# ANALIZA 3

## I SMER

VEŽBE

## VIŠESTRUKI INTEGRALI - NASTAVAK

① $$I = \iint_D \frac{x^2+y^2}{xy}\,dx\,dy$$

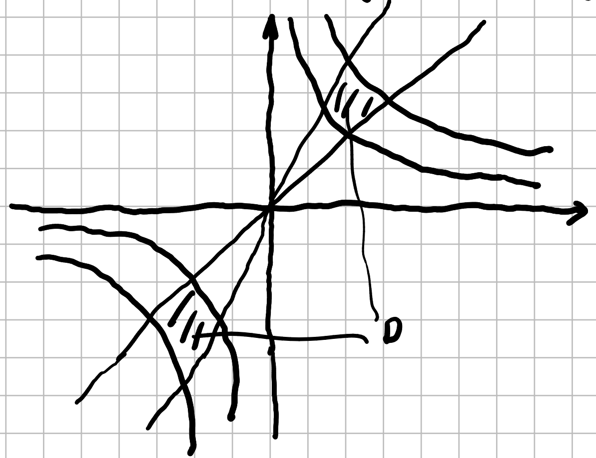
$D: \quad 1 \le xy \le 2 \qquad y^2 - 5xy + y^2 \le 0$

$\Downarrow \qquad\qquad \Updownarrow$

$(x>0 \wedge y>0)$ ili $(x<0 \wedge y<0) \qquad (y-x)(y-4x) \le 0$

$\Updownarrow$

$$\left(\frac{y}{x}-1\right)\left(\frac{y}{x}-4\right) \le 0$$




$D = D_1 \cup D_2$

$D_1 = D \cap \{x>0,\ y>0\}$

$D_2 = D \cap \{x<0,\ y<0\}$

Hoćemo smenu:

$u = xy \qquad v = \frac{y}{x}$

Ova smena nije bijekcija na $D$, $(x,y)$ i $(-x,-y)$ imaju istu sliku, ali jeste bijekcija na $D_1$ i $D_2$

$$f(x,y) = \frac{x^2+y^2}{xy} = f(-x,-y)$$

$\mu(D_1) = \mu(D_2)$ — ISTA POVRŠINA

$$I = \iint_D \frac{x^2+y^2}{xy}\,dx\,dy = \iint_{D_1} \frac{x^2+y^2}{xy}\,dx\,dy + \iint_{D_2} \frac{x^2+y^2}{xy}\,dx\,dy =$$

$$= 2\iint_{D_1} \frac{x^2+y^2}{xy}\,dx\,dy$$

NA $D_1$ UVODIMO SMENU:

$$u = xy \qquad v = \frac{y}{x}$$

$$J^{-1} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \underbrace{2\frac{y}{x}}_{=2v} \neq 0$$

$$J = \frac{1}{2v}$$

GRANICE:

$$1 \le u \le 2 \qquad\qquad (v-1)(v-4) \le 0$$

$$1 \le v \le 4$$

$$f(x,y) = \frac{x^2+y^2}{xy} = \frac{x}{y} + \frac{y}{x} = \frac{1}{v} + v$$

$$I = \cancel{2}\iint_{\substack{1\le u\le 2\\ 1\le v\le 4}} \underbrace{\left(\frac{1}{v}+v\right)\frac{1}{\cancel{2}v}}_{\text{NE ZAVISI OD } u}\,du\,dv$$

$$I = \int_1^2 du \int_1^4 \left(\frac{1}{v^2}+1\right) dv$$

$$= u\Big|_1^2 \left(-\frac{1}{v}+v\right)\Big|_1^4$$

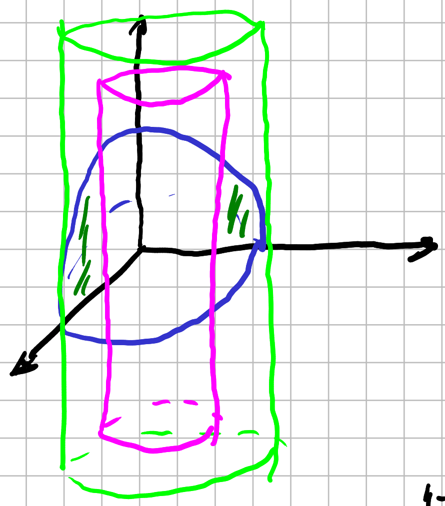
$$= (2-1)\left(-\frac{1}{4}+4+\frac{1}{1}-1\right) = \boxed{\frac{15}{4}}$$

(2) ODREDITI ZAPREMINU TELA OGRANIČENOG POVRŠIMA

$$x^2+y^2 = 1 \qquad x^2+y^2 = 4 \quad \text{— CILINDRI}$$

$$z = 0 \qquad\qquad z = 4-(x^2+y^2)$$

RAVAN $\qquad\qquad$ PARABOLOID



$$V(T) = \iiint_T dx\,dy\,dz = \iint_D \int_0^{4-x^2-y^2} dz\; dx\,dy =$$

$$= \iint_D (4 - x^2 - y^2)\, dx\,dy$$

D - PROJEKCIJA NA RAVAN $z = 0$

$D: \quad 1 \leq x^2 + y^2 \leq 4$

POLARNE KOORDINATE

$1 \leq \rho^2 \leq 4 \qquad 0 \leq \varphi \leq 2\pi$

$1 \leq \rho \leq 2$

$$V(T) = \iint_{\substack{1 \leq \rho \leq 2 \\ 0 \leq \varphi \leq 2\pi}} (4 - \rho^2)\, \rho \, d\rho \, d\varphi =$$

$$= \int_0^{2\pi} d\varphi \int_1^2 (4\rho - \rho^3)\, d\rho =$$

$$= \varphi \Big|_0^{2\pi} \left(2\rho^2 - \frac{1}{4}\rho^4\right)\Big|_1^2$$

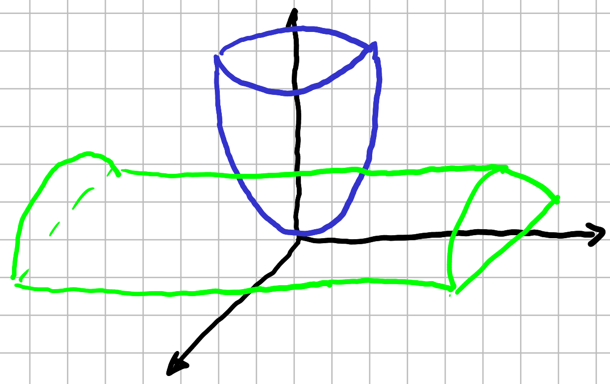
$$= 2\pi \left(2 \cdot 2^2 - \frac{1}{4} 2^4 - 2 \cdot 1^2 + \frac{1}{4} 1^4\right)$$

$$= 2\pi \left(8 - 4 - 2 + \frac{1}{4}\right) = 2\pi \frac{9}{4}$$

$$\boxed{V(T) = \frac{9\pi}{2}}$$

3 ODREDITI ZAPREMINU TELA OGRANIČENOG POVRŠIMA

$z = 1 - x^2 \qquad z = x^2 + y^2$



$z = x^2 + y^2$ - DONJA GRANICA ZA $z$

$z = 1 - x^2$ - GORNJA GRANICA ZA $z$

$$V(T) = \iiint\limits_T dx\,dy\,dz = \iint\limits_D \int_{x^2+y^2}^{1-x^2} dz\,dx\,dy$$

$$= \iint\limits_D (1 - 2x^2 - y^2)\,dx\,dy$$

D - PROJEKCIJA NA RAVAN $z=0$ ?

KAKO ODREDITI D ?

GORNJA GRANICA VEĆA OD DONJE :

$$1 - x^2 \geq x^2 + y^2$$

$$2x^2 + y^2 \leq 1$$ - OBLAST OGRANIČENA ELIPSOM

$$(\sqrt{2}x)^2 + y^2 \leq 1$$

MODIFIKOVANE POLARNE KOORDINATE

$$\sqrt{2}x = \rho\cos\varphi \qquad x = \frac{1}{\sqrt{2}}\rho\cos\varphi$$

$$y = \rho\sin\varphi$$

$$0 \leq \rho \leq 1 \qquad 0 \leq \varphi \leq 2\pi$$

$$J = \begin{vmatrix} \frac{1}{\sqrt{2}}\cos\varphi & -\frac{1}{\sqrt{2}}\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} = \frac{1}{\sqrt{2}}\rho$$

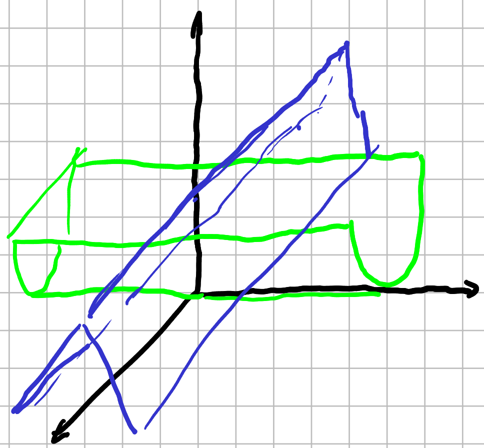
$$V(T) = \iint\limits_{\substack{0\leq\rho\leq1 \\ 0\leq\varphi\leq2\pi}} (1-\rho^2)\frac{1}{\sqrt{2}}\rho\,d\rho\,d\varphi =$$

$$= \frac{1}{\sqrt{2}}\int_0^{2\pi} d\varphi \int_0^1 (\rho - \rho^3)\,d\rho = \frac{1}{\sqrt{2}}\varphi\Big|_0^{2\pi}\left(\frac{1}{2}\rho^2 - \frac{1}{4}\rho^4\right)\Big|_0^1$$

$$V(T) = \sqrt{2}\pi \cdot \frac{1}{4} = \boxed{\frac{\pi}{2\sqrt{2}}}$$

④ ODREDITI ZAPREMINU TELA OGRANIČENOG POVRŠIMA

$z = x^2 \qquad z = 1 - |y|$



$z = x^2$ - DONJA GRANICA $\quad z = 1 - |y|$ - GORNJA GRANICA

MOŽEMO UPOTREBITI SIMETRIJU U ODNOSU NA RAVAN $y = 0$ (DA NE BISMO IMALI APSOLUTNO)

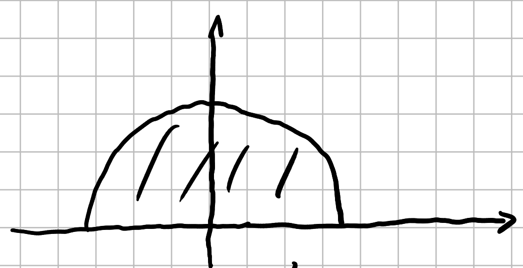
$$T_1 = T \cap \{ y \geq 0 \}$$

$$V(T) = 2\, V(T_1) = 2 \iiint_{T_1} dx\, dy\, dz =$$

$$= 2 \iint_D \int_{x^2}^{1-y} dz\, dx\, dy$$

$$= 2 \iint_D (1 - y - x^2)\, dx\, dy$$

$$D : \quad 0 \leq y \leq 1 - x^2$$



$$V(T) = 2 \int_{-1}^{1} \int_{0}^{1-x^2} (1 - y - x^2)\, dy\, dx =$$

$$= 2 \int_{-1}^{1} \left( (1-x^2)y - \frac{1}{2} y^2 \right) \Big|_0^{1-x^2} dx =$$

$$= 2\int_{-1}^{1}\left((1-x^2)^2 - \frac{1}{2}(1-x^2)^2\right) dx$$

$$= \int_{-1}^{1}(1-x^2)^2\, dx \underset{\nearrow}{=} 2\int_{0}^{1}(1-2x^2+x^4)\, dx$$

PARNOST

$$= 2\left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^1 = 2\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$V(T) = \frac{16}{15}$$

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$$\iiint_T \sqrt{x^2+y^2+z^2}\, dx\, dy\, dz$$

$$T: \quad x^2 + y^2 + z^2 = z$$

$$x^2 + y^2 + z^2 - z = 0$$

$$x^2 + y^2 + z^2 - z + \frac{1}{4} = \frac{1}{4}$$

$$x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

OVO JE SFERA SA CENTROM U $(0, 0, \frac{1}{2})$ I POLUPREČNIKA $\frac{1}{2}$.

$$\left|z - \frac{1}{2}\right| = \sqrt{\frac{1}{4} - x^2 - y^2}$$

$$\frac{1}{2} - \sqrt{\frac{1}{4}-x^2-y^2} \le z \le \frac{1}{2} + \sqrt{\frac{1}{4} - x^2 - y^2}$$

$$\vdots$$

KOMPLIKOVANO

LAKŠE SA SFERNIM KOORDINATAMA

$$x = \rho\cos\varphi\sin\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\theta$$

$$|J| = \rho^2\sin\theta$$

ODREĐUJEMO GRANICE

$$x^2 + y^2 + z^2 = \rho^2$$

$$\rho^2 = \rho\cos\theta$$

$$0 \le \rho \le \cos\theta$$

$$\Rightarrow \cos\theta \geq 0 \Rightarrow \theta \in \left[0, \frac{\pi}{2}\right]$$

$\theta$ JE UGAO SA POZITIVNIM DELOM Z-OSE

$$\iiint_T \sqrt{x^2+y^2+z^2}\, dx\, dy\, dz = \iiint_{\substack{0\le\rho\le\cos\theta \\ 0\le\theta\le\pi/2,\ 0\le\varphi\le 2\pi}} \rho\, \rho^2\sin\theta\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} d\varphi \iint_{\substack{0\le\rho\le\cos\theta \\ 0\le\theta\le\frac{\pi}{2}}} \rho^3 \sin\theta \, d\rho \, d\theta =$$

$$= \varphi\Big|_0^{2\pi} \int_0^{\pi/2} \int_0^{\cos\theta} \rho^3 \sin\theta \, d\rho \, d\theta =$$

$$= 2\pi \int_0^{\pi/2} \sin\theta \, \frac{1}{4} \rho^4 \Big|_0^{\cos\theta} d\theta =$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \sin\theta \cos^4\theta \, d\theta =$$

$$= \left[ \begin{array}{l} t = \cos\theta \\ dt = -\sin\theta \, d\theta \end{array} \quad \begin{array}{c|c|c} \theta & 0 & \pi/2 \\ \hline t & 1 & 0 \end{array} \right]$$

$$= \frac{\pi}{2} \int_0^1 t^4 \, dt = \frac{\pi}{2} \frac{1}{5} t^5 \Big|_0^1 = \boxed{\frac{\pi}{10}}$$

(6) IZRAČUNATI :

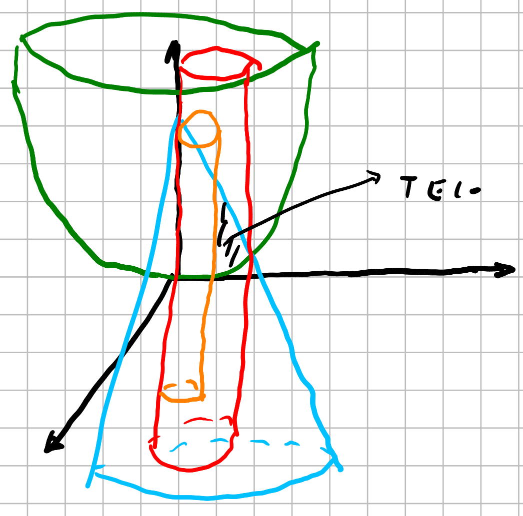
$$I_1 = \iiint_T x \, dx \, dy \, dz \qquad I_2 = \iiint_T y \, dx \, dy \, dz$$

AKO JE TELO $T$ OGRANIČENO POVRŠIMA:

$$10 - z = \sqrt{x^2+y^2} \qquad 5z = x^2+y^2 \qquad x^2+y^2 = y \quad ; \quad x^2+y^2 = 2y$$

KONUS — PARABOLOID — CILINDRI

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4} \qquad x^2 + (y-1)^2 = 1$$



TELO MORA BITI OGRANIČENO SVIM POVRŠIMA

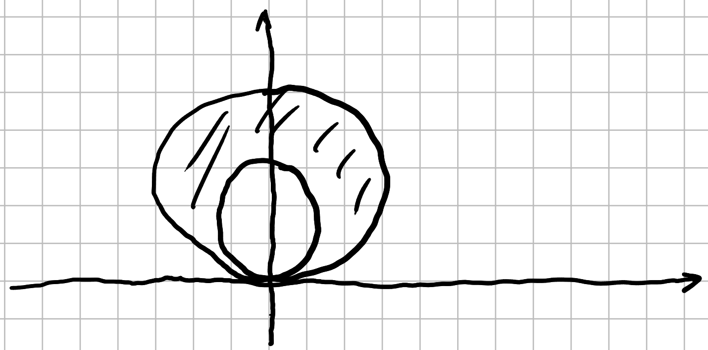
PARABOLOID JE ISPOD KONUSA

$$\frac{x^2+y^2}{5} \le z \le 10 - \sqrt{x^2+y^2}$$

$$I_1 = \iiint_T x \, dx \, dy \, dz = \iint_D x \int_{10-\sqrt{x^2+y^2}}^{\frac{x^2+y^2}{5}} dz \, dx \, dy =$$

$$= \iint_D \underbrace{x\left(\frac{x^2+y^2}{5} - 10 + \sqrt{x^2+y^2}\right)}_{\varphi(x,y)} dx\, dy$$

$$D: \quad x^2 + \left(y-\frac{1}{2}\right)^2 \geq \frac{1}{2} \qquad x^2+(y-1)^2 \leq 1$$



$\varphi(-x, y) = -\varphi(x,y)$ NEPARNA PO $x$

D JE SIMETRIČAN U ODNOSU NA $x$

$\Rightarrow \quad I_1 = 0$

(INTEGRAL NEPARNE FUNKCIJE NA SIMETRIČNOM DOMENU JE UVEK 0)

$$I_2 = \iint_D y\left(\frac{x^2+y^2}{5} - 10 + \sqrt{x^2+y^2}\right) dx\, dy$$

POLARNE KOORDINATE

$$x^2 + \left(y - \frac{1}{2}\right)^2 \geq \frac{1}{2} \qquad x^2 + (y-1)^2 \leq 1$$

$$\rho^2\cos^2\varphi + \rho^2\sin^2\varphi - \rho\sin\varphi + \cancel{\frac{1}{4}} \geq \cancel{\frac{1}{4}}$$

$$\rho^{\cancel{2}} - \cancel{\rho}\sin\varphi \geq 0$$

$$\rho \geq \sin\varphi$$

$$\rho^2\cos^2\varphi + \rho^2\sin^2\varphi - 2\rho\sin\varphi + \cancel{1} \leq \cancel{1}$$

$$\rho^2 - 2\rho\sin\varphi \leq 1$$

$$\rho \leq 2\sin\varphi$$

$\sin\varphi \geq 0$

$$\sin\varphi \leq \rho \leq 2\sin\varphi \qquad 0 \leq \varphi \leq \pi$$

$$I_2 = \iint \rho\sin\varphi\left(\frac{1}{5}\rho^2 - 10 + \rho\right)\rho\, d\rho\, d\varphi$$

$$= \int_0^{\pi} \sin\varphi \int_{\sin\varphi}^{2\sin\varphi} \left(\frac{1}{5}\rho^4 - 10\rho^2 + \rho^3\right) d\rho =$$

$$= \int_0^{\pi} \sin\varphi \left(\rho^5 - \frac{10}{3}\rho^3 + \frac{1}{4}\rho^4\right)\Big|_{\sin\varphi}^{2\sin\varphi} d\varphi =$$

$$= \int_0^{\pi} \sin\varphi \Big(32\sin^5\varphi - \frac{10}{3}8\sin^3\varphi + 4\sin^4\varphi -$$

$$- \sin^5\varphi + \frac{10}{3}\sin^3\varphi - \frac{1}{4}\sin^4\varphi\Big) d\varphi$$

$$= 31\int_0^{\pi}\sin^6\varphi \, d\varphi + \frac{15}{4}\int_0^{\pi}\sin^5\varphi \, d\varphi - \frac{70}{3}\int_0^{\pi}\sin^4\varphi \, d\varphi = \dots$$

7. NAĆI POVRŠINU DELA RAVNI OGRANIČENOG KRIVOM

$$(x^2+y^2)^2 = a(x^3 - 3xy^2), \quad a>0$$

POLARNE KOORDINATE

$$\rho^{\cancel{4}} = a(\cancel{\rho^3}\cos^3\varphi - 3\cancel{\rho^3}\cos\varphi\sin^2\varphi)$$

$$\rho = a\,(\underbrace{\cos^3\varphi - 3\cos\varphi\sin^2\varphi}_{=\cos 3\varphi})$$

$$\rho = a\cos 3\varphi$$

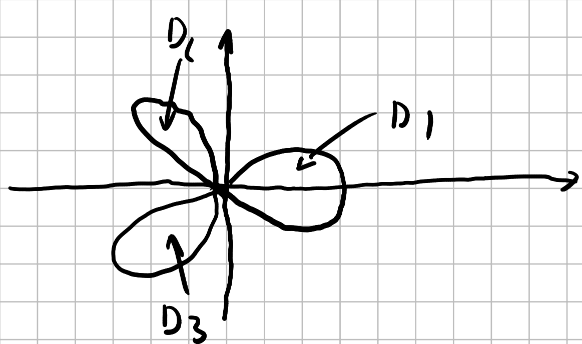
AKO BI UZELI $\rho \geq a\cos 3\varphi$ DOBILI BISMO NEOGRANIČENU OBLAST

$$0 \leq \rho \leq a\cos 3\varphi$$

USLOV ZA $\varphi$ : $\cos 3\varphi \geq 0$

$$-\frac{\pi}{2} + k\pi \leq 3\varphi \leq \frac{\pi}{2} + k\pi$$

$$-\frac{\pi}{6} + \frac{2k\pi}{3} \leq \varphi \leq \frac{\pi}{6} + \frac{2k\pi}{3} \qquad k \in \{0,1,2\}$$




$$D = D_1 \cup D_2 \cup D_3$$

D SE SASTOJI OD 3 JEDNAKA DELA

DOVOLJNO JE IZRAČUNATI POVRŠINU JEDNOG

$$P(D) = 3\,P(D_1) = 3 \iint_{D_1} dx\,dy =$$

$$= 3 \int_{-\pi/6}^{\pi/6} \int_0^{a\cos 3\varphi} \rho \, d\rho \, d\varphi =$$

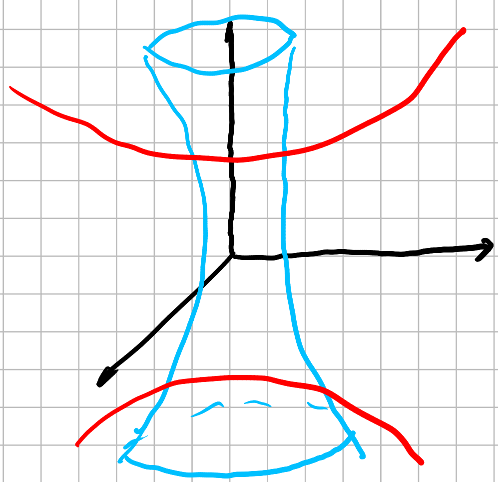
$$= 3 \int_{-\pi/6}^{\pi/6} \frac{1}{2} \rho^2 \Big|_0^{a\cos 3\varphi} d\rho \, d\varphi =$$

$$= \frac{3}{2} a^2 \int_{-\pi/6}^{\pi/6} \cos^2 3\varphi \, d\varphi = \ldots$$

8 ODREDITI ZAPREMINU TELA OGRANIČENOG POVRŠINAMA:

$$x^2 + y^2 - z^2 = 3 \qquad 5z^2 = x^2 + y^2 + 1$$

JEDNOGRANI HIPERBOLOID DVOGRANI HIPERBOLOID

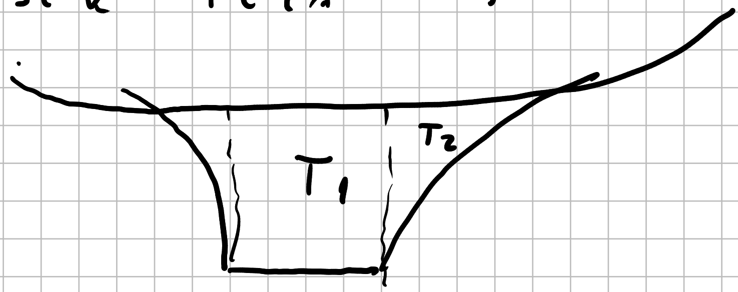


MOŽEMO UOČITI DA JE TELO $T$ SIMETRIČNO U ODNOSU NA RAVAN $z = 0$.

$$T' = T \cap \{ z \geq 0 \}$$

$$V(T) = 2\,V(T')$$

PRESEK TELA $T'$




NIJE ISTA GRANICA ZA $z$ PA DELIMO

$T'$ NA $T_1$ i $T_2$

NA $T_1$ : $0 \leq z^2 \leq \frac{x^2+y^2+1}{5}$

NA $T_2$ : $x^2+y^2-3 \leq z^2 \leq \frac{x^2+y^2+1}{5}$

PRESEK JEDNOGRANOG SA RAVNI $z=0$

$$x^2+y^2=3 \quad \Rightarrow \quad D_1: \quad 0 \leq \rho \leq \sqrt{3}$$

PRESEK DVA HIPERBOLOIDA

$$\underbrace{x^2+y^2}_{5z^2-1}-z^2=3 \qquad 5z^2=x^2+y^2+1$$

$$5z^2-1-z^2=3$$

$$z^2=1 \quad \Rightarrow \quad z=1 \qquad (\text{UZELI SMO } z\geq 0)$$

$$x^2+y^2=4$$

$$D_2: \quad \sqrt{3} \leq \rho \leq 2$$

$$V(T) = 2\,V(T') = 2\,(V(T_1)+V(T_2))$$

$$= 2\left(\iiint_{T_1} dx\,dy\,dz + \iiint_{T_2} dx\,dy\,dz\right) =$$

$$= 2\left(\iint_{D_1}\int_0^{\sqrt{\frac{x^2+y^2+1}{5}}} dz\,dx\,dy + \iint_{D_2}\int_{\sqrt{x^2+y^2-3}}^{\sqrt{\frac{x^2+y^2+1}{5}}} dz\,dx\,dy\right)$$

$$= 2\left(\iint_{D_1}\sqrt{\frac{x^2+y^2+1}{5}}\,dx\,dy + \iint_{D_2}\left(\sqrt{\frac{x^2+y^2+1}{5}}-\sqrt{x^2+y^2-3}\right)dx\,dy\right)$$

$$= 2\left( \iint\limits_{\substack{0\le\rho\le\sqrt{3}\\ 0\le\varphi\le 2\pi}} \sqrt{\frac{\rho^2+1}{5}}\,\rho\, d\rho\, d\varphi + \iint\limits_{\substack{\sqrt{3}\le\rho\le 2\\ 0\le\varphi\le 2\pi}} \left(\sqrt{\frac{\rho^2+1}{5}} - \sqrt{\rho^2-3}\right)\rho\, d\rho\, d\varphi \right)$$

$$= 2\int_0^{2\pi} d\varphi \left( \frac{1}{\sqrt{5}} \int_0^2 \sqrt{\rho^2+1}\,\rho\, d\rho - \int_{\sqrt{3}}^{2} \sqrt{\rho^2-3}\,\rho\, d\rho \right)$$

$$t^2 = \rho^2+1 \qquad \cancel{2}t\,dt = \cancel{2}\rho\, d\rho \qquad \begin{array}{c|c|c} \rho & 0 & 2 \\ \hline t & 1 & 5 \end{array}$$

$$s^2 = \rho^2-3 \qquad 2s\,ds = 2\rho\, d\rho \qquad \begin{array}{c|c|c} \rho & \sqrt{3} & 2 \\ \hline s & 0 & 1 \end{array}$$

$$= 2\cdot 2\pi \left( \frac{1}{\sqrt{5}} \int_1^5 t^2\, dt - \int_0^1 s^2\, ds \right) =$$

$$= 4\pi \left( \frac{1}{\sqrt{5}}\, \frac{1}{3} t^3 \Big|_0^5 - \frac{1}{3} s^3 \Big|_0^1 \right) =$$

$$= \boxed{\frac{4\pi}{3}\left( 25\sqrt{5} - \frac{1}{3} \right)}$$

## DOMAĆI:

① $\iint\limits_D (x^2+y^2)\, dx\, dy$

$D: \quad 1 \le x^2+y^2 \le 4$

② ODREDITI ZAPREMINU TELA OGRANIČENOG POVRŠIMA

$z = 0 \qquad z = x^2+y^2 \qquad y = x^2 \qquad y = 1$

③ ODREDITI ZAPREMINU TELA OGRANIČENOG SA

$$\sqrt{\frac{x}{2}} + \sqrt{\frac{y}{3}} + \sqrt{\frac{z}{15}} = 1$$

④ ODREDITI ZAPREMINU TELA OGRANIČENOG SA

$az = x^2+y^2, \quad a>0 \qquad z = \sqrt{x^2+y^2}$

(5) $\iint\limits_D xy \, dx \, dy$

D OGRANIČEN SA $xy = 1$ $x + y = \frac{5}{2}$

(6) $\iint\limits_D e^{x^2+y^2} \, dx \, dy$

D : $1 \leq x^2 + y^2 \leq 9$ , $x \geq 0$ , $y \geq \frac{x}{\sqrt{3}}$

(7) $\iint\limits_D (x+y) \, dx \, dy$

D : $x^2 + y^2 \leq x + y$

(8) ZAMENITI POREDAK INTEGRACIJE

$$\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} \, dy \, dx$$