ANALIZA 3

I SMER

VEŽBE

DIFERENCIJABILNOST VEKTORSKE FUNKCIJE

$$F : \underset{\mathbb{R}^n}{\underset{\cap}{A}} \longrightarrow \mathbb{R}^m$$

A - OTVOREN

$$F(x_1, x_2, \dots, x_n) = (f_1(x_1, \dots, x_n), f_2(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$$

$$X_0 \in A$$

F JE DIFERENCIJABILNA U x_0 AKO POSTOJI LINEARNO PRESLIKAVANJE L TDJ.

$$F(X_0 + h) - F(X_0) = Lh + o(h)$$

(X_0 I L SU VEKTORI DUŽINE n)

SVAKOM LINEARNOM PRESLIKAVANJU L MOŽEMO PRIDRUŽITI MATRICU A_L

$$A_L h = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} = \begin{bmatrix} a_{11} h_1 + \dots + a_{1n} h_n \\ \vdots \\ a_{m1} h_1 + \dots + a_{mn} h_n \end{bmatrix}$$

POKAZUJE SE DA JE

$$A_L = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

A_L SE NAZIVA JAKOBIJEVA MATRICA ILI MATRICA IZVODA.

L SE NAZIVA IZVOD FUNKCIJE F I OZNAčava sa dF ili F'

Ukoliko je $m=n$ može se izračunati i $\det A_L = J$ i naziva se Jakobijan preslikavanja F.

(1)

$$F : \mathbb{R}^3 \to \mathbb{R}^2$$

$$F(x, y, z) = \left(z e^{xy^2} , \sin(xz) - 3y \right)$$

Odrediti Jakobijevu matricu preslikavanja F u tački $(1,1,1)$.

$$f_1(x, y, z) = z e^{xy^2}$$

$$\frac{\partial f_1}{\partial x} = z e^{xy^2} \qquad \frac{\partial f_1}{\partial y} = 2zy e^{xy^2} \qquad \frac{\partial f_1}{\partial z} = e^{xy^2}$$

$$f_2(x, y, z) = \sin(xz) - 3y$$

$$\frac{\partial f_2}{\partial x} = z \cos(xz) \qquad \frac{\partial f_2}{\partial y} = -3 \qquad \frac{\partial f_2}{\partial z} = x \cos(xz)$$

$$A_L = \begin{bmatrix} z e^{xy^2} & 2zy e^{xy^2} & e^{xy^2} \\ z \cos(xz) & -3 & x \cos(xz) \end{bmatrix}$$

$$A_L((1,1,1)) = \begin{bmatrix} e & 2e & e \\ \cos 1 & -3 & \cos 1 \end{bmatrix}$$

(2) Odrediti Jakobijevu matricu i Jakobijan prelaska sa cilindričnih koordinata na pravougle.

$$F : A \to \mathbb{R}^3$$

$$F(\rho, \varphi, \theta) = (\underbrace{\rho \cos \varphi}_{f_1 = x} , \underbrace{\rho \sin \varphi}_{f_2 = y} , \underbrace{z}_{f_3 = z})$$

A: $\rho \geq 0$, $0 \leq \varphi < 2\pi$

$$dF = \begin{bmatrix} \cos \varphi & -\rho \sin \varphi & 0 \\ \sin \varphi & \rho \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$J = \det dF = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

RAZVOJ PO TREĆOJ VRSTI

$$= \rho\cos^2\varphi + \rho\sin^2\varphi = \rho$$

$$J = \rho$$

3) ODREDITI JAKOBIJEVU MATRICU I JAKOBIJAN PRELASKA SA SFERNIH KOORDINATA NA PRAVOUGLE.

$$F : A \to \mathbb{R}^3$$

$$F(\rho,\varphi,\theta) = (\underbrace{\rho\cos\varphi\sin\theta}_{f_1}, \underbrace{\rho\sin\varphi\sin\theta}_{f_2}, \underbrace{\rho\cos\theta}_{f_3})$$

$$A: \quad 0 \le \rho \qquad 0 \le \varphi < 2\pi \qquad 0 \le \theta < \pi$$

$$dF = \begin{bmatrix} \cos\varphi\sin\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\theta & 0 & -\rho\sin\theta \end{bmatrix}$$

Jakobijan računamo razvojem determinante po trećoj vrsti.

$$J = \cos\theta \begin{vmatrix} -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix} - \rho\sin\theta \begin{vmatrix} \cos\varphi\sin\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta \end{vmatrix}$$

$$= \cos\theta\left(-\rho^2\sin^2\varphi\sin\theta\cos\theta - \rho^2\cos^2\varphi\cos\theta\sin\theta\right)$$

$$- \rho\sin\theta\left(\rho\cos^2\varphi\sin^2\theta + \rho\sin^2\varphi\sin^2\theta\right)$$

$$= -\rho^2\sin\theta\cos^2\theta\,\overbrace{(\sin^2\varphi + \cos^2\varphi)}^{=1} - \rho^2\sin^3\theta\,\overbrace{(\cos^2\varphi + \sin^2\varphi)}^{=1}$$

$$= -\rho^2\sin\theta\cos^2\theta - \rho^2\sin^3\theta$$

$$= -\rho^2\sin\theta(\cos^2\theta + \sin^2\theta) = -\rho^2\sin\theta$$

$$|J| = \rho^2\sin\theta$$

IZVOD SLOŽENE FUNKCIJE

$$F : \underset{\mathbb{R}^m}{\overset{\cap}{A}} \to \mathbb{R}^n$$

$$G : B \to \mathbb{R}^k \qquad B \subseteq F[A] \subseteq \mathbb{R}^n$$

$$d(G \circ F) = ?$$

$$d(G \circ F)(x_0) = dG(F(x_0)) \cdot dF(x_0)$$

MNOŽENJE MATRICA

④

$$f : \mathbb{R}^3 \to \mathbb{R} \qquad u : \mathbb{R}^2 \to \mathbb{R}$$

$$u(x,y) = f(x^2+y^2,\ x^2-y^2,\ 2xy)$$

$$du = ?$$

$$u = f \circ g \qquad g(x,y) = (x^2+y^2,\ x^2-y^2,\ 2xy)$$

$$(x,y) \xmapsto{g} (x^2+y^2,\ x^2-y^2,\ 2xy) \xmapsto{f} f(x^2+y^2, x^2-y^2, 2xy)$$

$$\underbrace{(x,y) \mapsto f(x^2+y^2, x^2-y^2, 2xy)}_{u}$$

$$du = df(g(x,y)) \cdot dg(x,y)$$

$$dg(x_0,y_0) = \begin{bmatrix} 2x_0 & 2y_0 \\ 2x_0 & -2y_0 \\ 2y_0 & 2x_0 \end{bmatrix}$$

$$df(x_0,y_0) = \left[\frac{\partial f}{\partial x}(g(x_0,y_0)),\ \frac{\partial f}{\partial y}(g(x_0,y_0)),\ \frac{\partial f}{\partial z}(g(x_0,y_0))\right]$$

$$du = \left[\frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z}\right] \begin{bmatrix} 2x & 2y \\ 2x & -2y \\ 2y & 2x \end{bmatrix}$$

$$du = \left[2x\frac{\partial f}{\partial x} + 2x\frac{\partial f}{\partial y} + 2y\frac{\partial f}{\partial z},\ 2y\frac{\partial f}{\partial x} - 2y\frac{\partial f}{\partial y} + 2x\frac{\partial f}{\partial z}\right]$$

TANGENTNA RAVAN

Regularna površ je skup

$$\mathcal{P} = \{ (x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = 0 \}$$

gde za funkciju $f : \mathbb{R}^3 \to \mathbb{R}$ važi

$$\forall (x,y,z) \in \mathcal{P} \qquad \nabla f(x,y,z) \neq 0$$

Unija svih tangenti na sve krive sa površi $\mathcal{P}$ koje prolaze kroz tačku $A \in \mathcal{P}$ naziva se tangentna ravan na površ $\mathcal{P}$ u tački A i označava sa $T_A \mathcal{P}$

Vektor normale na $T_A \mathcal{P}$ je $\nabla f(A)$.

5. Odrediti tangentnu ravan na površ

$$x^2 + 2y^2 + 3z^2 = 36$$

u tački $A(1,2,3)$

$$f(x,y,z) = x^2 + 2y^2 + 3z^2 - 36$$

$$\nabla f = (2x, 4y, 6z)$$

$$\nabla f(A) = (2, 8, 18) = 2(1,4,9)$$

Možemo uzeti da je $\vec{n}_{T_A \mathcal{P}} = (1,4,9)$

$$T_A \mathcal{P} : \quad x + 4y + 9z + D = 0$$

D određujemo iz uslova $A \in \mathcal{P}$

$$1 + 4 \cdot 2 + 9 \cdot 3 = D$$

$$\Rightarrow D = -36$$

$$T_A \mathcal{P} : \quad x + 4y + 9z - 36 = 0$$

IZVODI VIŠEG REDA:

$$f : \underset{\mathbb{R}^2}{\overset{\cap}{A}} \to \mathbb{R}$$

A - OTVOREN

IZVODI DRUGOG REDA FUNKCIJE f SU IZVODI FUNKCIJA $\frac{\partial f}{\partial x}$ i $\frac{\partial f}{\partial y}$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \qquad \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

INDUKTIVNO SE DEFINIŠU IZVODI VIŠEG REDA.

ANALOGNO MOŽEMO DEFINISATI IZVODE VIŠEG REDA FUNKCIJE TRI PROMENLJIVE (ILI VIŠE)

$\frac{\partial^2 f}{\partial x\, \partial y}$ i $\frac{\partial^2 f}{\partial y\, \partial x}$ SE NAZIVAJU I MEŠOVITI IZVODI

KORISTE SE I OZNAKE:

$$f''_{xx},\ f''_{yy},\ f''_{xy},\ f''_{yx}$$

6. $$f(x,y) = \arcsin\sqrt{\frac{x}{y}}$$

NAĆI MEŠOVITE DRUGE PARCIJALNE IZVODE

ŠTA JE DOMEN f?

- $y \neq 0$
- $\frac{x}{y} \geq 0 \Rightarrow x$ i y ISTOG ZNAKA
- $\sqrt{\frac{x}{y}} \leq 1 \Rightarrow \left|\frac{x}{y}\right| \leq 1 \Rightarrow |x| \leq |y|$

$$D_f = \Big(\underbrace{\{(x,y)\in\mathbb{R}^2 \mid 0 \leq x \leq y\}}_{D_1} \cup \{(x,y)\in\mathbb{R}^2 \mid y \leq x \leq 0\}\Big) \setminus \{(0,0)\}$$

RAČUNAĆEMO IZVODE ZA SKUP D_1.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \arcsin\sqrt{\frac{x}{y}} = \frac{1}{\sqrt{1-\frac{x}{y}}} \; \frac{1}{2\sqrt{\frac{x}{y}}} \; \frac{1}{y} =$$

$$= \frac{1}{\frac{\sqrt{y-x}}{\cancel{\sqrt{y}}}} \; \frac{1}{\frac{2\sqrt{x}}{\cancel{\sqrt{y}}}} \; \frac{1}{\cancel{y}} = \frac{1}{2\sqrt{x}\,\sqrt{y-x}}$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{1}{2\sqrt{x}\,\sqrt{y-x}}\right) = \frac{1}{2\sqrt{x}} \; \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{y-x}}\right) =$$

$$= \frac{1}{2\sqrt{x}} \; \frac{-\frac{1}{2\sqrt{y-x}}}{y-x} = \frac{-1}{4\sqrt{x}\;\sqrt{(y-x)^3}}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\arcsin\sqrt{\frac{x}{y}}\right) = \frac{1}{\sqrt{1-\frac{x}{y}}} \; \frac{1}{2\sqrt{\frac{x}{y}}} \; \frac{-x}{y^2}$$

$$= \frac{1}{\frac{\sqrt{y-x}}{\cancel{\sqrt{y}}}} \; \frac{1}{2\,\frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{y}}}} \; \frac{-\cancel{x}}{y^{\cancel{2}}} = \frac{-\sqrt{x}}{2y\sqrt{y-x}}$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{-\sqrt{x}}{2y\,\sqrt{y-x}}\right) = -\frac{1}{2y} \; \frac{\partial}{\partial x}\left(\frac{\sqrt{x}}{\sqrt{y-x}}\right)$$

$$= -\frac{1}{2y} \; \frac{\frac{1}{2\sqrt{x}}\sqrt{y-x} - \sqrt{x}\,\frac{1}{2\sqrt{y-x}}\,(-1)}{y-x} \quad \Big/ \cdot \frac{\sqrt{y-x}}{\sqrt{y-x}}$$

$$= -\frac{1}{2y} \; \frac{\frac{1}{2\sqrt{x}}(y-x) + \frac{\sqrt{x}}{2}}{\sqrt{(y-x)^3}} \quad \Big/ \cdot \frac{\sqrt{x}}{\sqrt{x}}$$

$$= \frac{-1}{4\cancel{y}} \; \frac{\cancel{y}\cancel{-x}\cancel{+x}}{\sqrt{(y-x)^3}\;\sqrt{x}} = \frac{-1}{4\sqrt{x}\;\sqrt{(y-x)^3}}$$

VIDIMO DA JE

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}$$

⑦

$$f(x,y)=\begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

DOKAZATI DA POSTOJE MEŠOVITI DRUGI PARCIJALNI IZVODI U TAČKI (0,0), ALI I DA SE RAZLIKUJU.

ODREĐUJEMO PRVE PARCIJALNE IZVODE

$(x,y)\neq(0,0)$

$$\frac{\partial f}{\partial x}=y\,\frac{x^2-y^2}{x^2+y^2}+xy\,\frac{2x(x^2+y^2)-(x^2-y^2)2x}{(x^2+y^2)^2}$$

$$=y\,\frac{x^2-y^2}{x^2+y^2}+xy\,\frac{\cancel{2x^3}+2xy^2-\cancel{2x^3}+2xy^2}{(x^2+y^2)^2}$$

$$=y\,\frac{x^2-y^2}{x^2+y^2}+\frac{4x^2y^3}{(x^2+y^2)^2}$$

IZ SIMETRIJE MOŽEMO ODMAH ZAKLJUČITI

$$\frac{\partial f}{\partial y}=x\,\frac{x^2-y^2}{x^2+y^2}-\frac{4y^2x^3}{(x^2+y^2)^2}$$

ZA TAČKU (0,0) MORAMO TRAŽITI PARCIJALNE IZVODE PO DEFINICIJI.

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to0}\frac{x\cdot0\,\frac{x^2-0^2}{x^2+0^2}-0}{x}=0$$

$$\frac{\partial f}{\partial y}(0,0)=0$$

$$\frac{\partial f}{\partial x}=\begin{cases} y\,\dfrac{x^2-y^2}{x^2+y^2}+\dfrac{4x^2y^3}{(x^2-y^2)^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) =$$

$$= \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{h\,\frac{0^2-h^2}{0^2+h^2} + \frac{4\,0^2 h^3}{(0^2+h^2)^2} - 0}{h} = -1$$

$$\frac{\partial f}{\partial y} = \begin{cases} x\,\frac{x^2-y^2}{x^2+y^2} - \frac{4y^2x^3}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) =$$

$$= \lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{h\,\frac{h^2-0^2}{h^2+0^2} - \frac{4\,0^2 h^3}{(h^2+0^2)^2} - 0}{h} = 1$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = -1 \neq \frac{\partial^2 f}{\partial x \partial y}(0,0) = 1$$

MEDUTIM OVO JE REDAK SLUČAJ DA SE MEŠOVITI PARCIJALNI IZVODI RAZLIKUJU JER VAŽI:

TEOREMA:

AKO FUNKCIJA f IMA DEFINISANE MEŠOVITE PARCIJALNE IZVODE U NEKOJ OKOLINI TAČKE A I AKO SU NE PREKIDNI U A ONDA SU I JEDNAKI U A.

Тејлоров полином

$$f : A \to \mathbb{R}$$
$$A \subseteq \mathbb{R}^2$$
$$(x_0, y_0) \in A$$

НЕКА ФУНКЦИЈА f ИМА СВЕ ПАРЦИЈАЛНЕ ИЗВОДЕ СТЕПЕНА n И НЕПРЕКИДНИ СУ.

ТЕЈЛОРОВ ПОЛИНОМ РЕДА n У ТАЧКИ (x_0, y_0)

$$P_n(x_0,y_0)(h,l) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\,h + \frac{\partial f}{\partial y}(x_0,y_0)\,l +$$
$$+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(x_0,y_0)\,h^2 + 2\,\frac{\partial^2 f}{\partial x\,\partial y}(x_0,y_0)\,h\,l + \frac{\partial^2 f}{\partial y^2}(x_0,y_0)\,l^2\right)$$
$$+ \cdots +$$
$$+ \frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}\frac{\partial^n f}{\partial x^{n-j}\,\partial y^j}(x_0,y_0)\,h^{n-j}\,l^j$$

8) НАПИСАТИ ТЕЈЛОРОВ ПОЛИНОМ СТЕПЕНА 3 У ТАЧКИ $(1,1)$ ФУНКЦИЈЕ

$$f(x,y) = x^y$$

$$\frac{\partial f}{\partial x} = y\,x^{y-1}$$

$$\frac{\partial f}{\partial y} = \log x\; x^y$$

$$\frac{\partial^2 f}{\partial x^2} = y(y-1)\,x^{y-2} \qquad \frac{\partial^2 f}{\partial y^2} = \log^2 x\; x^y$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{1}{x}\,x^y + \log x\; y\,x^{y-1} = x^{y-1}(1 + y\log x)$$

$$\parallel$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = x^{y-1} + y\,x^{y-1}\log x = x^{y-1}(1 + y\log x)$$

$$\frac{\partial^3 f}{\partial x^3} = y(y-1)(y-2)\, x^{y-3} \qquad \frac{\partial^3 f}{\partial y^3} = \log^3 x \; x^y$$

$$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial}{\partial y}\left(x^{y-1}(1+y\log x)\right)$$

$$= \log x \; x^{y-1}(1+y\log x) + x^{y-1}\log x$$

$$= \log x \; x^{y-1}(2+y\log x)$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial}{\partial x}\left(x^{y-1}(1+y\log x)\right)$$

$$= (y-1)\, x^{y-2}(1+y\log x) + x^{y-1}\frac{y}{x}$$

TREBAJU NAM VREDNOSTI SVIH OVIH IZVODA U TAČKI $(1,1)$ $\quad f(1,1)=1$

$$\frac{\partial f}{\partial x}(1,1) = 1 \qquad \frac{\partial x}{\partial y}(1,1) = 0$$

$$\frac{\partial^2 f}{\partial x^2}(1,1) = 0 \qquad \frac{\partial^2 f}{\partial y^2}(1,1) = 0 \qquad \frac{\partial^2 f}{\partial x \partial y}(1,1) = 1$$

$$\frac{\partial^3 f}{\partial x^3}(1,1) = 0 \qquad \frac{\partial^3 f}{\partial y^3}(1,1) = 0 \qquad \frac{\partial^3 f}{\partial x^2 \partial y}(1,1) = 1 \qquad \frac{\partial^3 f}{\partial x \partial y^2}(1,1) = 0$$

ZAKLJUČUJEMO DA JE:

$$P_{3(1,1)}(h,l) = 1 + h + hl + \frac{1}{2}h^2 l$$