MIT в лаборатории Project MAC, а впоследствии

распространялась на коммерческой основе. Это была первая всеобъемлющая система символьной математики и одна из ранних систем, основанных на знаниях. Многие из идей, появившихся в Macsyma впоследствии были заимствованы такими системами как Mathematica, Maple, и другими.
exponential functions – экспоненциальная функция, показательная функция,
logarithm – логарифм
logarithm to the base e – натуральный логарифм
logarithm to the base ten – десятичный логарифм
taking the logarithm – взятие логарифма
to take a logarithm – брать логарифм, логарифмировать
to take the logarithm of a number – находить логарифм числа
inverse trigonometric function – обратная тригонометрическая функция
elementary function – элементарная функция
roots of a polynomial – корни полинома (многочлена)
Risch algorithm – Алгоритм Риша, алгоритм для аналитического взятия неопределённых интегралов, использующий методы дифференциальной алгебры. Он базируется на типе интегрируемой функции и на методах интегрирования рациональных функций, корней, логарифмов, и экспоненциальных функций. Назван в честь Роберта Генри Риша. Сам Риш, который разработал алгоритм в 1968 году, называл его «разрешающей процедурой», поскольку метод решает, является ли первообразная от функции элементарной функцией.
radical – радикал, знак корня, корень
Legendre function – функция Лежандра
hypergeometric function – гипергеометрическая функция
Gamma function – гамма-функция, Γ-функция определяетс как $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$, где x – действительное число больше нуля.
Incomplete Gamma function – неполная гамма-функция
linear differential equations – линейное дифференциальное уравнение

Grammar, Lexical, Translation and Speaking Exercises

Task 1. Answer the questions.

How can the improper integral be defined?
Give the definition to the line intergral.
What is the difference between the scalar field and the vector field?
What is a differential form in mathematics?
What do the techniques of summation and computation imply?

Task 2. Match the columns.

1) a function of a space whose value at each point is a scalar quantity; 2) the integral, taken along a line, of any function that has a continuously varying value along that line;	a) scalar field b) integral line c) tensor d) Stokes'

3) a mathematical object analogous to but more general than a vector, represented by an array of components that are functions of the coordinates of a space; 4) a theorem proposing that the surface integral of the curl of a function over any surface bounded by a closed path is equal to the line integral of a particular vector function round that path.	theorem

Task 3. Translate from Russian into English.

Определённый интеграл называется несобственным, если выполняется, по крайней мере, одно из следующих условий:

Область интегрирования является бесконечной. Например, является бесконечным интервалом $[a, +\infty)$.

Функция $f(x)$ является неограниченной в окрестности некоторых точек области интегрирования.

Если интервал $[a, b]$ конечный, и функция интегрируема по Риману, то значение несобственного интеграла совпадает с значением определённого интеграла.

Task 4. Translate the types of logarithms into Russian and give the definition to the each type.

binary logarithm	modulus of logarithm
common logarithm	monotonic logarithm
compositional logarithm	multivalued logarithm
continuous logarithm	Napierian logarithm
double logarithm	natural logarithm
hyperbolic logarithm	negative logarithm
integral logarithm	seven-place logarithm
inverse logarithm	subtracted logarithm
iterated logarithm	sum logarithm
leading logarithm	

Task 5. Translate the deffinions into Russian and memorize how the symbols are pronounced.

$f: U \to V$	f from U to V
$f(x)$	f of x
$x \to f(x)$	x mape to $f(x)$
of class C^k	of class C^k
of class C^∞	of class C infinity
the Lebesgue spaces L^p, L^∞	the Lebesgue spaces Lp , L infinity
the Sobolev spaces H^k, $W^{k,p}$	the Sobolev spaces H k , W k p

Task 6. Ask special questions using the words in parenthesis.

1. An arc is usually named by its endpoints. (how) 2. A chord is a line segment connecting any two points on the circle. (what) 3. They attended the lectures on geometry twice a week. (how often) 4. Most mathematical proofs can be given in many different ways. (how) 5. In geometry we separate all geometric figures into two groups: plane figures and space figures or solids. (how many) 6. Later you ought to do some measurements to check your calculations. (when) 7. We are already familiar with the basic concepts of geometry through our high school studies of maths. (what) 8. The points of geometry have no size and no dimensions. (what) 9. Numbers became abstract when we began to reason about their nature and enumerate their properties through arithmetical and logical operations. (when) 10. A straight line extends indefinitely only in one direction. (where) 11. Every math problem must be settled either in the form of a direct answer to the question, or by the proof of the impossibility of its solution. (how) 12. The Greeks were able to carry out many constructions with two tools. (how many) 13. The theory in question was developed successively by different scientists. (who)

Task 7. Choose the correct variant of translation.

1. We expect them to solve this problem.

a. Мы ожидаем, что они решат эту задачу.

b. Мы ждали, что они решат эту задачу.

c. Мы ждем их, пока они решат эту задачу.

2. They are believed to have done their best.

a. Они верят, что сделали все возможное.

b. Полагают, что они сделали все возможное.

c. Полагали, что они сделали все возможное.

3. They appear to have known all about the set theory.

a. Они появляются, чтобы узнать все о теории множеств.

b. Они пришли и узнали все о теории множеств.

c. Оказывается, они узнали все о теории множеств.

4. What made the students do the test quickly?

a. Что сделали студенты, чтобы выполнить тест быстро?

b. Что заставляет студентов выполнять тест быстро?

c. Что заставило студентов выполнять тест быстро?

5. First-year students are thought to show very good results at the exams.

a. Первокурсники, как считают, показывают очень хорошие *результаты на экзамене.*

b. Считают, что первокурсники хотят показать очень хорошие результаты на экзамене.

c. Считали, что первокурсники покажут очень хорошие результаты на экзамене.

UNIT 15

Text 15. Vector calculus

Vector calculus (or **vector analysis**) is a branch of mathematics concerned with differentiation and integration of vector fields, primarily in 3-dimensional Euclidean space $\mathbb{R}^3$. The term "vector calculus" is sometimes used as a synonym for the broader subject of multivariable calculus, which includes vector calculus as well as partial differentiation and multiple integration. Vector calculus plays an important role in differential geometry and in the study of partial differential equations. It is used extensively in physics and engineering, especially in the description of electromagnetic fields, gravitational fields and fluid flow.

Vector calculus was developed from quaternion analysis by J. Willard Gibbs and Oliver Heaviside near the end of the 19th century, and most of the notation and terminology was established by Gibbs and Edwin Bidwell Wilson in their 1901 book, *Vector Analysis* . In the conventional form using cross products, vector calculus does not generalize to higher dimensions, while the alternative approach of geometric algebra, which uses exterior products does generalize, as discussed below.

Scalar fields. A scalar field associates a scalar value to every point in a space. The scalar may either be a mathematical number or a physical quantity. Examples of scalar fields in applications include the temperature distribution throughout space, the pressure distribution in a fluid, and spin-zero quantum fields, such as the Higgs field. These fields are the subject of scalar field theory.

Vector fields. A vector field is an assignment of a vector to each point in a subset of space. A vector field in the plane, for instance, can be visualized as a collection of arrows with a given magnitude and direction each attached to a point in the plane. Vector fields are often used to model, for example, the speed and direction of a moving fluid throughout space, or the strength and direction of some force, such as the magnetic or gravitational force, as it changes from point to point.

Vectors and pseudovectors. In more advanced treatments, one further distinguishes pseudovector fields and pseudoscalar fields, which are identical to vector fields and scalar fields except that they change sign under an orientation-reversing map: for example, the curl of a vector field is a pseudovector field, and if one reflects a vector field, the curl points in the opposite direction. This distinction is clarified and elaborated in geometric algebra, as described below.

Vector operations. Algebraic operations. The algebraic (non-differential) operations in vector calculus are referred to as **vector algebra**, being defined for a vector space and then globally applied to a vector field, and consist of:

scalar multiplication multiplication of a scalar field and a vector field, yielding a vector field: $a\mathbf{v}$;

vector addition addition of two vector fields, yielding a vector field: $\mathbf{v}_1 + \mathbf{v}_2$;

dot product multiplication of two vector fields, yielding a scalar field: $\mathbf{v}_1 \cdot \mathbf{v}_2$;

cross product multiplication of two vector fields, yielding a vector field: $\mathbf{v}_1 \times \mathbf{v}_2$

There are also two triple products: scalar triple product the dot product of a vector and a cross product of two vectors: $\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)$;

vector triple product the cross product of a vector and a cross product of two vectors: $\mathbf{v}_1 \times (\mathbf{v}_2 \times \mathbf{v}_3)$ or $(\mathbf{v}_3 \times \mathbf{v}_2) \times \mathbf{v}_1$; although these are less often used as basic operations, as they can be expressed in terms of the dot and cross products.

Differential operations. Vector calculus studies various differential operators defined on scalar or vector fields, which are typically expressed in terms of the del operator (∇), also known as "nabla". The five most important differential operations in vector calculus are:

Operation	Notation	Description	Domain/Range
Gradient	$\operatorname{grad}(f) = \nabla f$	Measures the rate and direction of change in a scalar field.	Maps scalar fields to vector fields.
Curl	$\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$	Measures the tendency to rotate about a point in a vector field.	Maps vector fields to (pseudo)vector fields.
Divergence	$\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$	Measures the scalar of a source or sink at a given point in a vector field.	Maps vector fields to scalar fields.
Vector Laplacian	$\nabla^2\mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F})$	Measures the difference between the value of the vector field with its average on	Maps between vector fields.

		infinitesimal balls.	
Laplacian	$\Delta f = \nabla^2 f = \nabla \cdot \nabla f$	Measures the difference between the value of the scalar field with its average on infinitesimal balls.	Maps between scalar fields.

The curl and divergence differ because the former uses a cross product and the latter a dot product, f denotes a scalar field and F denotes a vector field. A quantity called the Jacobian is useful for studying functions when both the domain and range of the function are multivariable, such as a change of variables during integration.

Theorems. Likewise, there are several important theorems related to these operators which generalize the fundamental theorem of calculus to higher dimensions:

Theorem	Statement	Description
Gradient theorem	$\int_{L[\mathbf{p}\to\mathbf{q}]\subset\mathbb{R}^n} \nabla\varphi \cdot d\mathbf{r} = \varphi(\mathbf{q}) - \varphi(\mathbf{p})$	The line integral through a gradient (vector) field equals the difference in its scalar field at the endpoints of the curve L.
Green's theorem	$\iint_{A\subset\mathbb{R}^2} \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) d\mathbf{A} = \oint_{\partial A} (L\,dx + M\,dy)$	The integral of the scalar curl of a vector field over some region in the plane equals the line integral of the vector field over the closed curve bounding the region oriented in the counterclockwise direction.
Stokes' theorem	$\iint_{\Sigma\subset\mathbb{R}^3} \nabla\times\mathbf{F}\cdot d\mathbf{\Sigma} = \oint_{\partial\Sigma} \mathbf{F}\cdot d\mathbf{r}$	The integral of the curl of a vector field over a surface in $\mathbb{R}^3$ equals the line integral of the vector field over the closed curve bounding the surface.
Divergence theorem	$\iiint_{V\subset\mathbb{R}^3} (\nabla\cdot\mathbf{F})\,d\mathbf{V} = \oiint_{\partial V} \mathbf{F}\cdot d\mathbf{S}$	The integral of the divergence of a vector field over some solid equals the integral of the flux through the closed surface bounding the solid.

Applications. Linear approximations. Linear approximations are used to replace complicated functions with linear functions that are almost the same. Given a differentiable function $f(x, y)$ with real values, one can approximate

$f(x,y)$ for (x,y) close to (a,b) by the formula $f(x,y) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$. The right-hand side is the equation of the plane tangent to the graph of $z = f(x,y)$ at (a,b).

Optimization. For a continuously differentiable function of several real variables, a point *P* (that is a set of values for the input variables, which is viewed as a point in $\mathbf{R}^n$) is **critical** if all of the partial derivatives of the function are zero at *P*, or, equivalently, if its gradient is zero. The critical values are the values of the function at the critical points.

If the function is smooth, or, at least twice continuously differentiable, a critical point may be either a local maximum, a local minimum or a saddle point. The different cases may be distinguished by considering the eigenvalues of the Hessian matrix of second derivatives.

By Fermat's theorem, all local maxima and minima of a differentiable function occur at critical points. Therefore, to find the local maxima and minima, it suffices, theoretically, to compute the zeros of the gradient and the eigenvalues of the Hessian matrix at these zeros.

Different 3-manifolds. Vector calculus is initially defined for Euclidean 3-space, $\mathbb{R}^3$, which has additional structure beyond simply being a 3-dimensional real vector space, namely: an inner product (the dot product), which gives a notion of length (and hence angle), and an orientation, which gives a notion of left-handed and right-handed. These structures give rise to a volume form, and also the cross product, which is used pervasively in vector calculus.

The gradient and divergence require only the inner product, while the curl and the cross product also requires the handedness of the coordinate system to be taken into account (see cross product and handedness for more detail).

Vector calculus can be defined on other 3-dimensional real vector spaces if they have an inner product (or more generally a symmetric nondegenerate form) and an orientation; note that this is less data than an isomorphism to Euclidean space, as it does not require a set of coordinates (a frame of reference), which reflects the fact that vector calculus is invariant under rotations (the special orthogonal group SO(3)).

More generally, vector calculus can be defined on any 3-dimensional oriented Riemannian manifold, or more generally pseudo-Riemannian manifold. This structure simply means that the tangent space at each point has an inner product (more generally, a symmetric nondegenerate form) and an orientation, or more globally that there is a symmetric nondegenerate metric tensor and an orientation, and works because vector calculus is defined in terms of tangent vectors at each point.

Mathematical terminology

Vector calculus – векторное исчисление
differentiation and integration of vector field – дифференцирование и интегрирование векторного поля
primarily – главным образом; первоначально
3-dimensional - трёхмерное (пространство)
Euclidean space – евклидово пространство, т.е. пространство, в котором местоположение каждой точки задано и расстояния между точками вычисляются как корень квадратный из суммы квадратов разностей координат по каждому измерению. В математике рассматриваются и неевклидовы пространства (non-Euclidean space), где это правило не выполняется
multivariable calculus – многовариантное исчисление
partial differentiation – определение частной производной
multiple integration – многократное интегрирование
differential geometry – дифференциальная геометрия
partial differential equation – частный дифференциал
electromagnetic field – электромагнитное поле, ЭМП
gravitational field – гравитационное поле
fluid flow – поток текучей среды, течение жидкости, течение жидкости или газа
quaternion – кватернион, четырёхчлен – гиперкомплексное число с тремя мнимыми единицами i, j, k, то есть: q = w+x*i+y*j+z*k, где w, x. y, и z - действительные числа. Кватернион используется для представления вращения объектов в трёхмерном пространстве - в САПР, в машинной графике, компьютерных играх и т. п.
geometric algebra – Алгебра Клиффорда - специального вида ассоциативная алгебра с единицей Cl(E,Q(,)) над некоторым коммутативным кольцом K (E - векторное пространство, в дальнейшем обобщении - свободный K-модуль) с некоторой операцией [«умножения»], совпадающей с заданной на E билинейной формой Q.
exterior product – внешнее произведение
scalar field – скалярное поле
mathematical number (in linear algebra = real numbers = scalars) - скалярная величина
physical quantity – физическая величина
spin-zero quantum fields – бесспиновое квантовое поле
scalar field theory – теория скалярного поля
vector field – векторное поле
magnetic or gravitational force – магнитная и гравитационная сила
vector and pseudovector – вектор и псевдовектор
scalar multiplication – скалярное умножение
vector addition – векторное сложение, сложение векторов
dot product – скалярное произведение (векторов)
cross product – векторное произведение
triple product – смешанное произведе́ние (тройное скалярное произведение) **(a, b, c**) векторов **a, b, c** – скалярное произведение вектора **a** на векторное произведение векторов **b** и **c**: **(a,b,c) = a · (b x c).**
scalar triple product – смешанное произведение (векторов)
vector triple product – двойное векторное произведение
differential operator – дифференциальный оператор
del operator – оператор используемый в векторном анализе; набла, символ ∇, символ δ
gradient – 1) градиент (a) дифференциальный оператор, b) градиент скалярного или векторного поля, c) скорость изменения какой-либо величины с расстоянием, d) кривая

зависимости скорости изменения какой-либо величины от расстояния, e) инструмент для заливки выделенной области несколькими цветами с плавными переходами между ними, инструмент для (многоцветной) градиентной заливки (в графических редакторах));
2) склон; уклон; наклон; 3) наклонная поверхность; наклонная плоскость.
curl – ротор или вихрь, т.е. векторный дифференциальный оператор над векторным полем. Обозначается (в русскоязычной литературе) или curl (в англоязычной литературе), а также как векторное умножение дифференциального оператора набла на векторное поле: $\nabla\times$.
Результат действия этого оператора на конкретное векторное поле F называется *ротором поля* F или просто *ротором* F и представляет собой новое векторное поле: $\operatorname{rot}\mathbf{F}\equiv\nabla\times\mathbf{F}$.
Поле rot F (длина и направление вектора rot F в каждой точке пространства) характеризует в некотором смысле вращательную составляющую поля Fсоответственно в каждой точке.
divergence - 1) дивергенция 2) несовпадение, несходство 3) расхождение 4) отклонение
Laplacian – оператор Лапласа, лапласиан
change of variables – замена переменных
function of several real variables – функция нескольких вещественных переменных
smooth function – гладкая функция или непрерывно дифференцируемая функция т.е. функция, имеющая непрерывную производную на всём множестве определения.
local maximum / local minimum – локальный максимум (наибольшее значение, которое принимает функция на некотором промежутке значений ее аргументов) / локальный минимум
saddle point – седловая точка, т.е. точка, в которой функция двух аргументов является одновременно максимумом относительно одной переменной и минимумом для другой
eigenvalue (EV) – собственное значение; характеристическое число (матрицы)
Hessian matrix – гессиан, матрица Гессе
maxima and minima of a differentiable function – экстре́мумы (лат. extremum – крайний), т.е. максимальное или минимальное значение функции на заданном множестве. Точка, в которой достигается экстремум, называется точкой экстремума. Соответственно, если достигается максимум – точка экстремума называется точкой **максимума**, а если минимум – **минимума** точкой. В математическом анализе выделяют также понятие локальный экстремум (соответственно минимум или максимум).
inner product (the dot product) – внутреннее [скалярное] произведение
notion of length – понятие протяженности
orientation - направленность
left-handed and right-handed – левосторонний, левовинтовой; вращающийся против часовой стрелки/ правосторонний, правовинтовой; вращающийся по часовой стрелке
give rise to – вызывать, иметь результаты
volume form – форма объёма
handedness of the coordinate system – киральность системы координат, хиральность системы координат handedness правое или левое направление (напр. вращения); направление по часовой стрелке или против неё
symmetric nondegenerate form – симметричная невырожденная форма
special orthogonal group SO(3) – специальная ортогональная группа
Riemannian manifold – риманово множество;
pseudo-Riemannian manifold – псевдориманово множество
tangent space – касательное пространство
metric tensor – метрический тензор

Grammar, Lexical, Translation and Speaking Exercises

Task 1. Use the correct form of the degrees of comparison.

1) We all use this method of research because it is ………………. (interesting) the one we followed.
2) I could solve quicker than he because the equation given to me was……….(easy)

the one he was given.
3) The remainder in this operation of division is …………….. (great) than 1.
4) The name of Leibnitz is …………….. (familiar) to us as that of Newton.
5) Laptops are ………… (powerful) microcomputers. We can choose either of them.
6) A mainframe is……… (large) and…………..(expensive) a microcomputer.
7) One of the ……… (important) reasons why computers are used so widely today is that almost every big problem can be solved by solving a number of little problems.
8) Even the …………(sophisticated) computer, no matter how good it is, must be told what to do.

Task 2. Translate the sentences and note the form of the Infinitive.

We consider these two phenomena to be of the same origin. I expect this law to hold for all similar cases. We understand this method to consist of several steps. They wanted us to establish a certain correspondence between these two facts. We assume the program to have been carefully developed. We suppose the particles to be generated at very high speed. We expect this sentence to be true. We know mathematics to have become man's second language. We expect a variable or a mathematical expression containing a variable to represent a number. We know two numbers to be relatively prime to each other if their greatest common factor is 1. We expect this solution to satisfy the given statement. Professor wants his students to attend classes regularly. The students saw their instructor draw (drawing) a line segment. We heard them discuss (discussing) similar questions. Professor wanted his postgraduate students to take part in his research. For a proper correspondence between these phenomena to be established they first have to be considered separately. For correct conclusions to be drawn all the conditions must be observed. It was impossible for the process to continue. I wonder if it is necessary for them to come. For you to begin the work now is very important. For the problem to be understood it must be read carefully.

Task 3. In the following sentences use the Complex Object.

Model: I expect ***that the article will be written***. – I expect ***the article to be written***.

I expect that these rules will be observed. I know that this work is of great importance. He expects that the situation will be analysed carefully. We believe that the machine has certain advantages. I thought she was ready. He expected that I knew the solution. We found that they were interested in the problem. I expect that she will understand me. We expected that he had completed the experiment. I knew that you had obtained similar results. I believed that they had closely cooperated with you. We found that she had studied the material properly. I suppose that he is involved in this discussion. I assume that they have applied the previously obtained data.

Task 4. Ask general questions following the model. Replace the nouns with

pronouns. Model: We expect ***the students*** to learn the material. - Do you expect ***them*** to learn the material?

We expect *the scientists* to establish an appropriate pattern. He expects *the student* to speak on the coordinate system. The students wanted *their tutor* to speak on number relations. We found *these statements* to be mathematically incorrect. I believe *this result* to be of some importance.

Task 5. Replace the Object Clause with the Complex Object. See the models.

Model: I want to label this number line with X. (he)

I want him to label this number line with X.

We expect to locate this distant object in the sky (she). I should like to draw both of the axes (he). He expects to speak about the importance of coordinating our research (they). I should like to interpret these facts correctly (he).

Task 6. Fill in the blanks with the necessary words and word combinations given bellow. Mind there are two extra ones.

1. In algebra a square matrix is …with real entries whose columns and rows are … . 2. The set of n × n orthogonal matrices forms a group O(n) known as … . 3. An orthogonal matrix is the real specialization of…. 4. Orthogonal matrices arise naturally from…. 5. Thus … – rotations, reflections, and their combinations – produce orthogonal matrices. 6. Linear algebra includes orthogonal transformations between …. 7. The inverse of every orthogonal matrix is again ... . 8. The orthogonal matrices whose …is +1 form … .	a) finite-dimensional linear isometries; b) a linear transformation; c) an orthogonal group; d) a unitary matrix; e) the special orthogonal group; f) determinant; g) orthogonal unit vectors; h) spaces; i) orthogonal; j) an orthogonal matrix; k) inner products l

Task 7. Memorize the following word combinations

a square matrix – квадратичная матрица

an orthogonal unit vector – ортогональный единичный вектор

an identity matrix – единичная матрица

a linear transformation – линейное преобразование

a unitary transformation – унитарное (единичное) преобразование

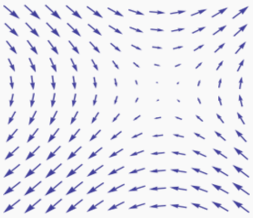
an orthogonal group – ортогональная группа

to bring to identity – привести к единице

finite-dimensional linear isometries – конечномерные линейные изометрии
an inner product – скалярное произведение
a bottom right entry – элемент матрицы, расположенный в нижнем правом углу таблицы
a matrix inverse – обратная матрица
simultaneous equations – совместные уравнения, система уравнений

UNIT 16

Text 16. Vector field

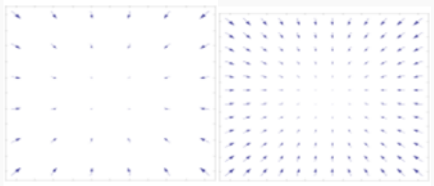
A portion of the vector field (sin *y*, sin *x*) 	In vector calculus, a vector field is an assignment of a vector to each point in a subset of space. A vector field in the plane, for instance, can be visualized as a collection of arrows with a given magnitude and direction each attached to a point in the plane. Vector fields are often used to model, for example, the speed and direction of a moving fluid throughout space, or the strength and direction of some force, such as the magnetic or gravitational force, as it changes from point to point.

The elements of differential and integral calculus extend to vector fields in a natural way. When a vector field represents force, the line integral of a vector field represents the work done by a force moving along a path, and under this interpretation conservation of energy is exhibited as a special case of the fundamental theorem of calculus. Vector fields can usefully be thought of as representing the velocity of a moving flow in space, and this physical intuition leads to notions such as the divergence (which represents the rate of change of volume of a flow) and curl (which represents the rotation of a flow).

In coordinates, a vector field on a domain in *n*-dimensional Euclidean space can be represented as a vector-valued function that associates an *n*-tuple of real numbers to each point of the domain. This representation of a vector field depends on the coordinate system, and there is a well-defined transformation law in passing from one coordinate system to the other. Vector fields are often discussed on open subsets of Euclidean space, but also make sense on other subsets such as surfaces, where they associate an arrow tangent to the surface at each point (a tangent vector).

More generally, vector fields are defined on differentiable manifolds, which are spaces that look like Euclidean space on small scales, but may have more complicated structure on larger scales. In this setting, a vector field gives a tangent vector at each point of the manifold (that is, a section of the tangent bundle to the manifold). Vector fields are one kind of tensor field.

Vector fields on subsets of Euclidean space



Two representations of the same vector field: **v**$(x, y) = -$**r**. The arrows depict the field at discrete points, however, the field exists everywhere.

Given a subset S in $\mathbf{R}^n$, a **vector field** is represented by a vector-valued function $V: S \to \mathbf{R}^n$ in standard Cartesian coordinates $(x_1, \ldots, x_n)$. If each component of V is continuous, then V is a continuous vector field, and more generally V is a C^k vector field if each component V is k times continuously differentiable. A vector field can be visualized as assigning a vector to individual points within an n-dimensional space. Given two C^k-vector fields V, W defined on S and a real valued C^k-function f defined on S, the two operations scalar multiplication and vector addition

$$(fV)(p) := f(p)V(p). \qquad (V + W)(p) := V(p) + W(p)$$

define the module of C^k-vector fields over the ring of C^k-functions.

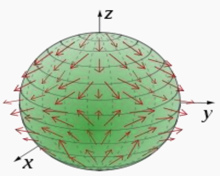
Coordinate transformation law. In physics, a vector is additionally distinguished by how its coordinates change when one measures the same vector with respect to a different background coordinate system. The transformation properties of vectors distinguish a vector as a geometrically distinct entity from a simple list of scalars, or from a covector. Thus, suppose that $(x_1,\ldots,x_n)$ is a choice of Cartesian coordinates, in terms of which the components of the vector V are $V_x = (V_{1,x}, \ldots, V_{n,x})$ and suppose that $(y_1,\ldots,y_n)$ are n functions of the x_i defining a different coordinate system. Then the components of the vector V in the new coordinates are required to satisfy the transformation law

$$V_{i,y} = \sum_{j=1}^{n} \frac{\partial y_i}{\partial x_j} V_{j,x}.$$

Such a transformation law is called contravariant. A similar transformation law characterizes vector fields in physics: specifically, a vector field is a specification of $n$functions in each coordinate system subject to the transformation law relating the different coordinate systems.

Vector fields are thus contrasted with scalar fields, which associate a number or *scalar* to every point in space, and are also contrasted with simple lists of scalar fields, which do not transform under coordinate changes.

Vector fields on manifolds



A vector field on a sphere

Given a differentiable manifold M, a **vector field** on M is an assignment of a tangent vector to each point in M. More precisely, a vector field F is a mapping from M into the tangent bundle TM so that $p \circ F$ is the identity mapping where p denotes the projection from TM to M. In other words, a vector field is a section of the tangent bundle.

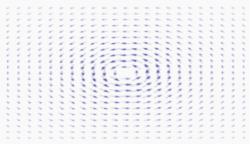
If the manifold M is smooth or analytic—that is, the change of coordinates is smooth (analytic) – then one can make sense of the notion of smooth (analytic) vector fields. The collection of all smooth vector fields on a smooth manifold M is often denoted by $\Gamma(TM)$ or $C^\infty(M,TM)$ (especially when thinking of vector fields as sections); the collection of all smooth vector fields is also denoted by $\mathfrak{X}(M)$ (a fraktur "X").

Examples.
The flow field around an airplane is a vector field in $\mathbf{R}^3$, here visualized by bubbles that follow the streamlines showing a wingtip vortex.

- A vector field for the movement of air on Earth will associate for every point on the surface of the Earth a vector with the wind speed and direction for that point. This can be drawn using arrows to represent the wind; the length (magnitude) of the arrow will be an indication of the wind speed. A "high" on the usual barometric pressure map would then act as a source (arrows pointing away), and a "low" would be a sink (arrows pointing towards), since air tends to move from high pressure areas to low pressure areas.
- Velocity field of a moving fluid. In this case, a velocity vector is associated to each point in the fluid.
- Streamlines, Streaklines and Pathlines are 3 types of lines that can be made from vector fields. They are : a) streaklines – as revealed in wind tunnels using smoke;

b) streamlines (or fieldlines) – as a line depicting the instantaneous field at a given time; c) pathlines – showing the path that a given particle (of zero mass) would follow.

* Magnetic fields. The fieldlines can be revealed using small iron filings.
* Maxwell's equations allow us to use a given set of initial conditions to deduce, for every point in Euclidean space, a magnitude and direction for the force experienced by a charged test particle at that point; the resulting vector field is the electromagnetic field.
* A gravitational field generated by any massive object is also a vector field. For example, the gravitational field vectors for a spherically symmetric body would all point towards the sphere's center with the magnitude of the vectors reducing as radial distance from the body increases.

Gradient field 	A vector field that has circulation about a point cannot be written as the gradient of a function. Vector fields can be constructed out of scalar fields using the gradient operator (denoted by the del: ∇).

A vector field V defined on a set S is called a **gradient field** or a **conservative field** if there exists a real-valued function (a scalar field) f on S such that

$$V = \nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n}\right).$$

The associated flow is called the **gradient flow**, and is used in the method of gradient descent. The path integral along any closed curve γ ($\gamma(0) = \gamma(1)$) in a gradient field is zero:

$$\oint_\gamma \langle V(x), \mathrm{d}x\rangle = \oint_\gamma \langle \nabla f(x), \mathrm{d}x\rangle = f(\gamma(1)) - f(\gamma(0)).$$

where the angular brackets and comma: ⟨, ⟩ denotes the inner product of two vectors (strictly speaking - the integrand $V(x)$ is a 1-form rather than a vector in the elementary sense).

Central field A C^∞-vector field over $\mathbf{R}^n \setminus \{0\}$ is called a **central field** if $V(T(p)) = T(V(p)) \qquad (T \in \mathrm{O}(n, \mathbf{R}))$ where $\mathrm{O}(n, \mathbf{R})$ is the orthogonal group. We say central fields are invariant under orthogonal transformations around 0.

The point 0 is called the **center** of the field.

Since orthogonal transformations are actually rotations and reflections, the invariance conditions mean that vectors of a central field are always directed towards, or away from, 0; this is an alternate (and simpler) definition. A central field is always a gradient field, since defining it on one semiaxis and integrating gives an antigradient.

Operations on vector fields. Line integral. A common technique in physics is to integrate a vector field along a curve, i.e. to determine its line integral. Given a particle in a gravitational vector field, where each vector represents the force acting on the particle at a given point in space, the line integral is the work done on the particle when it travels along a certain path.

The line integral is constructed analogously to the Riemann integral and it exists if the curve is rectifiable (has finite length) and the vector field is continuous.

Given a vector field V and a curve γ parametrized by $[a, b]$ (where a and b are real) the line integral is defined as

$$\int_\gamma \langle V(x), \mathrm{d}x\rangle = \int_a^b \langle V(\gamma(t)), \gamma'(t)\ \mathrm{d}t\rangle.$$

Divergence. The divergence of a vector field on Euclidean space is a function (or scalar field). In three-dimensions, the divergence is defined by

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z},$$

with the obvious generalization to arbitrary dimensions. The divergence at a point represents the degree to which a small volume around the point is a source or a sink for the vector flow, a result which is made precise by the divergence theorem.

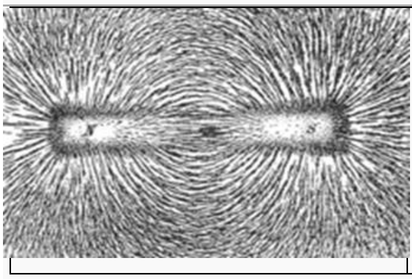
The divergence can also be defined on a Riemannian manifold, that is, a manifold with a Riemannian metric that measures the length of vectors.

Curl. The curl is an operation which takes a vector field and produces another vector field. The curl is defined only in three-dimensions, but some properties of the curl can be captured in higher dimensions with the exterior derivative. In three-dimensions, it is defined by

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{e}_1 - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\mathbf{e}_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{e}_3.$$

The curl measures the density of the angular momentum of the vector flow at a point, that is, the amount to which the flow circulates around a fixed axis. This intuitive description is made precise by Stokes' theorem.

Index of a vector field. The index of a vector field is a way of describing the behaviour of a vector field around an isolated zero (i.e. non-singular point) which can distinguish saddles from sources and sinks. Take a small sphere around the zero so that no other zeros are included. A map from this sphere to a unit sphere of dimensions $n-1$ can be constructed by dividing each vector by its length to form a unit length vector which can then be mapped to the unit sphere. The index of the vector field at the point is the degree of this map. The index of the vector field is the sum of the indices of each zero. The index will be zero around any non singular point, it is +1 around sources and sinks and -1 around saddles. In two dimensions the index is equivalent to the winding number. For an ordinary sphere in three dimension space it can be shown that the index of any vector field on the sphere must be two, this leads to the hairy ball theorem which shows that every such vector field must have a zero. This theorem generalises to the Poincaré–Hopf theorem which relates the index to the Euler characteristic of the space.



History. Magnetic field lines of an iron bar (magnetic dipole). Vector fields arose originally in classical field theory in 19th century physics, specifically in magnetism. They were formalized by Michael Faraday, in his concept of *lines of force*, who emphasized that the field *itself* should be an object of study, which it has become throughout physics in the form of field theory.

In addition to the magnetic field, other phenomena that were modeled as vector fields by Faraday include the electrical field and light field.

Flow curves. Consider the flow of a fluid through a region of space. At any given time, any point of the fluid has a particular velocity associated with it; thus there is a vector field associated to any flow. The converse is also true: it is possible to associate a flow to a vector field having that vector field as its velocity.

Given a vector field *V* defined on *S*, one defines curves $\gamma(t)$ on *S* such that for each t in an interval I : $\gamma'(t) = V(\gamma(t))$.

By the Picard–Lindelöf theorem, if V is Lipschitz continuous there is a *unique* C^1-curve γ_x for each point x in S so that $\gamma_x(0) = x$

$$\gamma'_x(t) = V(\gamma_x(t)) \qquad (t \in (-\epsilon, +\epsilon) \subset \mathbf{R}).$$

The curves γ_x are called **flow curves** of the vector field V and partition S into equivalence classes. It is not always possible to extend the interval $(-\varepsilon, +\varepsilon)$ to the whole real number line. The flow may for example reach the edge of S in a finite time. In two or three dimensions one can visualize the vector field as giving rise to a flow on S. If we drop a particle into this flow at a point p it will move along the curve γ_p in the flow depending on the initial point p. If p is a stationary point of V then the particle will remain at p.

Typical applications are streamline in fluid, geodesic flow, and one-parameter subgroups and the exponential map in Lie groups.

Difference between scalar and vector field. The difference between a scalar and vector field is not that a scalar is just one number while a vector is several numbers. The difference is in how their coordinates respond to coordinate transformations. A scalar *is* a coordinate whereas a vector *can be described* by coordinates, but it *is not* the collection of its coordinates.

Example 1. This example is about 2-dimensional Euclidean space ($\mathbf{R}^2$) where we examine Euclidean (x, y) and polar (r, θ) coordinates (which are undefined at the origin). Thus $x = r \cos \theta$ and $y = r \sin \theta$ and also $r^2 = x^2 + y^2$, $\cos \theta = x/(x^2 + y^2)^{1/2}$ and $\sin \theta = y/(x^2 + y^2)^{1/2}$. Suppose we have a scalar field which is given by the constant function 1, and a vector field which attaches a vector in the r-direction with length 1 to each point. More precisely, they are given by the functions

$$s_{\text{polar}} : (r, \theta) \mapsto 1, \quad v_{\text{polar}} : (r, \theta) \mapsto (1, 0).$$

Let us convert these fields to Euclidean coordinates. The vector of length 1 in the r-direction has the x coordinate $\cos \theta$ and the y coordinate $\sin \theta$. Thus in Euclidean coordinates the same fields are described by the functions: $s_{\text{Euclidean}} : (x, y) \mapsto 1,$

$$v_{\text{Euclidean}} : (x, y) \mapsto (\cos\theta, \sin\theta) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right).$$

We see that while the scalar field remains the same, the vector field now looks different. The same holds even in the 1-dimensional case, as illustrated by the next example.

Example 2. Consider the 1-dimensional Euclidean space $\mathbf{R}$ with its standard Euclidean coordinate x. Suppose we have a scalar field and a vector field which are both given in the x coordinate by the constant function 1,

$$s_{\text{Euclidean}} : x \mapsto 1, \quad v_{\text{Euclidean}} : x \mapsto 1.$$

Thus, we have a scalar field which has the value 1 everywhere and a vector field which attaches a vector in the *x*-direction with magnitude 1 unit of *x* to each point.

Now consider the coordinate $\xi := 2x$. If *x* changes one unit then ξ changes 2 units. Thus this vector field has a magnitude of 2 in units of ξ. Therefore, in the ξ coordinate the scalar field and the vector field are described by the functions

$s_{\mathbf{unusual}} : \xi \mapsto 1, \quad v_{\mathbf{unusual}} : \xi \mapsto 2$ which are different.

Mathematical terminology

vector field – векторное поле
extend to vector fields – распростаняется на векторные поля
line integral – линейный интеграл
conservation of energy – сохранение энергии
vector-valued function – векторнозначная функция
transformation law – закон преобразования
open subset of Euclidean space – открытое подмножество евклидового пространства
surface – площадь; поверхность
tangent vector – касательный вектор; тангенциальный вектор
differentiable manifold – дифференцируемое многообразие
tangent bundle to the manifold - касательное расслоение многообразия
tensor field – тензорное поле
vector-valued function – векторнозначная функция
continuously differentiable – непрерывно дифференцируемый
transformation properties of vectors – свойства преобразования векторов
covector – ковектор
contravariant – контравариантный
differentiable manifold – дифференцируемое многообразие
wingtip vortex – вихрь на конце крыла
magnitude – 1) величина; значение (величины); 2) амплитуда; 3) абсолютное значение, модуль (числа) ; длина, модуль (вектора)
barometric pressure – барометрическое давление
velocity - скорость
velocity vector – вектор скорости
streamline – линия обтекания, линия потока, линия течения
streakline, pathline – траектория
Magnetic field – магнитное поле
Maxwell's equations – уравнения Максвелла
electromagnetic field – электромагнитное поле
gravitational field – гравитационное поле
gradient field – градиентное поле
conservative field – консервативное поле, потенциальное (безвихревое) поле
gradient descent – градиентный (наибыстрейший) спуск, алгоритм градиентного спуска инкрементный алгоритм оптимизации, или поиска оптимального решения, где приближение к

локальному минимуму функции идёт шагами, пропорциональными обратной величине градиента этой функции в текущей точке.

path integral – градиентный (наибыстрейший) спуск, алгоритм градиентного спуска или поиска оптимального решения, где приближение к локальному минимуму функции идёт шагами, пропорциональными обратной величине градиента этой функции в текущей точке.

closed curve – замкнутая кривая

orthogonal group – ортогональные группа

invariant – инвариантный, постоянный, неизменяющийся; инвариант

orthogonal transformation – ортогональное преобразование

parametrize – параметризовать, оцифровывать (напр. геометрическую фигуру в САПР)

exterior derivative – внешняя производная

angular momentum – момент количества движения; момент импульса; угловой момент; вращательный момент

winding number – порядок кривой

hairy ball theorem – теорема о причёсывании ежа утверждает, что не существует непрерывного касательного векторного поля на сфере, которое нигде не обращается в ноль. Иначе говоря, если f – непрерывная функция, задающая касательный к сфере вектор в каждой её точке, то существует хотя бы одна точка p такая, что f(p)=0.

Poincaré–Hopf theorem – Теорема Пуанкаре о векторном поле (также теорема Пуанкаре-Хопфа, теорема об индексе) – одна из теорем, относящаяся к области дифференциальной топологии и теории динамических систем.

Euler characteristic – Эйлерова характеристика или характеристика Эйлера- Пуанкаре – характеристика топологического пространства. Эйлерова характеристика пространства X обычно обозначается $\chi(X)$.

magnetic dipole – магнитный диполь

magnetism – магнетизм физическое явление, при котором материалы (например, железо, некоторые виды стали и природный минерал магнетит - магнитный железняк) оказывают притягивающую или отталкивающую силу на другие материалы на расстоянии

light field – световое поле

Picard–Lindelöf theorem – Теорема Пикара-Линделефа – математическая теорема о существовании и единстве решения задачи Коши для обычного диференциального уравнения первого порядка.

Lipschitz continuous – Липшицево отображение (названо в честь Рудольфа Липшица) - отображение $f\colon X \to Y$ между двумя метрическими пространствами, применение которого увеличивает расстояния не более, чем в некоторую константу раз. А именно, отображение f метрического пространства $(X,\ \rho_X)$ в метрическое пространство $(Y,\ \rho_Y)$ называется липшицевым, если найдётся некоторая константа L (константа Липшица этого отображения), такая, что $\rho_Y(f(x),\ f(y)) \leqslant L \cdot \rho_X(x,\ y)$ при любых $x,\ y \in X$. Это условие называют условием Липшица.

equivalence classes – классы эквивалентности

real number line – вещественная числовая ось

geodesic flow – геодезический поток

one-parameter subgroup – однопараметрическая подгруппа

Lie group – группа Ли над полем K ($K = \mathbb{R}$ или $\mathbb{C}$) называется группа G, снабжённая структурой дифференцируемого(гладкого) многообразия над K, причём отображения mul и inv, определённые так:

$\mathrm{mul}\colon G \times G \to G;\ \ \mathrm{mul}(x, y) = xy$, $\mathrm{inv}\colon G \to G;\ \ \mathrm{inv}\ x = x^{-1}$

являются гладкими (в случае поля $\mathbb{C}$ требуют голоморфности введённых отображений).

Grammar, Lexical, Translation and Speaking Exercises

Task 1. Name the Complex Subject and the predicate in every sentence. Scientists are sure to find a reliable method of detecting errors. The hypothesis proved to be based on the wrong assumption. All the circumstances do not seem to have been properly observed. Certain mistakes appear to have occurred. A proper interpretation of this fact is likely to be obtained. The equipment we were interested in happened to be produced on the line at this factory. Only a century ago the atom was believed to be indivisible. The operator is sure to find errors in the program presented. This question is sure to arise. The computation is expected to have been carried out. Such a mistake is unlikely to have remained unnoticed. This major occasion is known to have caused a lot of argument. This phenomenon does not seem to obey the general law. This solution is believed to be obviously absurd. The preparatory work proved to be very slow and difficult.

Task 2. Change the sentences according to the model.

Model: It ***is believed*** that ***he*** is a reliable business partner.
He is believed to be a reliable business partner.

It is expected that they will detect the error. It is believed that he is very accurate in making calculations. It is known that they have foreseen all the possible mistakes. It is likely that he has given them explicit instructions. It is unlikely that they have supplied this lab with such complex equipment. It appears that they are unable to account for this absurd situation. It seems that he is an intelligent researcher. It happened so that the error was quickly detected.

Task 3. Translate from Russian into English.

Предполагают, что студент знает этот математический закон. Этот подход, полагают, даст определенные преимущества. Кажется, он изменил свою точку зрения. Кажется, этот факт уже объяснили соответствующим образом. Вычисление оказалось очень сложным. Так случилось, что мой преподаватель прочел эту статью. Имеются сведения, что они согласны с нашей теорией. Этот принцип оказался противоположен принципу, приведенному выше в этом исследовании. Можно предположить, что каждая дробь представляет собой частное ее числителя и знаменателя. Эти законы применимы ко всем видам экспонента: положительного и отрицательного, дробного и целого. Ожидают, что они обнаружат и устранят эту ошибку в ближайшем будущем. Полагают, что он очень точен в расчетах. Известно, что этот ученный предвидел возможные ошибки. Говорят, что эту теорему доказали сто лет тому назад. Кажется, этот закон справедлив для всех производных.

Task 4. Translate from Russian into English.

Оказывается, они не могут объяснить эту абсурдную ситуацию. Известно, что здание университета было построено 150 лет назад. Кажется, что он опытный работник. Так случилось, что компьютер был заражен вирусом, и вся информация была уничтожена. Он, по-видимому, удовлетворен результатом своей работы. Она, кажется, знает этот предмет очень хорошо. Он оказался хорошим математиком. Эта задача оказалась очень сложной. Известно, что он придерживается другого мнения по этому вопросу. Ожидают, что договор будет подписан украинскими и российскими представителями послезавтра. Считают, что он один из лучших математиков нашего университета. Вряд ли он примет участие в этой научной работе. Он, видимо, устал. Они, без сомнения, забыли о своем обещании. Я случайно зашел в ваш офис, когда твой начальник просматривал все резюме. Это соглашение, вероятно, будет заключено в ближайшем будущем. Погода, вероятно, изменится. Он, наверняка, придет вовремя. Они, безусловно, согласятся принять участие в международной конференции. Документы, наверное, будут отправлены без опоздания. Лекция, без сомнения, будет интересной. Оказалось, что твоя мама права. Заместитель директора, вероятно, вернется на следующей неделе. Представители обеих сторон, оказывается, готовы вести переговоры по этому вопросу. Их план, похоже, не будет обсуждаться на совете директоров во вторник. Этот вопрос, наверное, не будут обсуждать в прессе. Вряд ли они прибудут к концу недели. Маловероятно, что городские власти готовы потратить большие средства на ремонт дорог. Очень вероятно, что они используют традиционные методы для решения этой задачи. Эти факты, едва ли, объяснят нам реальное положение дел в правительстве. Ожидали, что члены комиссии придут к соглашению.

Task 5. Translate from English into Russian using the Model.

Model: $\log_2 x$ – log to the base 2 of x (логарифм числа X по основанию 2)

Logarithm product rule $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$ The logarithm of the multiplication of x and y is the sum of logarithm of x and logarithm of y.

Logarithm quotient rule $\log_b(x / y) = \log_b(x) - \log_b(y)$ The logarithm of the division of x and y is the difference of logarithm of x and logarithm of y.

Logarithm power rule $\log_b(x^y) = y \cdot \log_b(x)$ The logarithm of x raised to the power of y is y times the logarithm of x.

Derivative of natural logarithm The derivative of the natural logarithm function is the reciprocal function. When $f(x) = \ln(x)$. The derivative of f(x) is: $f'(x) = 1 / x$

Integral of natural logarithm

The integral of the natural logarithm function is given by: When $f(x) = \ln(x)$

The integral of f(x) is: $\int f(x)dx = \int \ln(x)dx = x \cdot (\ln(x) - 1) + C$

Ln of 0 The natural logarithm of zero is undefined: ln(0) is undefined

The limit near 0 of the natural logarithm of x, when x approaches zero, is minus infinity: $\lim_{x \to 0^+} \ln(x) = -\infty$

Ln of 1 The natural logarithm of one is zero: ln(1) = 0

Ln of infinity

The limit of natural logarithm of infinity, when x approaches infinity is equal to infinity: $\lim \ln(x) = \infty$, when $x \to \infty$

Task 6. Learn and pronounce the following rules.

Rule name	Rule
Logarithm product rule	$\log_b(x \cdot y) = \log_b(x) + \log_b(y)$
Logarithm quotient rule	$\log_b(x / y) = \log_b(x) - \log_b(y)$
Logarithm power rule	$\log_b(x^y) = y \cdot \log_b(x)$
Logarithm base switch rule	$\log_b(c) = 1 / \log_c(b)$
Logarithm base change rule	$\log_b(x) = \log_c(x) / \log_c(b)$
Derivative of logarithm	$f(x) = \log_b(x) \Rightarrow f'(x) = 1 / (x \ln(b))$
Integral of logarithm	$\int \log_b(x)\, dx = x \cdot (\log_b(x) - 1 / \ln(b)) + C$
Logarithm of negative number	$\log_b(x)$ is undefined when $x \leq 0$
Logarithm of 0	$\log_b(0)$ is undefined
	$\lim_{x \to 0^+} \log_b(x) = -\infty$
Logarithm of 1	$\log_b(1) = 0$
Logarithm of the base	$\log_b(b) = 1$

Logarithm of infinity	lim $\log_b(\infty) = \infty$, when $x \to \infty$

Logarithm product rule

The logarithm of the multiplication of x and y is the sum of logarithm of x and logarithm of y. $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$

Logarithm quotient rule

The logarithm of the division of x and y is the difference of logarithm of x and logarithm of y.

$\log_b(x / y) = \log_b(x) - \log_b(y)$

Logarithm power rule

The logarithm of x raised to the power of y is y times the logarithm of x.

$\log_b(x^y) = y \cdot \log_b(x)$

For example: $\log_{10}(2^8) = 8 \cdot \log_{10}(2)$

Logarithm base switch rule

The base b logarithm of c is 1 divided by the base c logarithm of b.

$\log_b(c) = 1 / \log_c(b)$

For example: $\log_2(8) = 1 / \log_8(2)$

Logarithm base change rule

The base b logarithm of x is base c logarithm of x divided by the base c logarithm of b.

$\log_b(x) = \log_c(x) / \log_c(b)$

For example, in order to calculate $\log_2(8)$ in calculator, we need to change the base to 10: $\log_2(8) = \log_{10}(8) / \log_{10}(2)$

Logarithm of negative number

The base b real logarithm of x when x<=0 is undefined when x is negative or equal to zero: $\log_b(x)$ is undefined when $x \le 0$

Logarithm of 0

The base b logarithm of zero is undefined: $\log_b(0)$ is undefined

The limit of the base b logarithm of x, when x approaches zero, is minus infinity:

$$\lim_{x \to 0^+} \log_b(x) = -\infty$$

Logarithm of 1

The base b logarithm of one is zero: $\log_b(1) = 0$

For example, teh base two logarithm of one is zero: $\log_2(1) = 0$

Logarithm of the base

The base b logarithm of b is one: $\log_b(b) = 1$

For example, the base two logarithm of two is one: $\log_2(2) = 1$

Logarithm of infinity

The limit of the base b logarithm of x, when x approaches infinity, is equal to infinity:

lim $\log_b(x) = \infty$, when $x \to \infty$

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