Analiza 3

I smer

Vežbe

Krivolinijski integrali - nastavak

Teorema:

V - oblast (otvoren i povezan skup)

$F \in C(V)$ - neprekidno vektorsko polje

Ekvivalentno je:

1) Postoji funkcija $u: V \to \mathbb{R}$ sa neprekidnim parcijalnim izvodima tdj:

$$\nabla u = F$$

2) $\int_{AB} \vec{F} \cdot d\vec{r}$, $A, B \in V$ ne zavisi od puta već samo od krajnjih tačaka

$$\int_{AB} \vec{F} \cdot d\vec{r} = u(B) - u(A)$$

3) $\oint_{\gamma} \vec{F} \cdot d\vec{r} = 0$ za svaku zatvorenu krivu

$$\gamma \subseteq V$$

① $$I = \oint_{\gamma} (x^2 + 2xy - y^2)\,dx + (x^2 - 2xy + y^2)\,dy$$

$$\gamma: 5(x-3)^2 + (y-4)^2 = 17$$

Možemo parametrizovati γ i računati kao ranije.

Radimo drugačije, tražimo funkciju u tdj

$$\nabla u = F = (x^2 + 2xy - y^2,\ x^2 - 2xy + y^2)$$

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

$$\frac{\partial u}{\partial x} = x^2 + 2xy - y^2$$

$$u = \int (x^2 + 2xy - y^2)\, dx$$

KONSTANTA PO x MOŽE ZAVISITI OD y

$$u = \frac{1}{3}x^3 + x^2 y - xy^2 + C(y)$$

$$\frac{\partial u}{\partial y} = x^2 - 2xy + C'(y)$$

$$\frac{\partial u}{\partial y} = x^2 - 2xy + y^2$$

$$C'(y) = y^2 \Rightarrow C(y) = \frac{1}{3}y^3 + A$$

KONSTANTU A MOŽEMO UZETI $A=0$ SVEJEDNO JE KOJU PRIMITIVNU FUNKCIJU BIRAMO.

$$u(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 + x^2 y - xy^2$$

$$\nabla u = F$$

PREMA TEOREMI INTEGRAL PO ZATVORENOJ KRIVOJ JE 0.

$$I = 0$$

(2) $$\oint_\gamma (x^2 - 2yz)\, dx + (y^2 - 2xz)\, dy + (z^2 - 2xy)\, dz$$

$\gamma = S^2 \cap \{x+y+z=1\}$ IZMEĐU TAČAKA

$A(1,0,0)$ i $B(0,1,0)$

[S^2 JE SFERA $x^2+y^2+z^2=1$]

$$F = (x^2 - 2yz,\ y^2 - 2xz,\ z^2 - 2xy)$$

TRAŽIMO u TDJ. $\nabla u = F$

$$\nabla u = (x^2 - 2yz,\ y^2 - 2xz,\ z^2 - 2xy)$$

$$\nabla u = \left(\frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y},\ \frac{\partial u}{\partial z}\right)$$

$$\frac{\partial u}{\partial x} = x^2 - 2yz \Rightarrow u = \frac{1}{3}x^3 - 2xyz + A(y,z)$$

$$\frac{\partial u}{\partial y} = y^2 - 2xz \quad \wedge \quad \frac{\partial u}{\partial y} = -2xz + \frac{\partial A}{\partial y}$$

$$\Rightarrow \frac{\partial A}{\partial y} = y^2 \Rightarrow A = \frac{1}{3}y^3 + B(z)$$

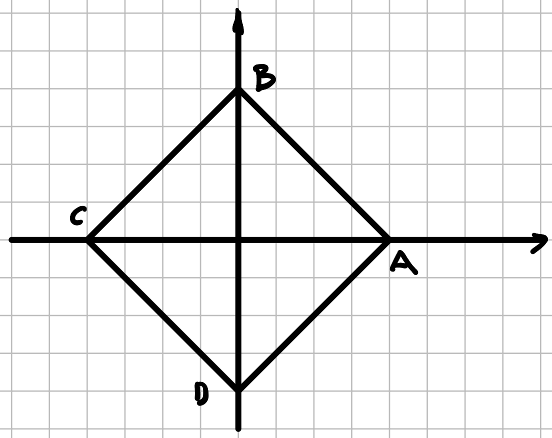
$$\frac{\partial u}{\partial z} = z^2 - 2xy \qquad \frac{\partial u}{\partial z} = -2xy + B'(z) \Rightarrow B(z) = \frac{1}{3}z^3$$

$$u(x,y,z) = \frac{1}{3}(x^3 + y^3 + z^3) - 2xyz$$

$$\oint_\gamma F\, dr = u(B) - u(A) = \frac{1}{3}(0^3 + 1^3 + 0^3) - 2\cdot 0\cdot 1\cdot 0 - \frac{1}{3}(1^3 + 0^3 + 0^3) - 2\cdot 1\cdot 0\cdot 0 = \boxed{0}$$

③ $\displaystyle\oint_{ABCDA} \frac{dx+dy}{|x|+|y|}$

ABCDA - KVADRAT SA TEMENIMA A (1,0), B (0,1), C (-1,0) i D (0,-1)



RAČUNAMO INTEGRAL PO SVAKOJ STRANICI

<u>AB</u> $\quad x, y \geq 0 \qquad x+y=1$

$y = 1-x \qquad dy = -dx \qquad 1 \rightsquigarrow x \rightsquigarrow 0$

$$\oint_{AB} \frac{dx+dy}{|x|+|y|} = \int_1^0 \frac{dx-dx}{\cancel{x}+1-\cancel{x}} = 0$$

<u>BC</u> $\quad x \leq 0, \ y \geq 0 \qquad y = x+1 \qquad 0 \rightsquigarrow x \rightsquigarrow -1$

$dy = dx$

$$\int_{BC} \frac{dx+dy}{|x|+|y|} = \int_0^{-1} \frac{2\,dx}{-x+x+1} = \int_0^{-1} 2\,dx = -2$$

<u>CD</u> $\quad x \leq 0 \quad y \leq 0 \qquad y = -x-1 \qquad -1 \rightsquigarrow x \rightsquigarrow 0$

$dy = -dx$

$$\int_{CD} \frac{dx+dy}{|x|+|y|} = \int_{-1}^{0} \frac{dx-dx}{-x+x+1} = 0$$

<u>DA</u> $\quad x \geq 0 \quad y \leq 0 \qquad y = x-1 \qquad 0 \rightsquigarrow x \rightsquigarrow 1$

$dy = dx$

$$\int_{DA} \frac{dx+dy}{|x|+|y|} = \int_0^1 \frac{2\,dx}{x-x+1} = \int_0^1 2\,dx = 2$$

④ $\displaystyle\oint_{B_\varepsilon} \frac{-y\,dx + x\,dy}{x^2+y^2} \qquad B_\varepsilon : \ x^2+y^2 = \varepsilon^2$

Parametrizujemo B_ε:

$x=\varepsilon\cos\varphi \qquad y=\varepsilon\sin\varphi \qquad 0\le\varphi\le 2\pi$

$dx=-\varepsilon\sin\varphi\,d\varphi \qquad dy=\varepsilon\cos\varphi\,d\varphi$

$$\oint_{B_\varepsilon}\frac{-y\,dx+x\,dy}{x^2+y^2}=\int_0^{2\pi}\frac{-\cancel{\varepsilon}\sin\varphi\,(-\cancel{\varepsilon}\sin\varphi\,d\varphi)+\cancel{\varepsilon}\cos\varphi\,\varepsilon\cos\varphi\,d\varphi}{\cancel{\varepsilon^2}}$$

$$=\int_0^{2\pi}(\sin^2\varphi+\cos^2\varphi)\,d\varphi=\int_0^{2\pi}d\varphi=\boxed{2\pi}$$

Vidimo da iako je B_ε zatvorena kriva integral nije nula. Sledeća teorema nam daje dovoljne uslove da integral ne zavisi od puta.

Teorema: (Grin)

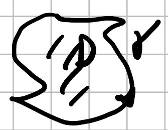
$D\subseteq\mathbb{R}^2$ - kompaktan

$\gamma=\partial D$ - deo po deo glatka kriva

$F=(P,Q)$ - vektorsko polje

$P,Q,\frac{\partial P}{\partial y},\frac{\partial Q}{\partial x}\in C(D)$

γ orijentisana tako da pri rastu parametra D ostaje sa leve strane



Tada važi:

$$\oint_\gamma P\,dx+Q\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

Posledica:

Ako je D prostopovezan

$\oint_\gamma P\,dx+Q\,dy$ ne zavisi od puta $\iff \frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$

Šta znači prostopovezan skup?

Neformalno: svaka zatvorena kriva može da se popuni neprekidnom površi.

U ravni možemo reći da nema „rupa"

⑤ $$\oint_L (x+y)\,dx - (x-y)\,dy$$

$$L: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

L - ELIPSA - OGRANIČAVA PROSTO POVEZANU OBLAST i SVE JE DEFINISANO U TOJ OBLASTI

$$F=(P,Q)=(x+y,\ -(x-y))$$

$$\frac{\partial P}{\partial y}=1 \qquad \frac{\partial Q}{\partial x}=-1$$

$$\oint_L (x+y)\,dx-(x-y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy =$$

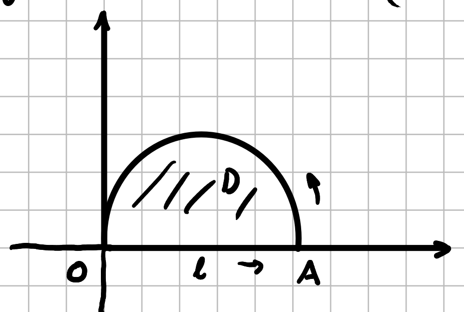
$$=\iint_D -2\,dx\,dy = -2\iint_D dx\,dy = -2\,P(D) = \boxed{-2\pi ab}$$

POVRŠINA OGRANIČENA ELIPSOM

⑥ $$\int_{\overset{\frown}{OA}} (e^x\sin y - Ay)\,dx + (e^x\cos y + Ax)\,dy \quad , A\in\mathbb{R}$$

$\overset{\frown}{OA}$ - GORNJA POLUKRUŽNICA $x^2+y^2=ax$

$$x^2+y^2=ax \iff \left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$$



$\overset{\frown}{OA}$ NE OGRANIČAVA OBLAST, MORAMO JE DOPUNITI SA DUŽI $OA=l$ DA BI PRIMENILI GRINOVU FORMULU.

DA BI D OSTALA SA LEVE STRANE U FORMULI RAČUNAMO PO LUKU $\overset{\frown}{AO}$.

$$\oint_{\overset{\frown}{AO}} P\,dx+Q\,dy + \int_l P\,dx+Q\,dy = \iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

$$\oint_{\partial A} P\,dx + Q\,dy = -\int_{\widehat{AO}} P\,dx + Q\,dy = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy + \int_l P\,dx + Q\,dy$$

$$P = e^x \sin y - Ay \qquad \frac{\partial P}{\partial y} = e^x \cos y - A$$

$$Q = e^x \cos y + A \qquad \frac{\partial Q}{\partial x} = e^x \cos y + A$$

$$\iint_D \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) dx\,dy = \iint_D \left(e^x \cos y + A - e^x \cos y + A\right) dx\,dy$$

$$= 2A \iint_D dx\,dy = 2A\; P(D) = 2A\, \frac{1}{2}\pi \left(\frac{a}{2}\right)^2$$

$$= \frac{\pi A a^2}{4}$$

$l: \quad y = 0 \quad dy = 0 \qquad 0 \leadsto x \longrightarrow a$

$$\int_l P\,dx + Q\,dy = \int_0^a (e^x \sin 0 + A \cdot 0)\,dx + (e^x \cos 0 + Ax)\cdot 0 = 0$$

$$\int_{\partial A} P\,dx + Q\,dy = \boxed{\frac{-\pi A a^2}{4}}$$

7 $\displaystyle\int_\gamma (3x^2 + y^2)\,dx + (3y^2 + 2xy)\,dy$

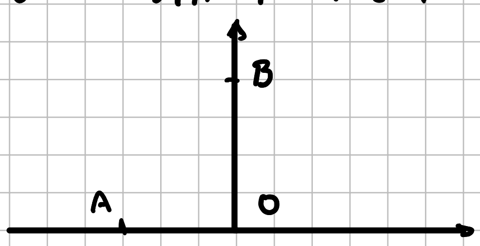
$\gamma: \quad y = (x-1)^2 (x+1)^2$

IZMEĐU TAČAKA $A(-1,0)$ i $B(0,1)$

$$F = (\underbrace{3x^2 + y^2}_{P},\ \underbrace{3y^2 + 2xy}_{Q})$$

$$\frac{\partial P}{\partial y} = 2y \qquad \frac{\partial Q}{\partial x} = 2y$$

$\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial y} \Rightarrow$ INTEGRAL NE ZAVISI OD PUTA PA MOŽEMO SPOJITI TAČKE A, B BILO KOJIM PUTEM KOJI NAMA NAJVIŠE ODGOVARA



NAJLAKŠE JE DA UBACIMO TAČKU $O(0,0)$ PA RAČUNAMO PO DUŽIMA AO I OB (NA OVIM DUŽIMA SE PO JEDNA PROMENLJIVA NE MENJA. MOGLI SMO I PARAMETRIZOVATI DUŽ AB)

$\underline{AO}$ $\quad y=0 \quad dy=0 \qquad -1 \rightsquigarrow x \rightsquigarrow 0$

$$\int_{AO} P\,dx + Q\,dy = \int_{-1}^{0} (3x^2+0^2)\,dx + (3\cdot 0^2 + 2x\cdot 0)\cdot 0$$

$$= \int_{-1}^{0} 3x^2\,dx = x^3\Big|_{-1}^{0} = 1$$

$\underline{OB}$ $\quad x=0 \quad dx=0 \qquad 0 \rightsquigarrow y \rightsquigarrow 1$

$$\int_{OB} P\,dx + Q\,dy = \int_{0}^{1} (3\cdot 0^2 + y^2)\cdot 0 + (3y^2 + 2\cdot 0\cdot y)\,dy$$

$$= \int_{0}^{1} 3y^2\,dy = y^3\Big|_{0}^{1} = 1$$

$$\int_{AB} P\,dx + Q\,dy = \int_{AO} P\,dx + Q\,dy + \int_{OB} P\,dx + Q\,dy = \boxed{2}$$

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$$\oint_{\gamma} \frac{-y\,dx + x\,dy}{x^2+y^2}$$

γ - PROIZVOLJNA ZATVORENA GLATKA KRIVA I $(0,0) \notin \gamma$

$$F = \Big(\underbrace{\frac{-y}{x^2+y^2}}_{P}, \underbrace{\frac{x}{x^2+y^2}}_{Q}\Big)$$

$$\frac{\partial P}{\partial y} = \frac{-(x^2+y^2) + y\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{x^2+y^2 - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

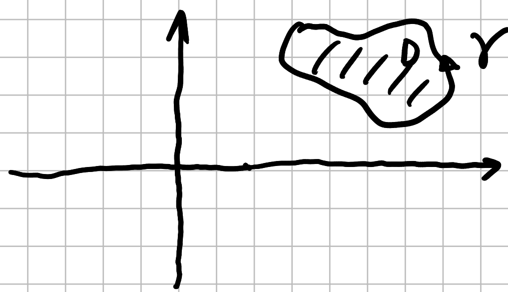
$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ PA JE INTEGRAL 0 AKO γ

OGRANIČAVA PROSTO POVEZANU OBLAST

$\gamma = \partial D$

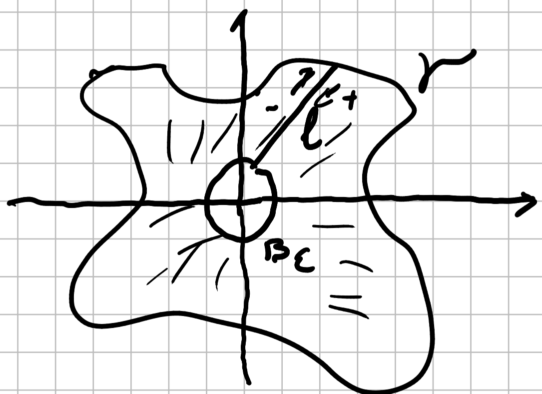
PROBLEM JE AKO $(0,0) \in D$, JER U TOJ TAČKI P i Q NISU DEFINISANI.

1) $(0,0) \notin D$



$$\oint_\gamma P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \boxed{0}$$

2) $(0,0) \in D$



ZA DOVOLJNO MALO ε MOŽEMO UBACITI $B_\varepsilon : x^2 + y^2 = \varepsilon^2$ U D, I NAPRAVITI ZASEK l IZMEĐU γ i B_ε.

SAD PRIMENJUJEMO GRINOVU FORMULU NA OBLAST IZMEĐU γ i B_ε, TAČKA $(0,0)$ NIJE VIŠE U TOJ OBLASTI.

$$\int_\gamma P\,dx + Q\,dy + \int_{l^+} P\,dx + Q\,dy + \int_{B_\varepsilon^-} P\,dx + Q\,dy + \int_{l^-} P\,dx + Q\,dy$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = 0$$

$$\int_{l^+} P\,dx + Q\,dy = -\int_{l^-} P\,dx + Q\,dy$$

SUPROTAN SMER

$$\Rightarrow \int_{\gamma} P\,dx + Q\,dy = \int_{B_\varepsilon} P\,dx + Q\,dy = 2\pi$$

ZADATAK 4

9 $\displaystyle\int_{\gamma} \frac{x\,dy - y\,dx}{x^2 - y^2}$

γ - KOMPLIKOVANA KRIVA KOJA SPAJA TAČKE $A(0,-1)$ i $B(1,0)$

$$P = \frac{x}{x^2 - y^2} \qquad Q = \frac{-y}{x^2 - y^2}$$

$$\frac{\partial P}{\partial y} = \frac{2xy}{(x^2-y^2)^2} \qquad \frac{\partial Q}{\partial x} = \frac{2xy}{(x^2-y^2)^2}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$ $\Rightarrow$ INTEGRAL NE ZAVISI OD PUTA

MOŽEMO SPOJITI TAČKE A i B PROIZVOLJNOM KRIVOM, SAMO NE SME DA SEČE PRAVE $y=x$ i $y=-x$ JER TU P i Q NISU DEFINISANI.

PARAMETRIZUJEMO DUŽ AB

$A = (1,2) \qquad B = (4,7)$

$\overrightarrow{AB} = B - A = (3,5)$

$$\frac{x-1}{3} = \frac{y-2}{5} = t$$

$$x = 3t+1 \qquad y = 5t+2 \qquad t \in [0,1]$$

PARAMETRIZACIJA DUŽI $\overrightarrow{AB}$

$$\int_{\gamma} \frac{x\,dy - y\,dx}{x^2 - y^2} = \int_0^1 \frac{(3t+1)5\,dt - (5t+2)3\,dt}{(3t+1)^2 - (5t+2)^2} =$$

$$\int_0^1 \frac{(15t+5-15t-6)\,dt}{9t^2+6t+1-25t^2-20t-4} =$$

$$= -\int_0^1 \frac{dt}{-16t^2-14t-3} = \int_0^1 \frac{dt}{16t^2+14t+3} = \dots$$

DOMAĆI :

① $\int_\gamma y\,dx + z\,dy + x\,dz$

$\gamma : S_1 \cap S_2$

$S_1 : \quad z = 1 - x^2$

$S_2 : \quad z = x^2 + y^2$

② $\int_{AB} x\,dx + y\,dy + z\,dz$

AB duž između $A(1,1,1)$ i $B(2,1,-1)$

③ $\int_\gamma z\,dx + x\,dy + y\,dz$

$\gamma : \quad z = x^2 + y^2 \quad \cap \quad 2|y| = 3 - z$

④ $\int_\gamma z\,ds$

$\gamma(t) = (t\cos t,\ t\sin t,\ t) \qquad 0 \le t \le \alpha$

⑤ $\int_\gamma \vec{F} \cdot d\vec{r}$

$F = \left(e^{x+y}\sin 2y + x + y,\ e^{x+y}(\sin 2y + 2\cos 2y) + x \right)$

$\gamma : \quad y = -\sqrt{2x - x^2}$ od $A(2,0)$ do $B(0,0)$

⑥ $\oint_L xy^2\,dy - x^2y\,dx$

$L : \quad x^2 + y^2 = r^2 \qquad r > 0$

⑦ $\vec{F} = \left(1 - \frac{1}{y} + \frac{y}{z},\ \frac{x}{y} + \frac{x}{y^2},\ \frac{-xy}{z^2}\right)$

NAĆI FUNKCIJU u TDJ. $\nabla u = F$

$$\int_{AB} \vec{F}\, d\vec{r} = ?$$

AB - PROIZVOLJNA KRIVA KOJA SPAJA TAČKE

$A(1,1,1)$ i $B(3,4,5)$.