ANALIZA 3

I SMER

VEŽBE

POVRŠINSKI INTEGRALI

DEFINICIJA:

$$r : D \longrightarrow \mathbb{R}^3 \quad (\mathbb{R}^n)$$
$$D \subseteq \mathbb{R}^2$$

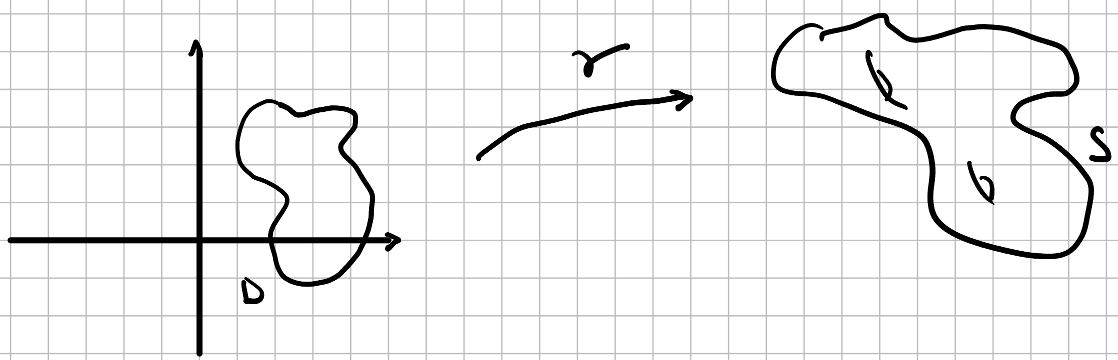
D - OTVOREN

r „1-1"

$$r, \frac{\partial r}{\partial u}, \frac{\partial r}{\partial v} \in C(D)$$

$S = r[D]$ JE PARAMETRIZOVANA POVRŠ

PARAMETRIZACIJA JE REGULARNA AKO SU $\frac{\partial r}{\partial u}$ i $\frac{\partial r}{\partial v}$ LINEARNO NEZAVISNI VEKTORI.
(TO MOŽEMO PROVERITI KAO $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \neq 0$)



PRIMERI:

1) $z = x^2 + y^2 \quad z \leq 4$ PARABOLOID

$x = u \quad y = v \quad z = u^2 + v^2$

$$\frac{\partial r}{\partial u} = (1, 0, 2u) \qquad \frac{\partial r}{\partial v} = (0, 1, 2v)$$

$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = (-2u, -2v, 1) \neq 0$$

PARAMETRIZACIJA JE REGULARNA

MOŽEMO IMATI I DRUGU PARAMETRIZACIJU

$x = \rho \cos\varphi \qquad y = \rho \sin\varphi \qquad z = \rho^2$

$r_\rho = (\cos\varphi, \sin\varphi, 2\rho) \qquad r_\varphi = (-\rho\sin\varphi, \rho\cos\varphi, 0)$

$r_\rho \times r_\varphi = (-2\rho^2\cos\varphi, -2\rho^2\sin\varphi, \rho)$

OVA PARAMETRIZACIJA JE REGULARNA ZA $\rho \neq 0$

2) $z = \frac{1}{2}\sqrt{x^2+y^2}$ - KONUS

$x = u \qquad y = v \qquad z = \frac{1}{2}\sqrt{u^2+v^2}$

$r_u = \left(1, 0, \frac{1}{2}\frac{u}{\sqrt{u^2+v^2}}\right) \qquad r_v = \left(0, 1, \frac{1}{2}\frac{v}{\sqrt{u^2+v^2}}\right)$

$r_u \times r_v = \left(-\frac{1}{2}\frac{u}{\sqrt{u^2+v^2}}, -\frac{1}{2}\frac{v}{\sqrt{u^2+v^2}}, 1\right) \neq 0$

PARAMETRIZACIJA JE REGULARNA.

POVRŠINSKI INTEGRAL PRVE VRSTE

VEKTOR NORMALE NA POVRŠ

$$d\vec{s} = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\, du\, dv$$

$$ds = \|d\vec{s}\|\, du\, dv = \left\|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right\| du\, dv$$

RAČUNAMO INTEGRAL

$\iint_S f\, ds \qquad f : S \to \mathbb{R}$

OSOBINE:

1) LINEARNOST

2) ADITIVNOST PO SKUPU

3) $\forall \xi \in S \quad f(\xi) \le g(\xi) \Rightarrow \iint_S f\, ds \le \iint_S g\, ds$

4) TEOREMA O SREDNJOJ VREDNOSTI

$f \in C(S) \Rightarrow \exists M \in S \quad \iint_S f\, ds = f(M)\, \underbrace{P(S)}_{\text{površina}}$

5) POVRŠINA S $\quad P(S) = \iint_S 1 \cdot ds$

① $I = \iint\limits_S (x+y+z)\, dS$

$S = \{(x,y,z) \in \mathbb{R}^3 \mid z \geq 0,\ x^2+y^2+z^2 = a^2\}$, $a>0$

$x = u$ $\qquad$ $u, v \in D = \{x^2+y^2 \leq a^2\}$

$y = v$

$z = \sqrt{a^2-u^2-v^2}$

$\frac{\partial r}{\partial u} = \left(1, 0, \frac{-u}{\sqrt{a^2-u^2-v^2}}\right) \qquad r_v = \left(0, 1, \frac{-v}{\sqrt{a^2-u^2-v^2}}\right)$

$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \left(\frac{u}{\sqrt{a^2-u^2-v^2}}, \frac{v}{\sqrt{a^2-u^2-v^2}}, 1\right)$

$dS = \sqrt{\frac{u^2}{a^2-u^2-v^2} + \frac{v^2}{a^2-u^2-v^2} + 1}\ du\, dv = \frac{a}{\sqrt{a^2-u^2-v^2}}\, du\, dv$

$I = \iint\limits_D \left(u+v+\sqrt{a^2-u^2-v^2}\right) \frac{a}{\sqrt{a^2-u^2-v^2}}\, du\, dv =$

$= a\iint\limits_D \frac{u}{\sqrt{a^2-u^2-v^2}}\, du\, dv + a\iint\limits_D \frac{v}{\sqrt{a^2-u^2-v^2}}\, du\, dv + a\iint\limits_D du\, dv$

POLARNE KOORDINATE

$= a \iint\limits_{\substack{0\leq \rho \leq a \\ 0 \leq \varphi \leq 2\pi}} \frac{\rho\cos\varphi}{\sqrt{a^2-\rho^2}}\, \rho\, d\rho\, d\varphi + a \iint\limits_{\substack{0\leq \rho \leq a \\ 0 \leq \varphi \leq 2\pi}} \frac{\rho\sin\varphi}{\sqrt{a^2-\rho^2}}\, \rho\, d\rho\, d\varphi + a P(D)$

$= a \int_0^{2\pi} \cancel{\cos\varphi\, d\varphi}^{\,0} \int_0^a \frac{\rho^2 d\rho}{\sqrt{a^2-\rho^2}} + a\int_0^a \frac{\rho^2}{\sqrt{a^2-\rho^2}}\, d\rho \int_0^{2\pi} \cancel{\sin\varphi\, d\varphi}^{\,0} + \pi a^3$

$$\boxed{I = \pi a^3}$$

AKO JE z FUNKCIJA x i y $\quad z = z(x,y)$

MOŽEMO UZETI x i y ZA PARAMETRE.

$\frac{\partial r}{\partial x} = \left(1, 0, \frac{\partial z}{\partial x}\right)$

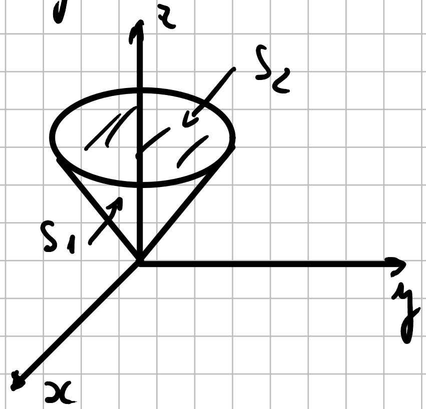
$\frac{\partial r}{\partial y} = \left(0, 1, \frac{\partial z}{\partial y}\right)$

$\frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} = \left(\frac{-\partial z}{\partial x}, \frac{-\partial z}{\partial y}, 1\right) \qquad dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dx\, dy$

② $\iint\limits_S (x^2+y^2)\,dS$

$S = \partial T$ (RUB TELA T)

$T: \quad \sqrt{x^2+y^2} \le z \le 1$



S_1 - OMOTAČ $\qquad S_2$ - BAZA

$$S = S_1 \cup S_2$$

$$\iint\limits_S (x^2+y^2)\,dS = \iint\limits_{S_1} (x^2+y^2)\,dS + \iint\limits_{S_2} (x^2+y^2)\,dS$$

$\underline{\underline{S_1}}$:

$$z = \sqrt{x^2+y^2}$$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$dS = \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,dx\,dy = \sqrt{2}\,dx\,dy$$

$$\iint\limits_{S_1} (x^2+y^2)\,dS = \iint\limits_D (x^2+y^2)\sqrt{2}\,dx\,dy =$$

$D: \ x^2+y^2 \le 1 \qquad$ POLARNE KOORDINATE

$$= \iint\limits_{\substack{0\le\rho\le 1\\ 0\le\varphi\le 2\pi}} \rho^2 \rho \sqrt{2}\,d\rho\,d\varphi = \sqrt{2}\int_0^1 \rho^3\,d\rho \int_0^{2\pi} d\varphi =$$

$$= \sqrt{2}\,\frac{1}{4}\rho^4\Big|_0^1 \varphi\Big|_0^{2\pi} = \sqrt{2}\,\frac{1}{4}\,2\pi = \frac{\pi\sqrt{2}}{2}$$

$\underline{\underline{S_2}}$

$$z = 1 \qquad \frac{\partial z}{\partial x} = 0 \qquad \frac{\partial z}{\partial y} = 0$$

$$dS = dx\,dy$$

$$\iint\limits_{S_2} (x^2+y^2)\,dS = \iint\limits_D (x^2+y^2)\,dx\,dy = \dots = \frac{\pi}{2}$$

$$\iint\limits_S (x^2+y^2)\,dS = \frac{\pi\sqrt{2}}{2} + \frac{\pi}{2} = \frac{\pi}{2}(\sqrt{2}+1)$$

③ $$I = \iint_S z \, dS$$

$$S: \quad x = u\cos v \qquad u \in [0, 2\pi]$$
$$y = u \sin v \qquad v \in [0, a]$$
$$z = v$$

$$\frac{\partial r}{\partial u} = (\cos v, \ \sin v, \ 0)$$
$$\frac{\partial r}{\partial v} = (-u\sin v, \ u\cos v, \ 1)$$
$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = (\sin v, \ -\cos v, \ u)$$

$$dS = \sqrt{\sin^2 v + \cos^2 v + u^2} \, du \, dv = \sqrt{1+u^2} \, du \, dv$$

$$I = \iint_S z \, dS = \iint_{\substack{0 \le u \le 2\pi \\ 0 \le v \le a}} v\sqrt{1+u^2} \, du \, dv =$$

$$= \int_0^a v \, dv \int_0^{2\pi} \sqrt{1+u^2} \, du = \frac{1}{2} v^2 \Big|_0^a \left(\frac{u}{2}\sqrt{u^2+1} + \frac{1}{2} \log|u+\sqrt{u^2+1}| \right) \Big|_0^{2\pi}$$

$$= \frac{1}{2} a^2 \left(\pi \sqrt{\pi^2+1} + \frac{1}{2} \log\left(2\pi + \sqrt{4\pi^2+1}\right) \right)$$

④ ODREDITI POVRŠINU OMOTAČA TELA OGRANIČENOG POVRŠIMA.

$S_1: \quad z = 1 + x^2 + y^2$ (PARABOLOID) $\qquad S_2: \quad z = \sqrt{4x^2 + 4y^2}$ (KONUS)

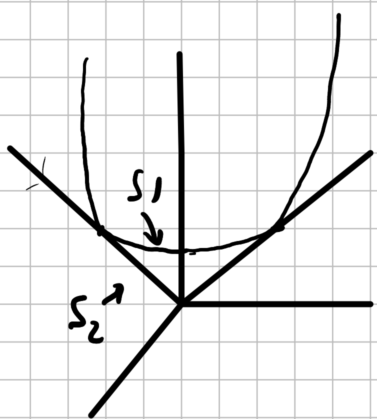
TRAŽIMO PRESEK OVIH POVRŠI

$S_2 \rightarrow \quad z = 2\sqrt{x^2+y^2} \qquad x^2 + y^2 = \frac{z^2}{4}$

$$z = 1 + \frac{z^2}{4} \quad / 4$$

$$z^2 - 4z + 4 = 0$$

$$z = 2$$



$$P(S) = P(S_1) + P(S_2) = \iint\limits_{S_1} dS + \iint\limits_{S_2} dS$$

$\underline{\underline{S_1}}$:

$$z = 1 + x^2 + y^2$$

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y$$

$$dS = \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy$$

$$P(S_1) = \iint\limits_{S_1} dS = \iint\limits_{D} \sqrt{1 + 4(x^2+y^2)}\, dx\, dy$$

D - PROJEKCIJA $\quad x^2 + y^2 \leq 1 \quad$ (PRESEK U $z=2$)

$$= \iint\limits_{\substack{0 \leq \rho \leq 1 \\ 0 \leq \varphi \leq 2\pi}} \sqrt{1+4\rho^2}\, \rho\, d\rho\, d\varphi = \int_0^{2\pi} d\varphi \int_0^1 \sqrt{1+4\rho^2}\rho\, d\rho =$$

SMENA: $\quad 1 + 4\rho^2 = t^2$

$\qquad 8\rho\, d\rho = 2t\, dt$

ρ	0	1
t	1	$\sqrt{5}$

$$= \varphi \Big|_0^{2\pi} \int_1^{\sqrt{5}} t\, \frac{1}{4} t\, dt = 2\pi\, \frac{1}{4}\, \frac{1}{3}\, t^3 \Big|_1^{\sqrt{5}} =$$

$$= \frac{\pi}{6}\left(5\sqrt{5} - 1\right)$$

$\underline{\underline{S_2}}$:

$$z = \sqrt{4x^2 + 4y^2}$$

$$\frac{\partial z}{\partial x} = \frac{4x}{\sqrt{4x^2+4y^2}} \qquad \frac{\partial z}{\partial y} = \frac{4y}{\sqrt{4x^2+4y^2}}$$

$$dS = \sqrt{1 + \frac{16x^2}{4x^2+4y^2} + \frac{16y^2}{4x^2+4y^2}}\, dx\, dy$$

$$dS = \sqrt{5}\, dx\, dy$$

$$P(S_2) = \iint\limits_{S_2} dx\, dy = \iint\limits_{D} \sqrt{5}\, dx\, dy = \sqrt{5}\, P(D) = \sqrt{5}\, \pi$$

$$P(S) = P(S_1) + P(S_2)$$

$$P(S) = \frac{\pi}{6}\left(5\sqrt{5} - 1\right) + \sqrt{5}\,\pi = \frac{\pi}{6}\left(11\sqrt{5} - 1\right)$$

POVRŠINSKI INTEGRAL DRUGE VRSTE

$F = (P, Q, R)$ – VEKTORSKO POLJE

$$\iint_{S^+} \vec{F} \cdot d\vec{S} = \iint_D (P, Q, R) \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right) du\, dv$$

OVDE JE ZA RAZLIKU OD INTEGRALA PRVE VRSTE BITNA ORIJENTACIJA

VAŽE LINEARNOST I ADITIVNOST PO SKUPU.

$$\iint_{S^+} \vec{F} \cdot d\vec{S} = \iint_{S^+} P\, dy\, dz + Q\, dz\, dx + R\, dx\, dy$$

1

$\vec{F} = (x, y, z)$

$S: x^2 + y^2 + z^2 = a^2$ SPOLJNA STRANA

$\iint_S \vec{F} \cdot d\vec{S} = ?$

PARAMETRIZACIJA NIJE REGULARNA KADA JE $v \in \{-\frac{\pi}{2}, \frac{\pi}{2}\}$

PARAMETRIZUJEMO SFERU

ZAŠTO OVO NE UTIČE NA RAČUN INTEGRALA?

$x = a \cos u \cos v \qquad u \in [0, 2\pi]$

$y = a \sin u \cos v \qquad v \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$z = a \sin v$

$$\frac{\partial r}{\partial u} = (-a \sin u \cos v,\ a \cos u \cos v,\ 0)$$

$$\frac{\partial r}{\partial v} = (-a \cos u \sin v,\ -a \sin u \sin v,\ a \cos v)$$

$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = (a^2 \cos u \cos^2 v,\ a^2 \sin u \cos^2 v,\ a^2 \sin v \cos v)$$

$$= a \cos v\, (a \cos u \cos v,\ a \sin u \cos v,\ a \sin v)$$

$$= a \cos v\, \vec{F}$$

$$\vec{F} \cdot d\vec{S} = \vec{F} \cdot a \cos v\, \vec{F} = a \cos v\, \|\vec{F}\|^2 = a^3 \cos v$$

$\sqrt{\vec{F} \cdot \vec{F}} = \|\vec{F}\|^2 \qquad \|\vec{F}\| = a$ – SFERA

$$\iint_S \vec{F}\, d\vec{S} = \iint_{\substack{0 \le u \le 2\pi \\ -\frac{\pi}{2} \le v \le \frac{\pi}{2}}} a^3 \cos v\, du\, dv = a^3 \int_0^{2\pi} du \int_{-\pi/2}^{\pi/2} \cos v\, dv = \boxed{4\pi a^3}$$

(2) $\iint\limits_S dx\,dy$

S : SPOLJNA STRANA SFERE $x^2+y^2+z^2=1$

HOĆEMO DA PROJEKTUJEMO NA RAVAN $z=0$, PA SFERU DELIMO NA GORNJU I DONJU POLOVINU GDE JE z FUNKCIJA x i y.

$$I=\iint\limits_S dx\,dy = \iint\limits_{S_1} dx\,dy + \iint\limits_{S_2} dx\,dy$$

PROJEKCIJA JE $D: x^2+y^2\leq 1$

VODIMO RAČUNA O ORIJENTACIJI. POZITIVNA JE AKO VEKTOR NORMALE NA POVRŠ IMA OŠTAR UGAO SA Z-OSOM, A NEGATIVNA AKO JE UGAO TUP.

$$\iint\limits_S dx\,dy = +\iint\limits_D dx\,dy - \iint\limits_D dx\,dy = 0$$

NEGATIVNA ORIJENTACIJA NA DONJOJ POLUSFERI

SPECIJALNO AKO JE z FUNKCIJA x i y

$S=\{(x,y,z(x,y)) \mid (x,y)\in D\}$

$$\iint\limits_S \vec{F}\,d\vec{S} = \iint\limits_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy =$$

$$= \iint\limits_D \left(\frac{-\partial z}{\partial x} P - \frac{\partial z}{\partial y} Q + R \right) dx\,dy$$

(3) $\iint\limits_S (y-z)\,dy\,dz + (z-x)\,dz\,dx + (x-y)\,dx\,dy$

S : $x^2+y^2=z^2$, $0\leq z\leq 1$

z OVDE JESTE FUNKCIJA

$z=\sqrt{x^2+y^2}$ $\qquad x^2+y^2\leq 1 : D$

$$P = y - z \qquad Q = z - x \qquad R = x - y$$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(\frac{-x}{\sqrt{x^2+y^2}} \left(y - \sqrt{x^2+y^2} \right) - \frac{y}{\sqrt{x^2+y^2}} \left(\sqrt{x^2+y^2} - x \right) + (x-y) \right) dx\,dy$$

$$= \iint_D \left(\cancel{\frac{-xy}{\sqrt{x^2+y^2}}} + x - y + \cancel{\frac{xy}{\sqrt{x^2+y^2}}} + x - y \right) dx\,dy$$

$$= 2 \iint_D (x-y)\, dx\,dy = \ldots = 0$$

4. NAĆI FLUKS VEKTORSKOG POLJA $\vec{F} = (x^2, y, z^2)$ KROZ SPOLJNU STRANU SFERE:

$$S: \quad (x-a)^2 + (y-b) + (z-c)^2 = R^2$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S x^2\, dy\,dz + y^2\, dz\,dy + z^2\, dx\,dy =$$

UVODIMO SMENU

$$\left. \begin{aligned} X &= x - a \\ Y &= y - b \\ Z &= z - c \end{aligned} \right\} \quad S_1 : X^2 + Y^2 + Z^2 = R^2$$

$$= \iint_{S_1} (X+a)^2\, dY\,dZ + (Y+b)^2\, dZ\,dX + (Z+C)^2\, dX\,dY$$

$$= \underbrace{\iint_{S_1} (X+a)^2\, dY\,dZ}_{I_1} + \underbrace{\iint_{S_1} (Y+b)^2\, dZ\,dX}_{I_2} + \underbrace{\iint_{S_1} (Z+C)^2\, dX\,dY}_{I_3}$$

$$I_3 = \iint_{S_1} (Z+C)^2\, dX\,dY$$

KAKO Z NIJE FUNKCIJA NA CELOJ SFERI MORAMO JE PODELITI NA GORNJU S_1' i DONJU S_1'' POLUSFERU.

$$I_3 = \iint_{S_1'} (z+c)^2\, dx\, dy + \iint_{S_1''} (z+c)^2\, dx\, dy$$

NA S_1' JE $z = \sqrt{R^2 - x^2 - y^2}$

NA S_1'' JE $z = -\sqrt{R^2 - x^2 - y^2}$

PROJEKCIJA NA RAVAN $z = 0$ JE

$$D: \quad x^2 + y^2 \leq R^2$$

$$I_3 = \iint_D \left(c + \sqrt{R^2 - x^2 - y^2}\right)^2 dx\, dy - \iint_D \left(c - \sqrt{R^2-x^2-y^2}\right)^2 dx\, dy$$

ZBOG ORIJENTACIJE

$$= \iint_D \Big(\cancel{c^2} + 2\sqrt{R^2-x^2-y^2}\, c + \cancel{R^2} - \cancel{x^2} - \cancel{y^2} -$$

$$- \cancel{c^2} + 2\sqrt{R^2-x^2-y^2}\, c - \cancel{R^2} + \cancel{x^2} + \cancel{y^2} \Big)\, dx\, dy$$

$$= 4c \iint_D \sqrt{R^2 - x^2 - y^2}\, dx\, dy$$

POLARNE KOORDINATE

$$= 4c \iint_{\substack{0 \leq \rho \leq R \\ 0 \leq \varphi \leq 2\pi}} \sqrt{R^2 - \rho^2}\, \rho\, d\rho\, d\varphi$$

$$= 4c \int_0^{2\pi} d\varphi \int_0^R \sqrt{R^2 - \rho^2}\, \rho\, d\rho =$$

$R^2 - \rho^2 = t^2$

$-2\rho\, d\rho = 2t\, dt$

ρ	0	R
t	R	0

$$= 4c\, \varphi \Big|_0^{2\pi} \int_0^R t^2\, dt$$

$$= 4c \cdot 2\pi\, \frac{1}{3} t^3 \Big|_0^R = \frac{8}{3} \pi c R^3$$

IZ SIMETRIJE IZRAZA (ILI RAČUNOM) MOŽEMO ZAKLJUČITI DA JE:

$$I_1 = \frac{8}{3} \pi a R^3 \qquad I_2 = \frac{8}{3} \pi b R^3$$

$$I = I_1 + I_2 + I_3 = \boxed{\frac{8}{3} \pi R^3 (a+b+c)}$$

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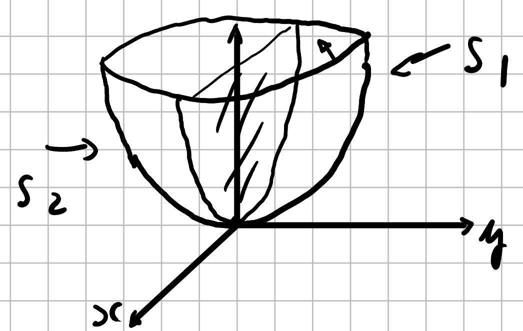
$$I = \iint_S y\, dz\, dx$$

S - GORNJA STRANA PARABOLOIDA

$$z = x^2 + y^2 \qquad 0 \le z \le 2$$



KAKO PROJEKTUJEMO NA RAVAN $y = 0$ $(dz\,dx)$ MORAMO PODELITI NA DELOVE GDE JE y FUNKCIJA x i z.

$$S_1 = S \cap \{y \ge 0\}$$
$$S_2 = S \cap \{y \le 0\}$$

NA S_1 JE $y = \sqrt{z - x^2}$ I ORIJENTACIJA JE „−" JER JE TUP UGAO NORMALE SA y-OSOM.

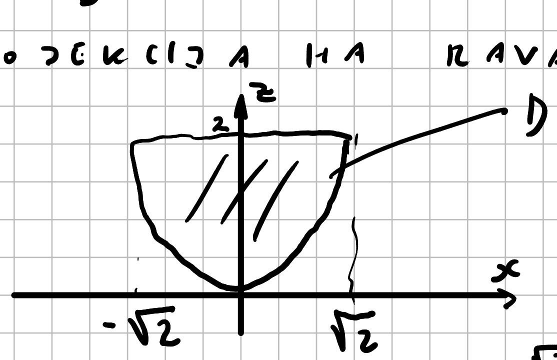
NA S_2 JE $y = -\sqrt{z - x^2}$ i ORIJENTACIJA JE „+".

$$\iint_S y\,dz\,dx = \iint_{S_1} y\,dz\,dx + \iint_{S_2} y\,dz\,dx$$

$$= -\iint_D \sqrt{z - x^2}\,dz\,dx + \iint_D -\sqrt{z - x^2}\,dz\,dx$$

$$= -2 \iint_D \sqrt{z - x^2}\,dz\,dx$$

D : PROJEKCIJA NA RAVAN $y = 0$



$$I = -2 \iint_D \sqrt{z - x^2}\,dz\,dx = -2 \int_{-\sqrt{2}}^{\sqrt{2}} \int_{x^2}^{2} \sqrt{z - x^2}\,dz\,dx =$$

$$= -2 \int_{-\sqrt{2}}^{\sqrt{2}} \frac{2}{3} (z - x^2)^{\frac{3}{2}} \Big|_{x^2}^{2} dx = \frac{-4}{3} \int_{-\sqrt{2}}^{\sqrt{2}} (2 - x^2)^{\frac{3}{2}} dx = \dots = -2\pi$$