ANALIZA 2

VEŽBE

DIFERENCIJALNE FORME

- PRIČALI SMO U $\mathbb{R}^3$

1-FORMA $\quad P\,dx + Q\,dy + R\,dz \qquad P, Q, R : \mathbb{R}^3 \to \mathbb{R}$

V - VEKTORSKI PROSTOR

w K-FORMA JE PRESLIKAVANJE

$$w : \underbrace{V \times V \times \cdots \times V}_{k} \to \mathbb{R}$$

K-LINEARNO I ANTISIMETRIČNO

LINEARNO PO SVAKOJ PROMENLJIVOJ

$$w(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_k) = -w(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_k)$$

AKO POSTOJE $i \neq j \quad x_i = x_j$

$\Rightarrow w = 0$

AKO POSTOJE DVA LINEARNO ZAVISNA $\Rightarrow w = 0$

SVAKA K-FORMA JE OBLIKA

$$\omega = \underset{\text{PO SVIM PERMUTACIJAMA}}{\sum} a_{i_1,\dots,i_k}\, dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

$$dx_{i_1} \wedge \dots \wedge dx_{i_k} (X_1, \dots, X_k) = \begin{vmatrix} dx_{i_1}(X_1) & \dots & dx_{i_k}(X_1) \\ \vdots & & \vdots \\ dx_{i_1}(X_k) & \dots & dx_{i_k}(X_k) \end{vmatrix}$$

SKUP SVIH K-DIFERENCIJALNIH FORMI NA OTVORENOM SKUPU V OZNAČAVAMO

$$\Omega^k(V)$$

$$\omega \in \Omega^k(V) \qquad \deg \omega = k$$

$$\dim V = n$$

$k > n$ $\quad \omega \in \Omega^k(V) \quad \omega = 0$

$k = n$ $\quad \omega$ = KONSTANTA PUTA DETERMINANTA

$k < n$ $\quad$ KOMBINACIJA MANJIH DETERMINANTI

$$\dim V = n \Rightarrow \dim \Omega^k(V) = \binom{n}{k}$$

OPERACIJE SA FORMAMA

1) SABIRANJE DVE FORME

$$\alpha \in \Omega^k(V), \quad \beta \in \Omega^k(V) \qquad \alpha + \beta \in \Omega^k(V)$$

2) SPOLJAŠNJE MNOŽENJE

$$\wedge : \Omega^k(V) \times \Omega^l(V) \to \Omega^{k+l}(V)$$

$$\left.\begin{matrix} \alpha \in \Omega^k(V) \\ \beta \in \Omega^l(V) \end{matrix}\right\} \quad \alpha \wedge \beta \in \Omega^{k+l}(V)$$

MNOŽIMO DISTRIBUTIVNO: $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$

$$\alpha = x^2\, dx + x e^z\, dy + \sin(zx)\, dz$$

$$\beta = \tan y\, dx + ch(yz)\, dy + \sin(xz)\, dz$$

$$\alpha \wedge \beta = x^2 \cancel{\tan y}\, dx \wedge dx + x^2\, ch(yz)\, dx \wedge dy + x^2 \sin(zx)\, dx \wedge dz$$
$$+ x e^z \tan y\, dy \wedge dx + x e^z \cancel{ch(yz)\, dy} \wedge dy + x e^z \sin(zx)\, dy \wedge dz$$
$$+ \sin(zx) \tan y\, dz \wedge dx + \sin(zx)\, ch(yz)\, dz \wedge dy + \cancel{(\sin(zx))^2}\, dz \wedge dz$$

$$= \left(x^2 ch(yz) - x e^z \tan y\right) dx \wedge dy +$$
$$+ \left(x^2 \sin(zx) - \sin(zx) \tan y\right) dx \wedge dz + \left(x e^z \sin(zx) - \sin(zx)\, ch(yz)\right) dy \wedge dz$$

FUNKCIJE SU 0-FORME

f - FUNKCIJA $\qquad f \wedge \omega = f\omega$

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3) SPOLJAŠNJE DIFERENCIRANJE

(UOPŠTENJE DIFERENCIJALA FUNKCIJE)

$$d : \Omega^k(V) \to \Omega^{k+1}(V)$$

- d JE LINEARNO PRESLIKAVANJE
- ZA $k=0$ d JE DIFERENCIJAL FUNKCIJE
- LAJBNICOVO PRAVILO
$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha}\, \alpha \wedge d\beta$$
- $d^2 = 0$

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4) UNUTRAŠNJE MNOŽENJE (SMANJUJE STEPEN)

α - k-FORMA

X - VEKTORSKO POLJE

$i(X)\alpha = X \lrcorner \alpha$ K-1 FORMA

$$(X \lrcorner \alpha)(X_2, X_3, \dots, X_k) = \alpha(X, X_2, \dots, X_k)$$

$$X \lrcorner (\alpha \wedge \beta) = (X \lrcorner \alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge (X \lrcorner \beta)$$

DEFINICIJA:

a) K-FORMA w JE TAČNA AKO POSTOJI K-1 FORMA η T.D.J. $d\eta = w$

η SE NAZIVA PRIMITIVNA FORMA

b) K-FORMA w JE ZATVORENA AKO JE

$$dw = 0$$

TAČNA $\Rightarrow$ ZATVORENA

$\not\Leftarrow$ - ZAVISI OD TOPOLOGIJE PROSTORA

$$\operatorname{Im}\{d : \Omega^{k-1}(V) \to \Omega^k(V)\} \subseteq \operatorname{Ker}\{d : \Omega^k(V) \to \Omega^{k+1}(V)\}$$

DOBIJAMO TAČAN NIZ:

$$0 \to \underset{\text{FUNKCIJE}}{\Omega^0(V)} \to \Omega^1(V) \to \Omega^2(V) \to \dots \to \Omega^n(V) \to 0$$

SVE SU ABELOVE GRUPE

$$H^k_{DR}(V) = \frac{\operatorname{Ker} d}{\operatorname{Im} d}$$

K-TA DE RAMOVA KOHOMOLOŠKA GRUPA

① $\Omega^0(V) = \mathbb{R} \iff V$ JE POVEZAN

$$\Omega^0(V) = \operatorname{Ker}\{df \mid f \in \mathcal{D}(V)\}$$

$$(-\infty, 0) \cup (0, +\infty)$$

$$f = c \Rightarrow df = 0$$

$df = 0 \Rightarrow f = c$ AKO JE PROSTOR POVEZAN

$H^0_{DR}(V)$ - BROJ KOMPONENTI POVEZANOSTI

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① ISPITATI ZATVORENOST I TAČNOST FORMI

a) $\xi = x\, dx + (xy)^2\, dy + yz\, dz$

$$d\xi = dx \wedge \cancel{dx} + 0 \cdot dy \wedge dx + 0 \cdot dz \wedge dx$$
$$2xy^2\, dx \wedge dy + 2x^2y\, \cancel{dy \wedge dy} + 0 \cdot dz \wedge dy$$
$$+ 0\, dx \wedge dz + z\, dy \wedge dz + y\, \cancel{dz \wedge dz}$$

$$= 2xy^2\, dx \wedge dy + z\, dy \wedge dz \neq 0$$

$\Rightarrow \xi$ NIJE ZATVORENA

$\Rightarrow \xi$ NIJE NI TAČNA

b) $\eta = yz\, dx + xz\, dy + xy\, dz$

$$d\eta = 0\, dx \wedge \cancel{dx} + z\, dy \wedge dx + y\, dz \wedge dx$$
$$+ z\, dx \wedge dy + 0\, \cancel{dy \wedge dy} + x\, dz \wedge dy)$$
$$+ y\, dx \wedge dz + x\, dy \wedge dz) + 0\, \cancel{dz \wedge dz}$$

$$= 0$$

$d\eta = 0 \Rightarrow \eta$ JE ZATVORENA FORMA

KAKO PROVERITI DA LI JE TAČNA?

PRETPOSTAVIMO DA JESTE

$df = \eta \qquad \eta = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$

$\frac{\partial f}{\partial x} = yz \qquad f = \int yz\, dx = xyz + C(y, z)$

$$\frac{\partial f}{\partial y} = xz \qquad \frac{\partial f}{\partial y} = xz + \frac{\partial C}{\partial y}$$

$$\Rightarrow \frac{\partial C}{\partial y} = 0 \quad \Rightarrow \quad C = C(z)$$

$$\frac{\partial f}{\partial z} = xy \qquad \frac{\partial f}{\partial z} = xy + \frac{\partial C}{\partial z}$$

$$\frac{\partial C}{\partial z} = 0 \qquad C - \text{KONSTANTA}$$

$$f = xyz$$

(3)

$$w = \underline{z^2\, dx \wedge dy} + (z^2 + y)\, \underline{dx} \wedge \underline{dz}$$

$$dw = 2z\, dz \wedge dx \wedge dy + dy \wedge dx \wedge dz$$

$$= (2z - 1)\, dx \wedge dy \wedge dz$$

(4)

$$w = 2x\underline{y}^2\, \underline{dx} \wedge \underline{dy} + z\, \underline{dy} \wedge \underline{dz}$$

$$dw = 0$$

$\Rightarrow$ w JESTE ZATVORENA

DA LI JE TAČNA?

PRETPOSTAVIMO DA JE TAČNA

$$w = d\eta$$

$$\eta = P\, dx + Q\, dy + R\, dz$$

$$d\eta = \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial P}{\partial z} dz \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy + \frac{\partial Q}{\partial z} dz \wedge dy + \frac{\partial R}{\partial x} dx \wedge dz + \frac{\partial R}{\partial y} dy \wedge dz$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) dx \wedge dz$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2xy^2 \qquad \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = z$$

$$\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} = 0$$

NE MOŽEMO IZ OVOGA ZAKLJUČITI

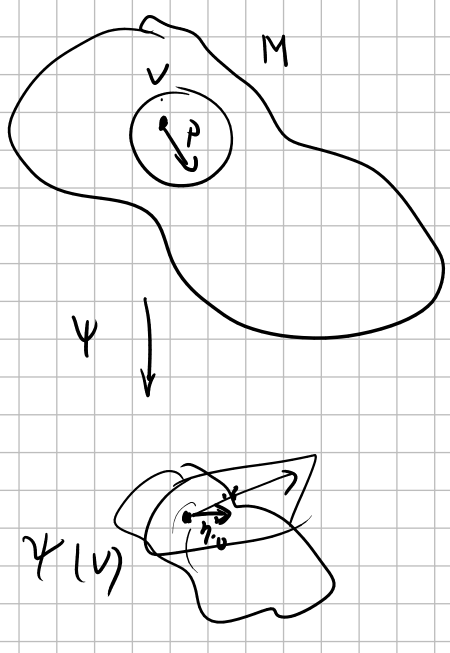
$$w = ye^z \, dx \wedge dy + z \sin x \, dy \wedge dz + (x+y) \, dz \wedge dx$$

$$dw = ye^z \, dz \wedge dx \wedge dy + z \cos x \, dx \wedge dy \wedge dz + dy \wedge dz \wedge dx$$

$$= (ye^z + z\cos x + 1) \, dx \wedge dy \wedge dz$$

⋮

DIFERENCIJALNE FORME NA MNOGOSTRUKOSTIMA



(V, ψ) KARTA U OKOLINI P

ψ_* - DIFERENCIJAL

$$\psi_* : T_P M \to T_{\psi(P)} \mathbb{R}^m$$

$$\psi_*(v) = w$$

η_0 - DIFERENCIJALNA FORMA U $\psi(V) \subseteq \mathbb{R}^m$

$$\eta_V = \psi^* \eta_0 \qquad \psi^* \text{ - PULLBACK}$$

η_0 - K-FORMA

$$\eta_0 |_{\psi(P)} = ?$$

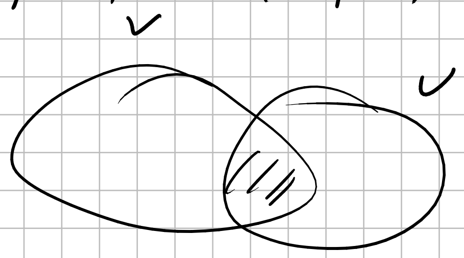
$(x_1, x_2, \ldots, x_k)$ - PROIZVOLJAN VEKTOR

$\eta_0|_{\psi(p)}(x_1, x_2, \ldots, x_k)$ - TO ZNAMO

$$\eta_V|_p(y_1, \ldots, y_k) = \eta_0|_{\psi(p)}(\psi_* y_1, \ldots, \psi_* y_k)$$

η_V - VIŠELINEARNO I ANTISIMETRIČNO PRESLIKAVANJE U T_pM.

$(V, \psi)_V$ (U, φ) DVE KARTE



$i: U \cap V \to U$ $j: U \cap V \to V$

$$i^* \eta_U = j^* \eta_V$$

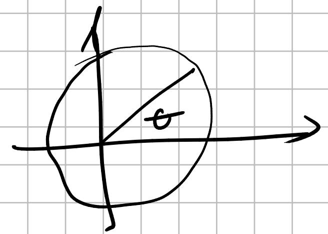
DIFERENCIJALNA FORMA NA MNOGOSTRUKOSTI JE FAMILIJA SAGLASNIH FORMI NA CELOM ATLASU.

DIFERENCIJAL JE ZADAT LOKALNO, PA GA JE DOVOLJNO DEFINISATI U KARTI

$$d\eta_V = \psi^*(d\eta_0)$$

PRIMER FORME KOJA JE ZATVORENA, A NIJE TAČNA:

$$M = S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$$



θ - UGAO KOD POLARNIH KOORDINATA

$[0, 2\pi)$ GLOBALNO NIJE NEPREKIDNO DEFINISANA FUNKCIJA

$$d\theta = d\left(\arctan \frac{y}{x}\right) =$$

$$= \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{-y}{x^2}\, dx + \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}\, dy$$

$$\underbrace{= \frac{-y\, dx}{x^2+y^2} + \frac{x\, dy}{x^2+y^2} = \omega}_{d\omega = 0\ ?}$$

$d\omega = d(d\theta)$ LOKALNO

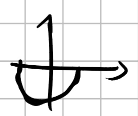
TAČNOST FORME JE LOKALNO SVOJSTVO

$d\theta$ - OZNAKA (NIJE IZVOD FUNKCIJE)

$d\theta$ NIJE TAČNA FORMA

PRETPOSTAVIMO DA JESTE

$$f : S^1 \to \mathbb{R}$$



$$\gamma(t) = (\cos t, \sin t) \quad t \in [0, 2\pi]$$

$$f(\gamma(2\pi)) - f(\gamma(\pi)) = \int_{\pi}^{2\pi} \frac{d}{ds} (f \cdot \gamma)(s)\, ds$$

$$= \int_{\pi}^{2\pi} (df \cdot \gamma(s))\, \gamma'(s)\, ds$$

$$= \int_{\pi}^{2\pi} (d\theta \circ \gamma(s))\, \gamma'(s)\, ds$$

$$= \int_{\pi}^{2\pi} \frac{d}{ds} (\theta \circ \gamma)\, ds$$

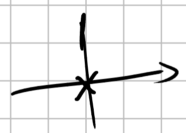
$$= \theta(\gamma(2\pi)) - \theta(\gamma(\pi)) = \pi$$

Isto će biti na gornjem polukrugu

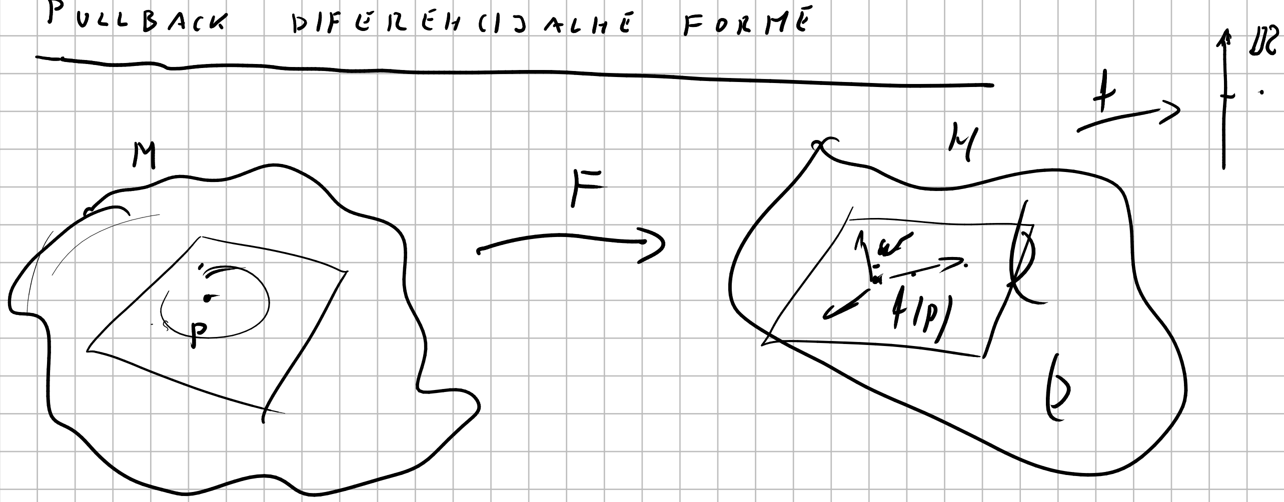
$$\theta(\gamma(\pi)) - \theta(\gamma(0)) = \pi$$

$$f(\gamma(2\pi)) - f(\gamma(0)) = 2\pi \quad \lightning \quad f(\gamma(2\pi)) = f(\gamma(0))$$

ω je zatvorena, a nije tačna u $\mathbb{R}^2 \setminus \{(0,0)\}$



Pullback diferencijalne forme



$$F_* : T_p M \to T_{f(p)} N$$

$$v \in T_pM, \quad v \rightsquigarrow F_*(v)$$

ω k-forma na N

$k=0 \qquad w=f \qquad$ FUNKCIJA

$$F^* f = f \circ F$$

$k \geq 1$

$p \in M$ PROIZVOLJNO $\qquad A_1, \ldots, A_k \in T_pM$

$$F^* w (A_1, \ldots, A_k) = ?$$

$$(F^* w)\big|_p (A_1, \ldots, A_k) = w_{F(p)} (F_* A_1, \ldots, F_* A_k)$$

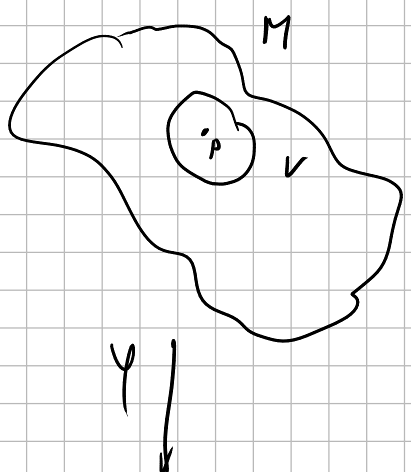
OVAKO JE DEFINISANO VIŠELINEARNO I ANTI-SIMETRIČNO PRESLIKAVANJE NA T_pM, $\forall p \in M$.

ANALIZA 2

VEŽBE

__PODSEĆANJE:__

DEFINISALI SMO DIFERENCIJALNE FORME NA MNOGOSTRUKOSTI.



(V, φ) - KARTA

η_0 - DIFERENCIJALNA FORMA NA $\varphi(V)$

$$\eta_V = \varphi^* \eta_0$$

(U, ψ) - DRUGA KARTA

$$\eta_U = \psi^* \eta_0$$

$$i : V \cap V \to V \qquad j : V \cap V \to U$$

FORME SU SAGLASNE:

$$i^* \eta_V = j^* \eta_U$$

DIFERENCIJALNA FORMA NA M JE FAMILIJA SAGLASNIH FORMI NA ATLASU.

$\Omega^k(V)$ - K-FORME NA V

$\Omega^0(V)$ - FUNKCIJE NA V

$\dim \Omega^k(V) = \binom{n}{k}$ U $\mathbb{R}^n$

OPERACIJE SA FORMAMA:

- SABIRANJE $+ : (\alpha, \beta) \longmapsto \alpha + \beta$

- SPOLJAŠNJI PROIZVOD

$$\wedge : \Omega^k(V) \times \Omega^l(V) \longrightarrow \Omega^{k+l}(V)$$

DISTRIBUTIVNO JE

$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$$

- SPOLJAŠNJE DIFERENCIRANJE

$$d : \Omega^k(V) \longrightarrow \Omega^{k+1}(V)$$

  - LINEARNO
  - ZA $k=0$ DIFERENCIJAL FUNKCIJE
  - LAJBNICOVO PRAVILO

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$$

- $d^2 = 0$

- UNUTRAŠNJE MNOŽENJE

  α - FORMA

  X - VEKTORSKO POLJE

$$i(X)\alpha = X \lrcorner \alpha$$

$$(X \lrcorner \alpha)(X_2, \dots X_k) = \alpha(X, X_2, \dots, X_k)$$

- K-FORMA JE TAČNA AKO IMA PRIMITIVNU, TJ. AKO POSTOJI K-1 FORMA η TDJ.

$$d\eta = \omega$$

- FORMA ω JE ZATVORENA AKO JE $d\omega = 0$

$d^2 = 0 \Rightarrow$ SVAKA TAČNA FORMA JE I ZATVORENA.

- VAŽAN PRIMER

$$M = S^1$$

θ - UGAO KOD POLARNIH KOORDINATA

$$d\theta = \frac{x\,dy}{x^2+y^2} - \frac{y\,dx}{x^2+y^2}$$

$d\theta$ JE ZATVORENA, A NIJE TAČNA

(ZATVORENOST JE LOKALNO SVOJSTVO, A TAČNOST GLOBALNO SVOJSTVO)

$$\mathrm{Im}\{d : \Omega^{k-1}(V) \to \Omega^k(V)\} \subseteq \mathrm{Ker}\{d : \Omega^k(V) \to \Omega^{k+1}(V)\}$$

$$0 \to \Omega^0(V) \to \Omega^1(V) \to \dots \to \Omega^n(V) \to 0 \to 0$$

$$H^k_{DR}(V) = \frac{\mathrm{Ker}\, d}{\mathrm{Im}\, d}$$

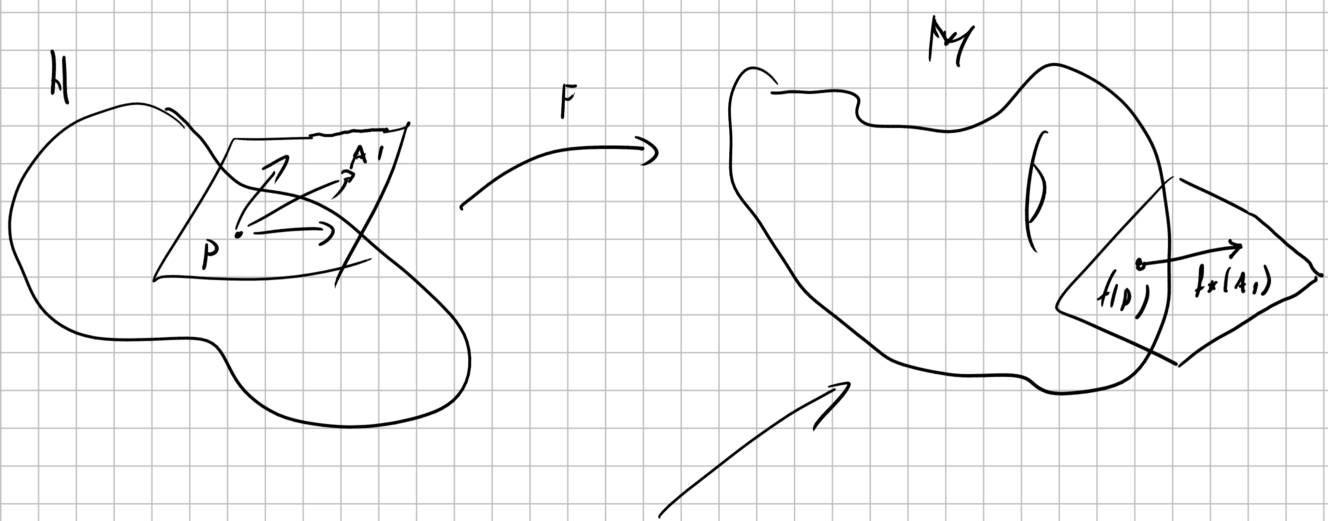
$H^0_{DR}(V)$ – BROJI KOMPONENTE POVEZANOSTI

$H^0_{DR}(V) = \mathbb{R}^k \Rightarrow k$ KOMPONENTI POVEZANOSTI

$H^1_{DR}(V) = ?$

PULLBACK FORME

$F : N \to M$



w k-FORMA NA M

$$w : \underbrace{T_pM \times \dots \times T_pM}_{k} \to \mathbb{R}$$

$F^* w$ k-FORMA NA N

1) $k=0$ ω ЈЕ ФУНКЦИЈА

$$F^*\omega = \omega\circ F$$

2) $k \geq 1$

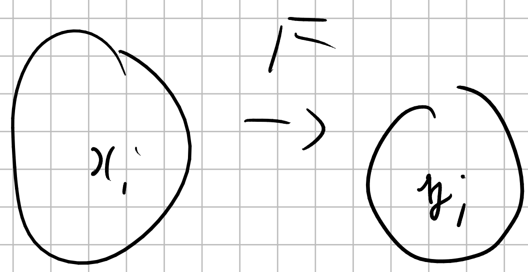
$p \in M$ - ПРОИЗВОЉНО

$A_1,\dots,A_k \in T_pM$ - ПРОИЗВОЉНИ

$$(F^*\omega)\big|_p(A_1,\dots,A_k) = \omega\big|_{F(p)}(F_*A_1,\dots,F_*A_k)$$

ПРИМЕР:

$$F:\mathbb{R}^n \longrightarrow \mathbb{R}^m$$



$$F=(F_1,\dots,F_m)$$

НА $\mathbb{R}^n$ $(x_1,\dots,x_n)$

НА $\mathbb{R}^m$ $(y_1,\dots,\underset{y_i}{\cdot},y_m)$

$\omega = dy_i$ 1-ФОРМА НА $\mathbb{R}^m$

$F^*\omega = ?$

$$F^* w = \sum_{j=1}^{n} \alpha_j \, dx_j$$

$$\boxed{F^* w = \sum_{j=1}^{n} \frac{\partial F_i}{\partial x_j} \, dx_j = dF_i}$$

$F^* dy_i$

OSOBINE PULL BACK-A

w GLATKA K-FORMA $\Rightarrow$ F^* GLATKA K-FORMA

$$F^* (w_1 + w_2) = F^* w_1 + F^* w_2$$

$$F^* (\alpha \wedge \beta) = F^* \alpha \wedge F^* \beta$$

$$F^* (dw) = d F^* w$$

$$\mathbb{1}^* = \mathbb{1} \qquad \mathbb{1}^* \alpha = \alpha$$

$$d(f \circ g) = df \circ dg$$

$$\Rightarrow (f \circ g)^* = g^* \circ f^*$$

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(1) $F : \mathbb{R}^3 \to \mathbb{R}^2$

$$F(x,y,z) = (\underbrace{x^2 + yz}_{u}, \underbrace{e^{xyz}}_{v})$$

$w = u v^3 \, du \wedge dv$ 2-FORMA NA $\mathbb{R}^2$

$F^* w = ?$

$$F^* w = (x^2+yz)\, e^{3xyz}\, d(x^2+yz) \wedge d(e^{xyz})$$

$$= (x^2+yz)\, e^{3xyz} \left(2x\,dx + z\,dy \wedge y\,dz \right) \wedge$$

$$\wedge \left(e^{xyz}\, yz\,dx \wedge e^{xyz}\, xz\,dy \wedge e^{xyz}\, xy\,dz \right)$$

$$= (x^2+yz)\, e^{3xyz} \Big(2x\, e^{xyz} xz\,dx \wedge dy + z\, e^{xyz}\, yz\,dy \wedge dx$$
$$2x\, e^{xyz}\, xy\,dx \wedge dz + y\, e^{xyz}\, yz\,dz \wedge dx$$
$$z\, e^{xyz}\, xy\,dy \wedge dz + y\, e^{xyz}\, xz\,dz \wedge dy \Big)$$

$$= (x^2+yz)\, e^{4xyz} \Big((2x^2 z - zy^2)\, dx \wedge dy$$
$$+ (2x^2 y - y^2 z)\, dx \wedge dz$$
$$+ (\cancel{zxy - xzy})\, dy \wedge dz \Big)$$

$$= (x^2+yz)\, e^{4xyz} \left(z(2x^2-y^2)\, dx \wedge dy + y(2x^2 - yz)\, dx \wedge dz \right)$$

(2)

$$\alpha : \mathbb{R}^2 \to \mathbb{R}^3$$

$$\alpha(u,v) = (\underbrace{uv}_{x}, \underbrace{u^2}_{y}, \underbrace{3u+v}_{z})$$

$$w = xy\,dx + 2z\,dy - y\,dz \quad \text{na } \mathbb{R}^3$$

$$x = uv \qquad y = u^2 \qquad z = 3u+v$$

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

$dx = v\,du + u\,dv$

$dy = 2u\,du$

$dz = 3\,du + dv$

$$\alpha^* w = u^3 v(v\,du + u\,dv) + 2(3u+v)\,2u\,du$$

$$- u^2(3\,du + dv)$$

$$= (u^3v^2 + 12u - 3u^2)\,du + (u^4 v - u^2)\,dv$$

DOMAĆI:

$f : \mathbb{R}^3 \to \mathbb{R}^2$

$$f(u,v,w) = (\underbrace{u-v}_{x}, \underbrace{v^2-w}_{y})$$

$$w = (2xy + x^2 + 1)\,dx + (x^2 - y)\,dy$$

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(4) $w = x\,dy - y\,dx + z\,dt - t\,dz \in \Omega^1(\mathbb{R}^4)$

$i : S^3 \to \mathbb{R}^4$

DOKAZATI DA

$i^* w \neq 0$ NA S^3

PROIZVOLJNO $\to P = (a, b, c, d) \in S^3$ $\quad a^2 + b^2 + c^2 + d^2 = 1$

GLEDAMO RESTRIKCIJU ω NA T_pS^3

$$v \in T_p S^3 \Rightarrow v \perp p$$

$$a v_x + b v_y + c v_z + d v_t = 0$$

$$v = (-b, a, -d, c) \in T_p S^3$$

$$(i^*(\alpha))|_p (v) = a(a) - b(-b) + c(c) - d(d) =$$

$$= a^2 + b^2 + c^2 + d^2 = 1 \neq 0$$

$\Rightarrow$ SLEDI ZAKLJUČAK

$i^*(\alpha)$ JE NENULA FORMA NA S^3

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(5)

$$\omega = \left(z \sin(xz) + \sqrt{x} + y^2 \cos x\right) dx +$$

$$+ \left(2y e^{\cos z} + 2y \sin x + \operatorname{arctan} y\right) dy +$$

$$+ \left(x \sin(xz) - y^2 \sin z\, e^{\cos z} + 3z e^{z}\right) dz$$

$$d\omega = 2y \cos x \, dy \wedge dx + 2y \cos x \, dx \wedge dy$$

$$+ (\sin(xz) + zx \cos xz) \, dz \wedge dx + (\sin(xz) + xz \cos(xz)) \, dx \wedge dz$$

$$+ 2y e^{\cos z}(-\sin z) \, dz \wedge dy + (2y \sin z \, e^{\cos z}) \, dy \wedge dz$$

$$= 0$$

$\Rightarrow$ FORMA ω JE ZATVORENA

VEKTORSKA POLJA

VEKTORSKO POLJE NA MNOGOSTRUKOSTI M JE PRESLIKAVANJE

$$X : M \to TM = \bigcup_{p \in M} T_p M$$

KOJE ZADOVOLJAVA $X(p) \in T_p M$

IMAMO KANONSKU PROJEKCIJU:

$$\pi : TM \to M \qquad \pi(X_p) = p$$

MOŽE SE DEFINISATI I SA:

$$\pi \circ X = \mathbb{1}_M$$

VEKTOR NABLA

f - FUNKCIJA

$f : U \to \mathbb{R}\ (\mathbb{C})$

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$$

$$\nabla \cdot f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) = \mathrm{GRAD} f$$

SKALAR PRETVARA U VEKTOR

$$\xi : V \to \mathbb{R}^k \ (\mathbb{C}^k)$$

ξ DIF. $\qquad V$ - OTVOREN

$\nabla \cdot \xi \rightsquigarrow$ SHVATAMO KAO SKALARNI PROIZVOD

$$\xi = (\xi_1, \ldots, \xi_k)$$

$$\nabla \cdot \xi = \frac{\partial \xi_1}{\partial x_1} + \frac{\partial \xi_2}{\partial x_2} + \ldots + \frac{\partial \xi_k}{\partial x_k}$$

DIVERGENCIJA

$$\operatorname{div} \xi = \nabla \cdot \xi$$

↓

PRETVARA VEKTOR U SKALAR

GRAD i div SU DEFINISANI U BILO KOJOJ DIMENZIJI.

U $\mathbb{R}^3$ MOŽEMO DEFINISATI ROTOR

$$\operatorname{rot} \xi = \nabla \times \xi$$

$$\xi = (P, Q, R)$$

$$\operatorname{rot} \xi = \nabla \times (P, Q, R) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\frac{\partial R}{\partial x} + \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

LAPLASIJAN

$$\Delta f = \nabla \cdot \nabla f$$

$$\Delta = \nabla^2$$

$$\operatorname{div} \circ \operatorname{rot} = 0$$

$\cancel{\operatorname{rot} \circ \operatorname{div}}$

НЕМА СМИСЛА

$$\xi = (P, Q, R)$$

$$\operatorname{rot} \xi$$

$$\shortparallel$$

$$(A, B, C)$$

$$\operatorname{div}(\operatorname{rot}(\xi)) = \operatorname{div}(A, B, C) = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}$$

$$\operatorname{div}(\operatorname{rot}(\xi)) = \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial R}{\partial x} + \frac{\partial P}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 R}{\partial x \partial y}} - \cancel{\frac{\partial^2 Q}{\partial x \partial z}} - \cancel{\frac{\partial^2 R}{\partial y \partial x}} + \cancel{\frac{\partial^2 P}{\partial y \partial z}} + \cancel{\frac{\partial^2 Q}{\partial z \partial x}} - \cancel{\frac{\partial^2 P}{\partial z \partial y}} = 0$$

ВЕКТОРСКО ПОЉЕ ЈЕ ПОТЕНЦИЈАЛНО

АКО ЈЕ ГРАДИЈЕНТ НЕКЕ ФУНКЦИЈЕ

$$\xi = \nabla f$$

f - ПОТЕНЦИЈАЛ

НЕОПХОДАН УСЛОВ ЈЕ $\operatorname{rot} \xi = 0$

ВЕКТОРСКО ПОЉЕ ЈЕ СОЛЕНОИДНО

АКО ПОСТОЈИ ЊЕГОВ ВЕКТОРСКИ ПОТЕНЦИЈАЛ

$$\xi = \operatorname{rot} \eta$$

НЕОПХОДАН УСЛОВ $\operatorname{div} \xi = 0$

ANALIZA 2

VEŽBE

VEKTORSKA POLJA

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$$ – OPERATOR NABLA

– GRADIJENTNO VEKTORSKO POLJE

$$\text{grad } f = \nabla \cdot f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

– DIVERGENCIJA:

$$\text{div}(P) = \frac{\partial P_1}{\partial x_1} + \frac{\partial P_2}{\partial x_2} + \dots + \frac{\partial P_n}{\partial x_n} = \nabla \cdot P$$

$P : \mathbb{R}^n \to \mathbb{R}^n \qquad P = (P_1, \dots, P_n)$

– $n = 3$

$X = (P, Q, R)$

$$\text{rot } X = \nabla \times X = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

GRAD : SKALAR $\to$ VEKTOR

div : VEKTOR $\to$ SKALAR

rot : VEKTOR $\to$ VEKTOR

rot ∘ div — NEMA SMISLA

rot ∘ GRAD = 0

div ∘ rot = 0

rot ∘ rot = ? (DOMAĆI)

$$(\operatorname{div} \circ \operatorname{GRAD})(f) = \nabla \cdot \nabla f = \nabla \cdot \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$$

$$= \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \dots + \frac{\partial^2 f}{\partial x_n^2} = \Delta(f) \text{ - LAPLASIJAN}$$

$$\Delta = \nabla^2$$

- VEKTORSKO POLJE X JE POTENCIJALNO AKO POSTOJI FUNKCIJA f TDJ.

$$\nabla f = X$$

NEOPHODAN USLOV: $\operatorname{rot} X = 0$

- VEKTORSKO POLJE JE SOLENOIDNO AKO POSTOJI VEKTORSKO POLJE Y TDJ:

$$\operatorname{rot} Y = X$$

Y - VEKTORSKI POTENCIJAL

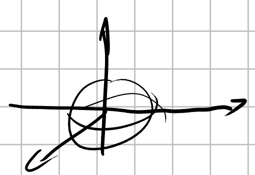
NEOPHODAN USLOV: $\operatorname{div} X = 0$

NEOPHODNI USLOVI SU I DOVOLJNI U PROSTO POVEZANIM PROSTORIMA (CEO $\mathbb{R}^n$ RECIMO)

---

(1) ODREDITI GRADIJENT

$$u = \varphi(r) \qquad r = \sqrt{x^2+y^2+z^2} \qquad \varphi \in D(\mathbb{R})$$



$u = \varphi(r)$

GRAD u : $\nabla \cdot u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right)$

$= \left(\varphi' \frac{x}{r}, \varphi' \frac{y}{r}, \varphi' \frac{z}{r} \right) = \varphi' \frac{\vec{r}}{r}$

$\parallel$

$\varphi' \frac{1}{r} (x, y, z)$

$\parallel$

$\vec{r}$

$u = \varphi(r) = \varphi\left(\sqrt{x^2+y^2+z^2} \right)$

$\frac{\partial u}{\partial x} = \varphi' \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \varphi' \frac{x}{r}$

GRAD $u = \varphi' \frac{\vec{r}}{r}$

---

(2) $f(x,y,z) = xyz$

$\nabla f(M) = ?$ $\qquad M = (-2, 3, 4)$

$\frac{\partial f}{\partial \vec{l}}(M) = ?$ $\qquad \vec{l} = (3, -4, 12)$

$\nabla f = (yz, xz, xy)$

$\nabla f(M) = (12, -8, -6)$

$\frac{\partial f}{\partial \vec{l}}(M) = \nabla f(M) \cdot \frac{\vec{l}}{\|\vec{l}\|} = (12, -8, -6) \cdot \frac{(3, -4, 12)}{13}$

$= \frac{36}{13} + \frac{32}{13} - \frac{72}{13} = \frac{-4}{13}$

---

(3) $\vec{X} = (x, y^2, z^3)$

$A = (-2, 4, 5)$

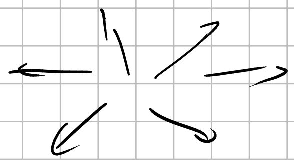
$\operatorname{div}(\vec{X})(A) = ?$

$$\operatorname{div}(\vec{x}) = 1 + 2y + 3z^2$$

$$\operatorname{div}(\vec{x})(A) = 1 + 2 \cdot 4 + 3 \cdot 5^2 = 84$$

---

(4) $\vec{X} = \varphi(r)\,\vec{r}$ – SFERNO VEKTORSKO POLJE



$$r = \sqrt{x^2+y^2+z^2}$$

$$\vec{r} = (x, y, z)$$

$$\varphi \in \mathcal{D}(\mathbb{R})$$

$$\operatorname{div}(\vec{X}) = ?$$

KAKVO TREBA DA JE φ DA POLJE BUDE POTENCIJALNO?

$$\vec{X} = \left(\varphi(\sqrt{x^2+y^2+z^2})\,x,\ \varphi(\sqrt{x^2+y^2+z^2})\,y,\ \varphi(\sqrt{x^2+y^2+z^2})\,z\right)$$

$$\operatorname{div}(\vec{x}) = \varphi + x\varphi'\frac{x}{r} + \varphi + y\varphi'\frac{y}{r} + \varphi + z\varphi'\frac{z}{r} =$$

$$= 3\varphi + \frac{\varphi'}{r}\underbrace{(x^2+y^2+z^2)}_{r^2} = 3\varphi + \varphi' \cdot r$$

$$\operatorname{div}\vec{X} = 3\varphi + \varphi' \cdot r$$

$$\varphi' = \frac{\partial\varphi}{\partial r}$$

DEFINISANO JE U PROSTO POVEZANOJ OBLASTI (??)

PA JE POLJE POTENCIJALNO AKKO $\operatorname{div}\vec{X} = 0$

$$\operatorname{div}\vec{X} = 0 \iff 3\varphi + \varphi' \cdot r = 0$$

$$3\varphi = -\frac{d\varphi}{dr}\,r$$

$$\frac{d\varphi}{\varphi} = -3\,\frac{dr}{r} \quad /\int$$

$$\log \varphi = -3 \log r + \log C$$

$$\log \varphi = \log (C \cdot r^{-3}) \quad / \, e$$

$$\boxed{\varphi = \frac{C}{r^3}} \quad ???$$

OVO NIJE DEFINISANO U $\mathbb{R}^3$.

---

(5)

$$\vec{X} = (z^2, x^2, y^2) \qquad A = (1,2,3)$$

$$\text{rot}(\vec{X})(A) = ?$$

$$\text{rot}\,\vec{X} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x^2 & y^2 \end{vmatrix} = (2y, 2z, 2x)$$

$$\text{rot}(\vec{X})(A) = (4, 6, 2)$$

---

(6)

$$\vec{X} = \varphi(r)\, \vec{r}$$

$$\text{rot}\,\vec{X} = ?$$

$$\text{rot}(\vec{X}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x\varphi(r) & y\varphi(r) & z\varphi(r) \end{vmatrix} =$$

$$= \left(\frac{\partial}{\partial y} z\varphi(r) - \frac{\partial}{\partial z} y\varphi(r), -\frac{\partial}{\partial x} z\varphi(r) + \frac{\partial}{\partial z} x\varphi(r), \frac{\partial}{\partial x} y\varphi(r) - \frac{\partial}{\partial y} x\varphi(r) \right)$$

$$= \left(z\varphi' \frac{y}{r} - y\varphi' \frac{z}{r}, -z\varphi' \frac{x}{r} + x\varphi' \frac{z}{r}, y\varphi' \frac{x}{r} - x\varphi' \frac{y}{r} \right)$$

$$= (0, 0, 0) = \vec{0}$$

⑦ $\vec{X} = (y^2, 2xy, z)$

$$\operatorname{rot} \vec{x} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy & z \end{vmatrix} = (0, 0, 2y-2y) = (0,0,0) = \vec{0}$$

NAĆI POTENCIJAL

$$\vec{X} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\frac{\partial f}{\partial x} = y^2 \implies f = xy^2 + C(y,z)$$

$$\frac{\partial f}{\partial y} = 2xy \qquad \frac{\partial f}{\partial y} = 2xy + \overset{=0}{\frac{\partial C}{\partial y}} \implies C = C(z)$$

$$\frac{\partial f}{\partial z} = z \qquad \boxed{f = xy^2 + \frac{1}{2} z^2 + C}$$

KAKVE OVO VEZE IMA SA FORMAMA?

- 0 $\Omega^0(\mathbb{R}^3) \cong C^\infty(\mathbb{R}^3, \mathbb{R})$
- 1 $\Omega^1(\mathbb{R}^3) \cong C^\infty(\mathbb{R}^3, \mathbb{R}^3)$

  $P\,dx + Q\,dy + R\,dz \rightsquigarrow (P, Q, R)$
- 2 $\Omega^2(\mathbb{R}^3) \cong C^\infty(\mathbb{R}^3, \mathbb{R}^3)$

  $P\,\underbrace{dy \wedge dz} + Q\,\underbrace{dz \wedge dx} + R\,\underbrace{dx \wedge dy} \rightsquigarrow (P, Q, R)$
- 3 $\Omega^3(\mathbb{R}^3) \cong C^\infty(\mathbb{R}^3, \mathbb{R})$

  $f\,dx \wedge dy \wedge dz$

$$
\begin{tikzcd}
0 \arrow[r] & \Omega^0(\mathbb{R}^3) \arrow[r, "d"] \arrow[d, phantom, "\cong"] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"] \arrow[d, phantom, "\cong"] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"] \arrow[d, phantom, "\cong"] & \Omega^3(\mathbb{R}^3) \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\
0 \arrow[r] & C^\infty(\mathbb{R}^3,\mathbb{R}) \arrow[r, "\nabla"', "\mathrm{GRAD}"] & C^\infty(\mathbb{R}^3,\mathbb{R}^3) \arrow[r, "\nabla"', "\mathrm{rot}"] & C^\infty(\mathbb{R}^3,\mathbb{R}^3) \arrow[r, "\nabla"', "\mathrm{div}"] & C^\infty(\mathbb{R}^3,\mathbb{R}) \arrow[r] & 0
\end{tikzcd}
$$

1) 2) 3) 4)

1) TRIVIJALNO

$$
\begin{array}{ccc}
0 & \longrightarrow & 0\,dx + 0\,dy + 0\,dz \\
\downarrow & & \cong \\
0 & \longrightarrow & 0 : \mathbb{R}^3 \to \mathbb{R}
\end{array}
$$

2) $f \in \Omega^0(\mathbb{R}^3)$ PROIZVOLJNO

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \cong \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$f \cong f \xrightarrow{\mathrm{GRAD}} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \quad // \quad \checkmark$$

3) $w = P\,dx + Q\,dy + R\,dz \in \Omega^1(\mathbb{R}^3)$ PROIZVOLJNI

$$dw = \frac{\partial P}{\partial y}dy\wedge dx + \frac{\partial P}{\partial z}dz\wedge dx + \frac{\partial Q}{\partial x}dx\wedge dy + \underline{\frac{\partial Q}{\partial z}dz\wedge dy}$$
$$+ \frac{\partial R}{\partial x}\overline{dx\wedge dz} + \underline{\frac{\partial R}{\partial y}dy\wedge dz}$$
$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dy\wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dz\wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\wedge dy$$
$$\cong$$
$$= \left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right), \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right), \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right)$$

$$w \cong (P, Q, R) \xrightarrow{\mathrm{rot}} \quad //$$

$$\mathrm{rot}(P,Q,R) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

4) $\omega = P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy \in \Omega^2(\mathbb{R}^3)$

$$d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz \cong \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\omega \rightsquigarrow (P, Q, R) \xrightarrow{\operatorname{div}} \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

CEO DIJAGRAM KOMUTIRA

$d^2 = 0 \Rightarrow \operatorname{rot} \circ \operatorname{GRAD} = 0$ i $\operatorname{div} \circ \operatorname{rot} = 0$

$f \in C^\infty(M) \rightsquigarrow \omega_f^0 = f$

$\vec{B} \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \rightsquigarrow \omega^1_{\vec{B}} = \langle \vec{B}, \cdot \rangle$

$n = 3$

MEŠOVITI PROIZVOD

$$\omega^2_{\vec{B}} = (\vec{B}, \cdot, \cdot) = \vec{B} \cdot (\cdot \times \cdot)$$

$$\omega^3_f = f\, dx\, dy\, dz$$

TVRĐENJE:

SVAKA FORMA U $\mathbb{R}^3$ JE DATOG OBLIKA

$\Omega^0(\mathbb{R}^3)$ i $\Omega^3(\mathbb{R}^3)$ TRIVIJALNO

$$A = \{ \omega^1_{\vec{B}} \mid \vec{B} \in C^\infty(\mathbb{R}^3, \mathbb{R}^3) \} \subseteq \Omega^1(\mathbb{R}^3)$$

$\dim A = 3 \quad = \quad \dim \Omega^1(\mathbb{R}^3) = 3$

$\Omega^2(\mathbb{R}^3) = \binom{3}{2} = 3$

MOŽEMO EKVIVALENTNO DEFINISATI

- GRADIJENT $\quad d\omega^0_f = \omega^1_{\text{grad} f}$
- ROTOR $\quad d\omega^1_{\vec{B}} = \omega^2_{\text{rot}\,\vec{B}}$
- DIVERGENCIJA $\quad d\omega^2_{\vec{B}} = \omega^3_{\text{div}\,\vec{B}}$

1/ NA OSNOVU PRETHODNOG DEFINICIJE SU DOBRE

2/ POKLAPAJU SE DEFINICIJE PO KOORDINATAMA

$$\omega^1_{\vec{B}} = \langle \vec{B}, \cdot \rangle = B_1\,dx + B_2\,dy + B_3\,dz$$

$$d\omega^1_{\vec{B}} = \omega^2_{\text{rot}\,\vec{B}}$$

$$\omega^1_{\vec{A}} \wedge \omega^1_{\vec{B}} = \omega^2_{\vec{A}\times\vec{B}}$$

$$\shortparallel$$

$$(A_1\,dx + A_2\,dy + A_3\,dz) \wedge (B_1\,dx + B_2\,dy + B_3\,dz) =$$

$$= \big((A_2 B_3 - A_3 B_2)\,dy\wedge dz + (A_3 B_1 - A_1 B_3)\,dz\wedge dx + (A_1 B_2 - A_2 B_1)\,dx\wedge dy \big)$$

$$= \omega^2_{\vec{A}\times\vec{B}}$$

$$\omega^1_{\vec{A}} \wedge \omega^2_{\vec{B}} = \omega^3_{\vec{A}\cdot\vec{B}}$$

$$(A_1\,dx + A_2\,dy + A_3\,dz) \wedge (B_1\,dy\wedge dz + B_2\,dz\wedge dx + B_3\,dx\wedge dy)$$

$$= \underbrace{(A_1 B_1 + A_2 B_2 + A_3 B_3)}_{\vec{A}\cdot\vec{B}}\,dx\wedge dy\wedge dz$$

ORIJENTACIJA

- U Euklidskom prostoru

Izbor jedne od dve klasa ekvivalencije baza prostora

$x_1, \dots, x_n$ i $y_1, \dots, y_n$ - baze

Baze su ekvivalentne ako za matricu prelaska A važi $\det A > 0$.

- Dualno možemo izabrati

n 1-formi

$$\mu_1, \mu_2, \dots, \mu_n$$

ili

jednu n-formu $\quad \dim \Omega^n(V) = 1$

Dve forme su ekvivalentne ako je

$$\alpha = c\beta \qquad c > 0$$

μ - forma orijentacije

$x_1, \dots, x_n$ - baza

Zadaju istu orijentaciju ako je

$$\mu(x_1, \dots, x_n) > 0$$

- Opštije

$U, V \in \mathcal{T}_{\mathbb{R}^n}$

μ - netrivijalna n-forma na V

$$f : U \to V \quad \text{glatko}$$

$$f^* \mu = \underset{\substack{\uparrow \\ \text{Jakobijan}}}{\det}(f_*)\, \mu$$

f difeomorfizam $\Rightarrow f^* \mu$ netrivijalna forma

$\det(f_*) > 0$ — DEFINIŠU ISTU ORIJENTACIJU

KAŽEMO DA f ČUVA ORIJENTACIJU

INTEGRACIJA NA MNOGOSTRUKOSTIMA

V - GLATKA MNOGOSTRUKOST PREKRIVENA JEDNOM KARTOM

(V, φ)

$\alpha \in \Omega^n(V)$

α - GLATKA n-FORMA SA KOMPAKTNIM NOSAČEM.

$$(\varphi^{-1})^* \alpha = a(x_1, \dots, x_n)\, dx_1 \wedge \dots \wedge dx_n$$

$a : \mathbb{R}^n \to \mathbb{R}$

GLATKA FUNKCIJA SA KOMPAKTNIM NOSAČEM.

$$\int_V \alpha = \int_{\mathbb{R}^n} a(x_1, \dots, x_n)\, dx_1 \wedge \dots \wedge dx_n$$

RIMANOV INTEGRAL

- NE ZAVISI OD IZBORA KARTE

- SPECIJALNO $n = 1$

J - INTERVAL

$\gamma : J \to \mathbb{R}^n$

PARAMETRIZOVANA KRIVA

$C = \gamma(J)$

NEPARAMETRIZOVANA KRIVA

$F : C \to \mathbb{R}^k$

VEKTORSKO POLJE

DEFINIŠE 1-FORMU

$$\alpha \in \Omega^1(C)$$

$$\alpha(X_p) = X_p \cdot F(p) \qquad X_p \in T_p M$$

KRIVOLINIJSKI INTEGRAL $\underline{\mathrm{II}}$ VRSTE:

$$\int_C \vec{F} \cdot d\vec{r}$$

AKO JE KRIVA ZATVORENA KORISTIMO OZNAKU

$$\oint_C \vec{F}\, d\vec{r}$$

- ORIJENTACIJA JE ZADATA PARAMETRIZACIJOM (RASTOM PARAMETRA)

$$\int_C \vec{F}\, d\vec{r} = \int_C \vec{F} \frac{d\vec{r}}{dt}\, dt$$

RIMANOV INTEGRAL

$$\vec{F} = (P, Q, R)$$
$$d\vec{r} = (dx, dy, dz)$$

$$\int_C P\, dx + Q\, dy + R\, dz$$

$$\int_C P\, dx = \int_C P(x(t), y(t), z(t)) \frac{dx}{dt}\, dt$$

OSOBINE:

- LINEARNOST

$$\int_\gamma (\alpha P_1 + \beta P_2)\, dx = \alpha \int_\gamma P_1\, dx + \beta \int_\gamma P_2\, dx$$

- ADITIVNOST PO SKUPU

C između A, B

$$\int_{AB} P\, dx = \int_{AC} P\, dx + \int_{CB} P\, dx$$

Opštije:

$$S = S_1 \cup S_2$$

$$\mu(S_1 \cap S_2) = 0$$

$$\int_S \omega = \int_{S_1} \omega + \int_{S_2} \omega$$

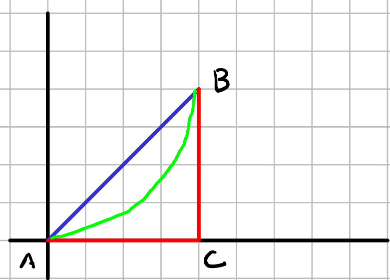
- ZAVISI OD ORIJENTACIJE

$$\int_{BA} P\, dx = -\int_{AB} P\, dx$$

ZADACI:

(1) $$\int_{AB} 2xy\, dx + x^2\, dy$$

$A = (0,0) \qquad B = (1,1)$



γ_1 : $\quad y = x \quad \Rightarrow \quad dy = dx$

$$\int_{AB} 2xy\, dx + x^2\, dy = \int_0^1 2x^2\, dx + \int_0^1 x^2\, dx$$

$$= 3\int_0^1 x^2\, dx = 3\left(\frac{1}{3}x^3\right)\Big|_0^1 = 1$$

$\underline{\gamma_2}$

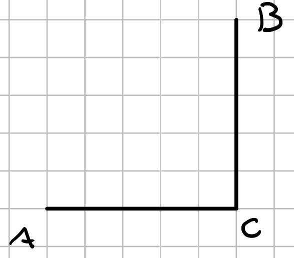
$$y = x^2 \qquad 0 \rightsquigarrow x \rightsquigarrow 1$$

$$dy = 2x\,dx$$

$$\int_{AB} 2xy\,dx + x^2\,dy = \int_0^1 2x^3\,dx + x^2\,2x\,dx$$

$$= 4\int_0^1 x^3\,dx = 4\left(\frac{1}{4}x^4\right)\Big|_0^1 = 1$$

$\underline{\gamma_3 :}$



$$\int_{AB} w = \int_{AC} w + \int_{CB} w$$

$x = t \qquad y = 0$

$0 < x \leq 1$

$x = 1 \qquad 0 \leq y \leq 1$

$dx = 0$

$$\int_{AB} w = \int_0^1 2x\cdot 0\,dx + \int_0^1 2y\underbrace{dx}_{0} + 1\,dy = \int_0^1 dy = 1$$

$$w = 2xy\,dx + x^2\,dy$$

$$dw = 2x\,dy\wedge dx + 2x\,dx\wedge dy = 0$$

$$dy\wedge dx = -\,dx\wedge dy$$

$$dx\wedge dx = 0 \qquad dy\wedge dy = 0$$

(2)

$$\oint_L (x+y)\,dx + (x-y)\,dy$$

$L: \quad (x-1)^2 + (y-1)^2 = 4$

$x = 1 + 2\cos t \qquad y = 1 + 2\sin t \qquad 0 \leq t \leq 2\pi$

$dx = -2\sin t\,dt \qquad dy = 2\cos t\,dt$

$$I = \int_0^{2\pi} (1+2\cos t + 1 + 2\sin t)(-2\sin t\,dt) +$$

$$(\cancel{1}+2\cos t - \cancel{1} - 2\sin t)\,2\cos t\,dt$$

$$= 4\int_0^{2\pi} \big(-\sin t - 2\cos t\sin t - \sin t - 2\sin^2 t +$$

$$+ \cos t + 4\cos^2 t - \cos t - 2\sin t\cos t\big)\,dt$$

$$= -8\int_0^{2\pi} \sin t\,dt - 4\int_0^{2\pi} \sin t\cos t\,dt + 4\int_0^{2\pi} (\cos^2 t - \sin^2 t)\,dt$$

$$= 8\cos t\Big|_0^{2\pi} - 2\int_0^{2\pi} \cancel{\sin 2t\,dt}^{\,0} + 4\int_0^{2\pi} \cancel{\cos 2t\,dt}^{\,0} = 0$$

---

(3)

$$\oint_L y^2\,dx + z^2\,dy + x^2\,dz$$

L – VIVIJAHIJEVA KRIVA

$L: \quad x^2 + y^2 + z^2 = a^2, \quad z \geq 0, \quad x^2 + y^2 = ax, \quad a > 0$

$$x^2 - ax + y^2 = 0$$

$$x^2 - 2\frac{a}{2}x + \frac{a^2}{4} + y^2 = \frac{a^2}{4}$$

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$$

$$y = \frac{a}{2}\sin t \qquad x = \frac{a}{2} + \frac{a}{2}\cos t \qquad 0 \le t \le 2\pi$$

$$dy = \frac{a}{2}\cos t \, dt \qquad dx = -\frac{a}{2}\sin t \, dt$$

$$z = \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - \left(\frac{a}{2} + \frac{a}{2}\cos t\right)^2 - \left(\frac{a}{2}\sin t\right)^2} =$$

$$= \sqrt{\frac{a^2}{2} - \frac{a^2}{2}\cos t} = a\sqrt{\frac{1-\cos t}{2}}$$

$$= a\left|\sin\frac{t}{2}\right| = a\sin\frac{t}{2}$$

$$\boxed{\begin{array}{l} t \in [0, 2\pi] \\ \frac{t}{2} \in [0, \pi] \end{array}}$$

$$dz = \frac{a}{2}\cos\frac{t}{2}$$

$$I = \int_0^{2\pi} \frac{a^2}{4}\sin^2 t\left(-\frac{a}{2}\sin t \, dt\right) + a^2\sin^2\frac{t}{2}\left(\frac{a}{2}\cos t \, dt\right)$$

$$+ \left(\frac{a}{2} + \frac{a}{2}\cos t\right)^2 \frac{a}{2}\cos\frac{t}{2} \, dt$$

$$= \ldots$$

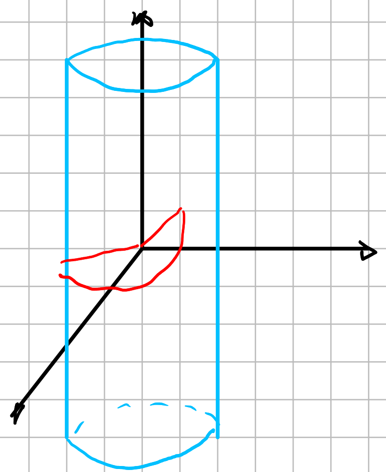
③ $$\int_\gamma \frac{xy}{x^2+y^2+1}\,dx + x\,dy + y\,dz$$

γ - PRESEK RAVNI $x+y+z=2$

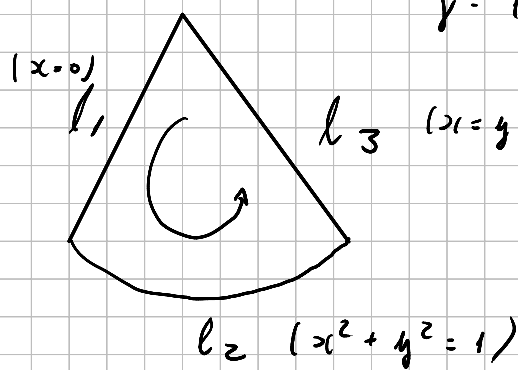
SA RUBOM TELA ODREĐENOG SA

$$x^2+y^2 \le 1 \qquad 0 \le x \le y$$

(ORIJENTACIJA SA SLIKE)



$\gamma = l_1 \cup l_2 \cup l_3$



l_1: $x=0$ $\quad y$ - PARAMETAR $\quad z=2-y \qquad 0\le y\le 1$

$dx=0 \qquad dy \qquad dz=-dy$

l_2: $x=\cos t \qquad y=\sin t \qquad z=2-\sin t-\cos t \qquad t\in[\frac{\pi}{4},\frac{\pi}{2}]$

$dx=-\sin t\,dt \qquad dy=\cos t\,dt \qquad dz=(\sin t-\cos t)\,dt$

l_3: $x=y$ - PARAMETAR $x \qquad z=2-2x \qquad 1 \rightsquigarrow x \rightsquigarrow 0$

$dx=dy \qquad dz=-2\,dx$

$$I=\int_\gamma \omega = \int_{l_1}\omega + \int_{l_2}\omega + \int_{l_3}\omega =$$

$$= \int_0^1 y(-dy) + \int_{\pi/4}^{\pi/2}\left(\frac{\sin t\cos t}{2}(-\sin t) + \cos^2 t + \sin t(\sin t-\cos t)\right)dt$$

$$+ \int_1^0 \left(\frac{x^2}{2x^2+1} + x + x(-2)\right)dx$$

$$= -\frac{y^2}{2}\Big|_0^1 - \frac{1}{2}\int_{\pi/4}^{\pi/2} \sin^2 t \, d(\sin t) + \int_{\pi/4}^{\pi/2} \underbrace{(\cos^2 t + \sin^2 t)}_{1}\,dt - \int_{\pi/4}^{\pi/2} \sin t \, d(\sin t)$$

$$-\frac{1}{2}\int_0^1 \frac{2x^2+1-1}{2x^2+1}\,dx + \int_0^1 x\,dx$$

$$= -\frac{1}{2} - \frac{1}{2}\,\frac{1}{3}\sin^3 t\Big|_{\pi/4}^{\pi/2} + t\Big|_{\pi/4}^{\pi/2} - \frac{1}{2}\sin^2 t\Big|_{\pi/4}^{\pi/2}$$

$$-\frac{1}{2}\int_0^1 dx + \int_0^1 \frac{dx}{1+2x^2} + \frac{1}{2}x^2\Big|_0^1$$

$$= -\frac{1}{2} - \frac{1}{6}\left(1 - \frac{\sqrt{2}}{4}\right) + \frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\left(1 - \frac{1}{2}\right) - \frac{1}{2} +$$

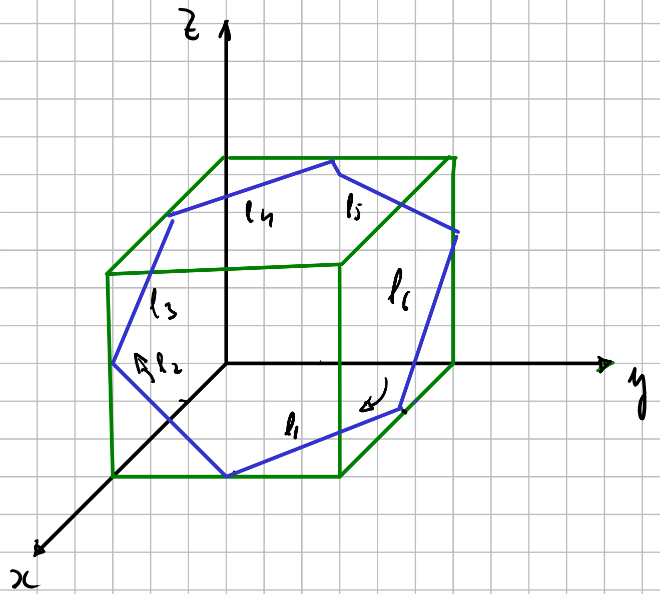
$$+ \frac{1}{\sqrt{2}}\arctan\sqrt{2}x\Big|_0^1 + \frac{1}{2} = \ldots$$

---

4. $$\oint_\gamma \underbrace{(y^2-z^2)\,dx + (z^2-x^2)\,dy + (x^2-y^2)\,dz}_{0}$$

$$\gamma : \partial\left([0,2]^3\right) \cap \quad x+y+z=3$$

POZITIVNO ORIJENTISANA GLEDANO IZ TAČKE $(-100, 0, 0)$



$z = 0 \Rightarrow x + y = 3$

$z = 2 \Rightarrow x + y = 1$

MORAMO DELITI NA 6 DUŽI

l_1 : $z = 0$ x - PARAMETAR $1 \le x \le 2$ $y = 3 - x$

$dz = 0 \qquad dx \qquad dy = -dx$

$$\int_{l_1} \omega = \int_1^2 (3-x)^2\,dx - x^2(-dx) = \int_1^2 \left(9 - 6x + 2x^2\right)dx$$

$$= \left(9x - 3x^2 + \frac{2}{3}x^3\right)\Big|_1^2 = 18 - 12 + \frac{2}{3}8 - 9 + 3 - \frac{2}{3} = \ldots$$

l_2: $x = 2$ $\quad$ y - PARAMETAR $\quad 1 \rightsquigarrow y \rightsquigarrow 0 \quad z = 1 - y$

$dx = 0 \qquad dy \qquad dz = -dy$

$$\int_{l_2} \omega = \int_1^0 (1-y)^2\,dy - y^2(-dy) =$$

$$= \int_0^1 \left(-1 + 2y - 2y^2\right)dy = \left(-y + y^2 - \frac{2}{3}y^3\right)\Big|_0^1 = -\frac{2}{3}$$

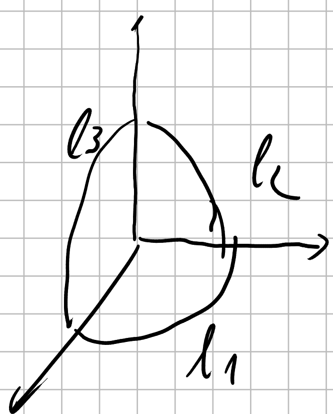
ZA DOMAĆI ZAVRŠITI NA OSTALIM DUŽIMA.

---

⑤ $$\oint_L (y^2 - z^2)\,dx + (z^2 - x^2)\,dy + (x^2 - y^2)\,dz$$

L - PRESEK SFERE S^2 SA KOORDINATNIM RAVNIMA U PRVOM OKTANTU

$$L: S^2 \cap \{x = 0,\ y \geq 0,\ z \geq 0\}$$
$$\cup$$
$$S^2 \cap \{y = 0,\ x \geq 0,\ z \geq 0\}$$
$$\cup$$
$$S^2 \cap \{z = 0,\ x \geq 0,\ y \geq 0\}$$



$x \rightsquigarrow y \rightsquigarrow z \rightsquigarrow x$ $\qquad$ IMAMO 3 ISTA INTEGRALA

$$I = 3\int_{l_1} w$$

PO SVIM l-OVIMA ISTO

$$w\Big|_{l_1} = y^2\,dx - x^2\,dy$$

$x = \cos t \qquad y = \sin t \qquad 0 \le t \le \frac{\pi}{2}$

$$I = 3\int_0^{\pi/2} \sin^2 t\,(-\sin t)\,dt - \cos^2 t \cos t\,dt = \ldots$$

---

(5) $\vec{F} = (x, y, z)$

AB DUŽ IZMEĐU $A(1,1,1)$ I $B(2,1,-1)$

$$\int_{AB} \vec{F}\,d\vec{r} = ?$$

PARAMETRIZUJEMO DUŽ AB

$\vec{AB} = (1, 0, -2)$

$$\frac{x-1}{1} = \frac{y-1}{0} = \frac{z-1}{-2} = t \qquad t \in [0,1]$$

$x = 1+t \qquad y = 1 \qquad z = 1-2t$

$dx = dt \qquad dy = 0 \qquad dz = -2\,dt$

$$\int_{AB} \vec{F}\,d\vec{r} = \int_{AB} x\,dx + y\,dy + z\,dz = \int_0^1 (1+t)\,dt + 0 + (1-2t)(-2\,dt)$$

$= \ldots$

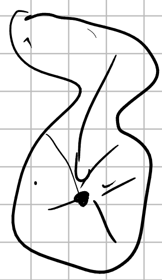
TEOREMA: (STOKS)

ω - GLATKA n-FORMA

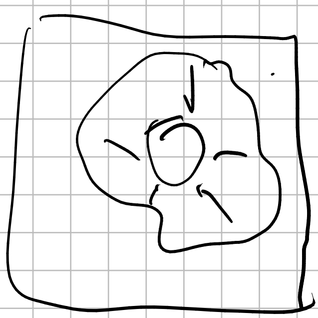
$\gamma = \partial S \qquad S$ - PROSTO POVEZANA OBLAST

$$\boxed{\int_\gamma \omega = \int_S d\omega}$$

S - PROSTOPOVEZANA



$\gamma : S^1 \to S$



ANALIZA 2

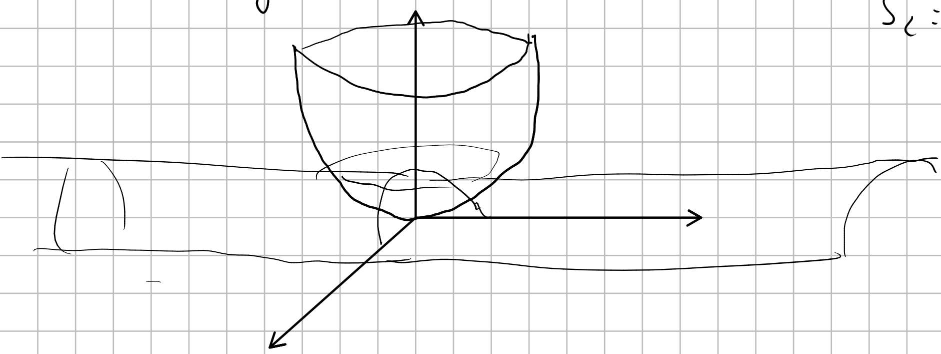
VEŽBE

(1) $\int_\gamma y\,dx + z\,dy + x\,dz$

$\gamma: S_1 \cap S_2$

$S_1: z = 1 - x^2$

$S_2: z = x^2 + y^2$



$1 - x^2 = x^2 + y^2$

$2x^2 + y^2 = 1$

$(\sqrt{2}x)^2 + y^2 = 1$

$\sqrt{2}x = \cos\varphi \qquad y = \sin\varphi \qquad \varphi \in [0, 2\pi)$

$x = \frac{1}{\sqrt{2}}\cos\varphi$

$dx = -\frac{1}{\sqrt{2}}\sin\varphi\,d\varphi \qquad dy = \cos\varphi\,d\varphi$

$z = 1 - x^2 = 1 - \frac{1}{2}\cos^2\varphi$

$dz = \cos\varphi\sin\varphi\,d\varphi$

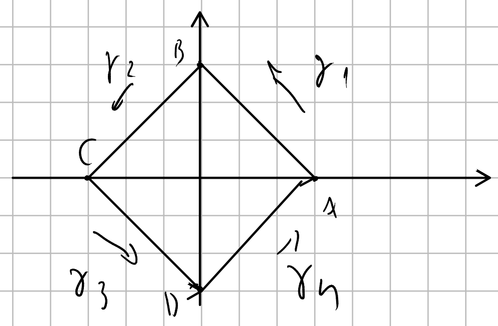
$$I = \int_0^{2\pi} \sin\varphi\left(-\frac{1}{\sqrt{2}}\right)\sin\varphi\,d\varphi + \left(1 - \frac{1}{2}\cos^2\varphi\right)\cos\varphi\,d\varphi + \frac{1}{\sqrt{2}}\cos\varphi\cos\varphi\sin\varphi\,d\varphi$$

$= \ldots$

(2) $\oint\limits_{ABCDA} \frac{dx+dy}{|x|+|y|}$

ABCDA - KVADRAT SA TEMENIMA

$A(1,0)$ $B(0,1)$ $C(-1,0)$ $D(0,-1)$



$I = \int\limits_{\gamma_1} w + \int\limits_{\gamma_2} w + \int\limits_{\gamma_3} w + \int\limits_{\gamma_4} w$

$w = \frac{dx+dy}{|x|+|y|}$

$\underline{\gamma_1}$ $x, y \geq 0$ $\int\limits_{\gamma_1} \frac{dx+dy}{x+y} = \int_1^0 \frac{dx-dx}{\cancel{x}+1-\cancel{x}} = 0$

$x+y=1$ $y=1-x$ $dy=-dx$ $1 \rightsquigarrow x \rightsquigarrow 0$

$\underline{\gamma_2}$ $x \leq 0$, $y \geq 0$ $0 \rightsquigarrow x \rightsquigarrow -1$

$y = x+1$ $dy = dx$

$\int\limits_{\gamma_2} w = \int_0^{-1} \frac{dx+dx}{-\cancel{x}+\cancel{x}+1} = 2\int_0^{-1} dx = -2$

$\underline{\gamma_3}$ $x \leq 0$, $y \leq 0$

$y = -x-1$ $-1 \rightsquigarrow x \rightsquigarrow 0$ $\int\limits_{\gamma_3} w = \int_{-1}^{0} \frac{dx-dx}{-\cancel{x}+\cancel{x}+1} = 0$

$dy = -dx$

$\underline{\gamma_4}$ $x \geq 0$, $y \leq 0$

$y = x-1$ $0 \rightsquigarrow x \rightsquigarrow 1$ $\int\limits_{\gamma_4} w = \int_0^1 \frac{2\,dx}{+x-x+1} = 2$

$dy = dx$

(3) $$\oint_\gamma \underbrace{(x^2+2xy-y^2)\,dx + (x^2-2xy+y^2)\,dy}_{w}$$

$$\gamma: \; 5(x-3)^2 + (y-4)^2 = 17$$

DA LI w IMA PRIMITIVNU?
(DA LI JE TAČNA?)

DA LI POSTOJI η TDJ. $d\eta = w$

$$\frac{\partial \eta}{\partial x} = x^2 + 2xy - y^2$$

$$\eta = \int (x^2+2xy-y^2)\,dx$$

$$\eta = \frac{1}{3}x^3 + x^2y - y^2x + C(y)$$

$$\frac{\partial \eta}{\partial y} = x^2 - 2xy + C'(y) = \frac{\partial \eta}{\partial y} = x^2 - 2xy + y^2$$

$$C'(y) = y^2 \qquad C(y) = \frac{1}{3}y^3 + D$$

$$\eta = \frac{1}{3}x^3 + x^2y - y^2x + \frac{1}{3}y^3$$

γ JE ZATVORENA KRIVA (POČETNA I KRAJNJA TAČKA JE ISTA, RECIMO M)

$$\int_\gamma w = \eta(M) - \eta(M) = 0$$

---

(4) $$\int_\gamma w$$

$$w = \underline{(x^2-yz)}\,dx + (y^2-2xz)\,dy + (z^2-2xy)\,dz$$

γ: $S^2 \cap x+y+z=1$ IZMEĐU TAČAKA $A(1,0,0)$ I $B(0,1,0)$

Tražimo η tdj. $d\eta = \omega$

$$\frac{\partial\eta}{\partial x} = x^2 - 2yz$$

$$\eta = \frac{1}{3}x^3 - 2xyz + C(y,z)$$

$$\frac{\partial\eta}{\partial y} = -2xz + C'_y \qquad \frac{\partial\eta}{\partial y} = y^2 - 2xz$$

$$\Rightarrow C = \frac{1}{3}y^3 + D(z)$$

$$\eta = \frac{1}{3}x^3 + \frac{1}{3}y^3 - 2xyz + D(z)$$

$$\frac{\partial\eta}{\partial z} = -2xy + D' \qquad \frac{\partial\eta}{\partial z} = z^2 - 2xy$$

$$D = \frac{1}{3}z^3$$

$$\eta = \frac{x^3+y^3+z^3}{3} - 2xyz$$

$$\int_\gamma \omega = \eta(A) - \eta(B) = \frac{1}{3} - \frac{1}{3} = 0$$

$$\int_\gamma \vec{F}\, d\vec{r}$$

Teorema:

V - oblast (povezan, otvoren)

$V \subseteq \mathbb{R}^3$

$F \in C(V)$

Ekvivalentno je sledeće:

1) Postoji funkcija $u: V \to \mathbb{R}$ sa nepre-kidnim parcijalnim izvodima tdj.:

$$\nabla u = \vec{F},$$

2) $\int_{\gamma} \vec{F}\, d\vec{r}$ NE ZAVISI OD PUTA VEĆ SAMO OD KRAJNJIH TAČAKA $(u(B) - u(A))$

3) $\oint_{C} \vec{F}\, d\vec{r} = 0 \qquad C \subseteq V$

KADA OVO VAŽI?

KADA JE ZATVORENA FORMA I TAČNA?

<u>HOMOTOPIJA:</u>

$$f_1 : X \to Y$$

$$f_2 : X \to Y$$

$$H(x,t) = t f_1(x) + (1-t) f_2(x)$$

$$H(x,0) = f_2(x) \qquad H(x,1) = f_1(x)$$

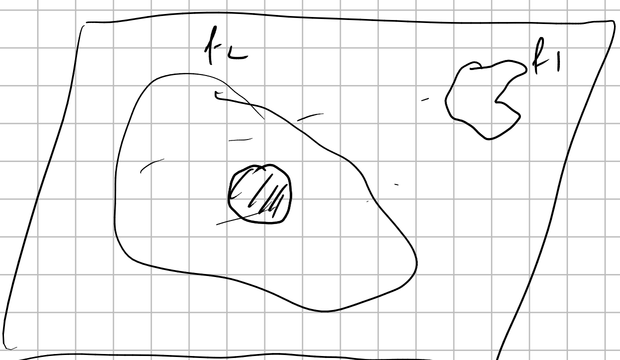
HOMOTOPIJA JE SVAKO PRESLIKAVANJE

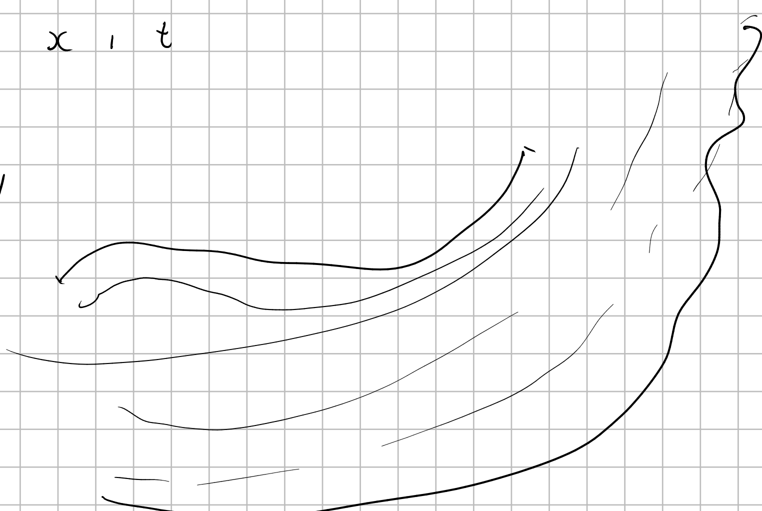
$$H : X \times [0,1] \to Y$$

H NEPREKIDNO PO x I t

$$H(x,0) = f_1(x)$$

$$H(x,1) = f_2(x)$$





$f_1 \simeq f_2$ – OZNAKA ZA HOMOTOPIJU

$X \simeq *$ – AKO POSTOJI HOMOTOPIJA

$*$ - TAČKA $\qquad id_X$ i $f(x) = *$

$X \simeq * \Rightarrow$ MOŽEMO PRIMENITI STOKSOVU TEOREMU

$$\int_{\partial X} \omega = \int_X d\omega \qquad \omega \text{ - NEPREKIDNA NA } X$$

(MOGUĆI SU I BLAŽI USLOVI)

ŠTA JE HOMOTOPNO TAČKI

$\mathbb{R}^n$, $B(x, r)$, KONVEKSNI SKUPOVI, ...

ŠTA NIJE HOMOTOPNO TAČKI

TORUS, SFERA, RAVAN SA „RUPAMA", ..

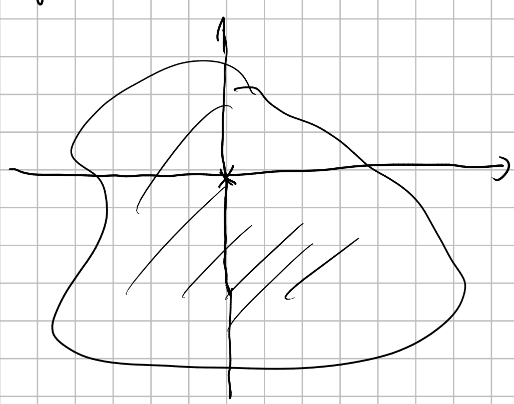
<u>GRINOVA FORMULA:</u>

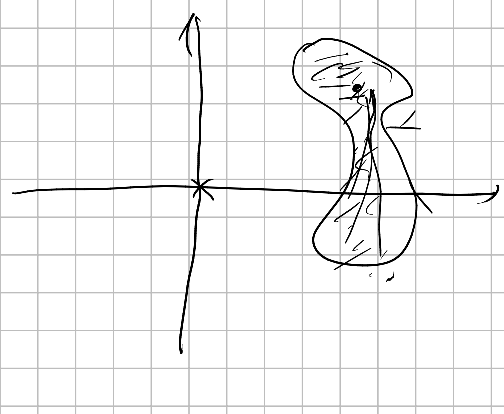
$$\int_{\gamma = \partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

RIMANOV INTEGRAL

D MORA DA BUDE „LEPA" OBLAST.

$$\oint_\gamma \frac{-y\,dx + x\,dy}{x^2 + y^2} \qquad (0,0) \notin \gamma$$





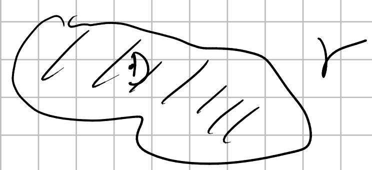
$$w = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$dw = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}\,dy \wedge dx + \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\,dx \wedge dy$$

$$= \frac{-y^2+x^2}{(x^2+y^2)^2}\,dx \wedge dy + \frac{y^2-x^2}{(x^2+y^2)^2}\,dx \wedge dy = 0$$

W JE ZATVORENA I BIĆE TAČNA U PROSTOPOVEZANOJ OBLASTI.

$\gamma = \partial D$

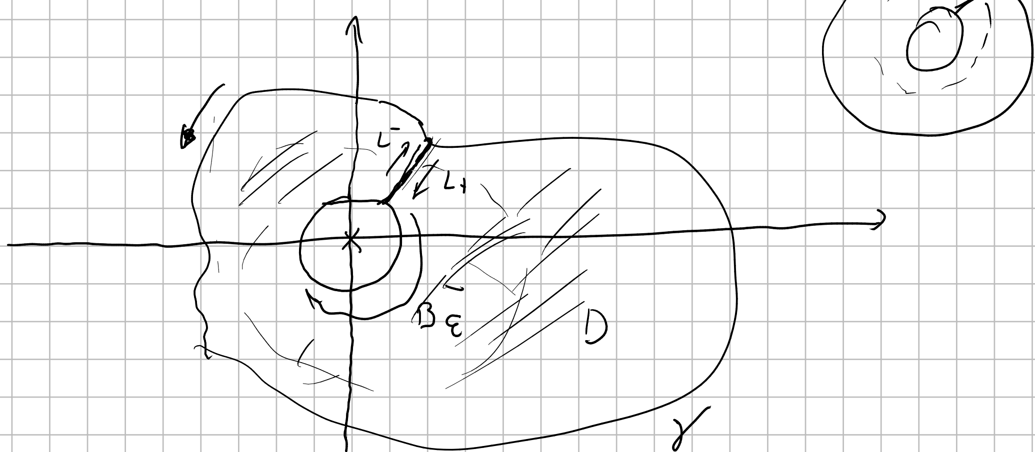


D - PROSTOPOVEZANA

$$\int_\gamma w = \iint_D dw = \iint_D 0 = 0$$

D JE PROSTOPOVEZANA AKO $(0,0) \notin D$

2) $(0,0) \in D$



$$\int_\gamma w + \cancel{\int_{L^+} w} + \int_{B_\varepsilon^-} w + \cancel{\int_{L^-} w} = \iint_D dw = 0$$

$$\int_\gamma w = \int_{B_\varepsilon^+} w$$

INTEGRAL NE ZAVISI OD IZBORA KRIVE KOJA OKRUŽUJE TAČKU $(0,0)$

$$\int_\gamma \omega = \int_{B_\varepsilon} \omega = \int_0^{2\pi} \frac{+\varepsilon^2 \sin^2\varphi + \varepsilon^2 \cos^2\varphi}{\varepsilon^2}\, d\varphi = \boxed{2\pi}$$

$$x = \varepsilon \cos\varphi \qquad y = \varepsilon \sin\varphi$$

$$dx = -\varepsilon \sin\varphi \, d\varphi \qquad dy = \varepsilon \cos\varphi \, d\varphi$$

ZAKLJUČAK:

$$\gamma = \partial D$$

1) $(0,0) \in D$ $\qquad \int_\gamma \omega = 2\pi$

2) $(0,0) \notin D$ $\qquad \int_\gamma \omega = 0$

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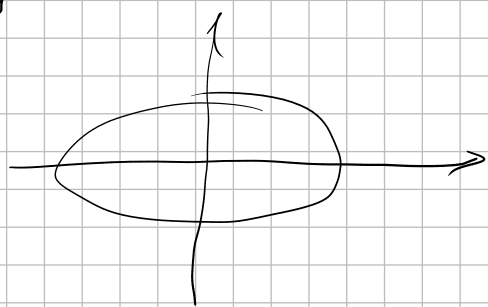
② $$\oint_L (x+y)\,dx - (x-y)\,dy$$

$$L: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



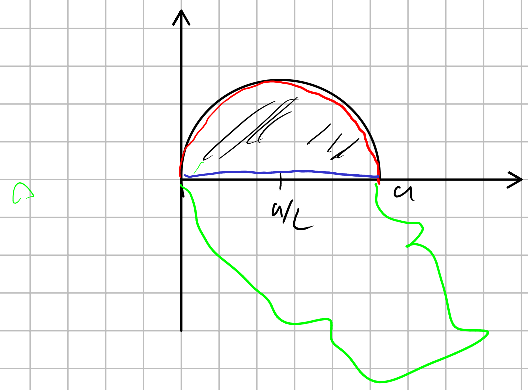
$$L = \partial D \qquad \pi_1(D) = 0$$
$$\Downarrow$$
$$H_1^{DR}(D) = 0$$

$$\oint_L (x+y)\,dx - (x-y)\,dy = \iint_D dy \wedge dx - dx \wedge dy$$

$$= -2 \iint_D dx\, dy = \boxed{-2\pi ab}$$

③ $I = \int\limits_{\widehat{OA}} (e^x \sin y - py)\,dx + (e^x \cos y + px)\,dy \quad p \in \mathbb{R}$

$\widehat{OA}$ - GORNJA POLU KRUŽNICA KRUGA $x^2 + y^2 = ax$



$x^2 + y^2 = ax$

$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$

l_1 - DUŽ OD $(0,0)$ DO $(a,0)$

$$\int\limits_{\widehat{OA}} w + \int\limits_{l_1} w = \iint\limits_D dw$$

$$dw = d\left((e^x \sin y - py)\,dx + (e^x \cos y + px)\,dy\right)$$

$$= (\cancel{e^x \cos y} - p)\,dy \wedge dx + (\cancel{e^x \cos y} + p)\,dx\,dy$$

$$= 2p\,dx\,dy$$

$$\iint\limits_D dw = \iint\limits_D 2p\,dx\,dy = 2p \iint\limits_D dx\,dy = 2p\,\frac{1}{4}\pi\left(\frac{a}{2}\right)^2$$

$$= p\pi\,\frac{a^2}{4}$$

$w|_{l_1} = ?$ $\quad w|_{l_1} = 0$ (JER JE $y = 0$, $dy = 0$)

$$\int\limits_{\widehat{AB}} w = \iint\limits_D dw = p\pi\,\frac{a^2}{4}$$

---

④ $\vec{F} = (3x^2 + y^2,\ 3y^2 + 2xy)$

$\gamma:\ y = (x-1)^2 (x+1)^2$

IZMEĐU TAČAKA $(-1,0)$ i $(0,1)$

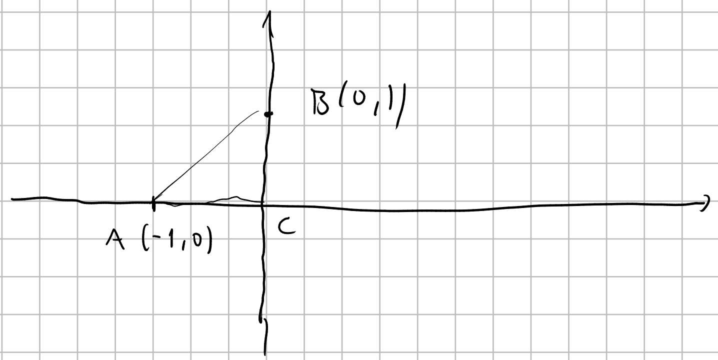
$$\int_\gamma \vec{F} \cdot d\vec{r}$$

$$I = \int_\gamma \underbrace{(3x^2 + y^2)\, dx + (3y^2 + 2xy)\, dy}_{\omega}$$

ω JE DEFINISANA SVUGDE (NEMA SINGULARITETA)

$$I = \iint_D 2y \; dy \wedge dx + 2y \; dx \wedge dy = 0$$

AKO JE FORMA TAČNA INTEGRAL NE ZAVISI OD PUTA.
OVDE JE TAČNA JER JE ZATVORENA U $\mathbb{R}^2$
i $\pi_1(\mathbb{R}^2)$ $(\pi_1(D) = 0)$



NE ZAVISI OD PUTA, PA JE:

$$I = \int_{AC} \omega + \int_{CB} \omega$$

$\omega|_{AC}$ $y = 0$ $dy = 0$

$\omega|_{AC} = 3x^2\, dx$

$$\int_{AC} \omega = \int_{-1}^{0} 3x^2\, dx = x^3 \Big|_{-1}^{0} = 1$$

$\omega|_{CB} = 3y^2\, dy$ $x = 0$ $dx = 0$

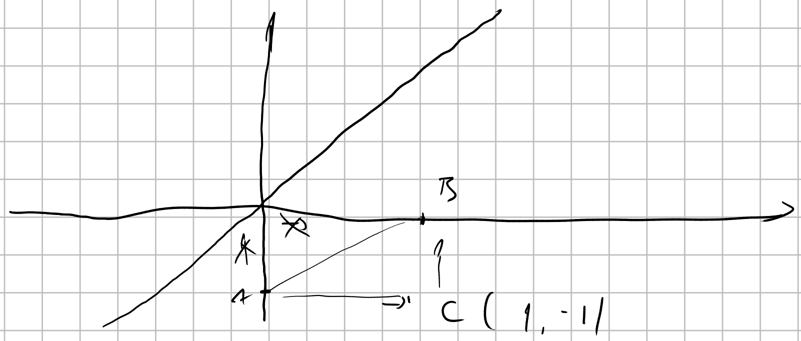
$$\int_{CB} \omega = \int_0^1 3y^2\,dy = y^3\Big|_0^1 = 1$$

$$\Rightarrow \quad \boxed{I = 2}$$

$$\int_\gamma \underbrace{\frac{x\,dy - y\,dx}{(x-y)^2}}_{\omega} \qquad A(0,-1) \text{ DO } B(1,0)$$

γ - KOMPLIKOVANA KRIVA

$d\omega = 0 \quad \Rightarrow$ INTEGRAL NE ZAVISI OD PUTA



$\omega\big|_{AC} = \frac{dx}{(x+1)^2} \qquad 0 \rightsquigarrow x \rightsquigarrow 1$

$\downarrow$

$y=-1 \quad dy = 0$

$$\int_{AC} \omega = \int_0^1 \frac{dx}{(x+1)^2} = -\frac{1}{x+1}\Big|_0^1 = -\frac{1}{2}+1 = \frac{1}{2}$$

$\omega\big|_{CB} = \frac{dy}{(1-y)^2}$

$\downarrow$

$x=1 \quad dx = 0$

$$\int_{CB} \omega = \int_{-1}^0 \frac{dy}{(1-y)^2} = \frac{1}{1-y}\Big|_{-1}^0 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\boxed{I = 1}$$

DRUGI NAČIN (DUŽ AB)

$$p: \quad \frac{x-0}{1} = \frac{y+1}{1} = t$$

$\overrightarrow{AB} = (1,1)$

$x = t \qquad y = t-1$

$dx = dt \qquad dy = dt \qquad - - - -$

A N A L I Z A 2

V E Ž B E

(1) $I = \oint\limits_L (y^2 - z^2)\, dx + 2yz\, dy - x^2\, dz$

$L: \quad x = t \quad y = t^2 \quad z = t^3 \qquad 0 \le t \le 1$

$dx = dt \quad dy = 2t\,dt \quad dz = 3t^2\, dt$

$$I = \int_0^1 (t^4 - t^6)\, dt + 2t^2 t^3 (2t\, dt) - t^2 (3t^2 dt)$$

$$= \int_0^1 (t^4 - t^6 + 4t^6 - 3t^4)\, dt$$

$$= \int_0^1 (3t^6 - 2t^4)\, dt$$

$$= \left(\frac{3}{7} t^7 - \frac{2}{5} t^5\right)\Big|_0^1 = \frac{3}{7} - \frac{2}{5}$$

---

(2) $I = \int\limits_\alpha \vec{F} \cdot d\vec{\alpha}$

$\vec{F} = (y^2 \cos x,\ 2y \sin x + e^z,\ y\, e^z)$

$\alpha(t) = (t \cos \pi t,\ t e^t,\ \log(1+t^2) \sin \pi t) \qquad 0 \le t \le 1$

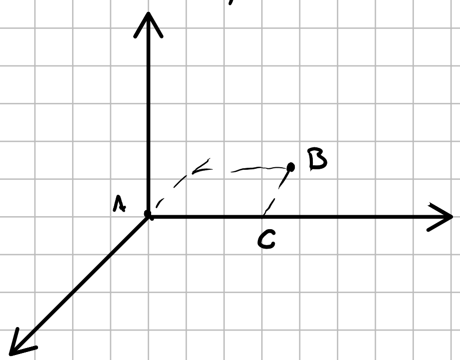
$$I = \int_{\alpha} \underbrace{y^2 \cos x \, dx + (2y \sin x + e^z)\, dy + y e^z \, dz}_{w}$$

$$dw = 2y \cos x \; dy \wedge dx + 2y \cos x \; dx \wedge dy +$$
$$+ e^z \; dz \wedge dy + e^z \; dy \wedge dz = 0$$

$dw = 0$, w JE DEFINISANA U $\mathbb{R}^3$

$\Rightarrow$ INTEGRAL NE ZAVISI OD PUTA

$A = \alpha(0) = (0,0,0)$ $\qquad$ $B = \alpha(1) = (-1, e, 0)$



$C = (0, e, 0)$

$$\int_{AB} w = \int_{AC} w + \int_{CB} w$$

<u>AC</u> $\quad x = 0 \quad z = 0 \quad 0 \le y \le e$

$\quad dx = 0 \quad dz = 0$

$w|_{AC} = dy$

$$\int_{AC} w = \int_0^e dy = e$$

<u>CB</u> $\quad 0 \rightsquigarrow x \rightsquigarrow -1 \quad y = e \quad z = 0$

$\quad dy = 0 \quad dz = 0$

$$w|_{CB} = e^2 \cos x$$

$$\int_{CB} w = \int_0^{-1} e^2 \cos x \, dx = e^2 \sin x \Big|_0^{-1} = e^2 \sin(-1)$$

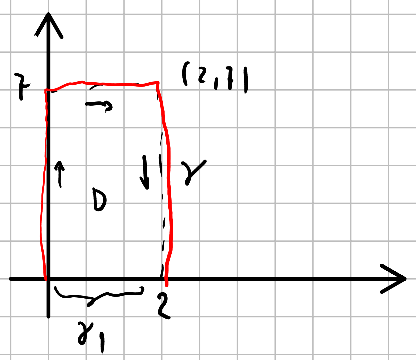
$$\int_{AB} w = \int_{AC} w + \int_{CD} w = e - e^2 \sin 1$$

---

(3) $f : \mathbb{R} \to \mathbb{R}$ $\qquad f \in C^\infty(\mathbb{R})$

$$I = \int_\gamma \underbrace{y f'(x)\, dx + (\operatorname{ch} x \log(2+x) + f(x))\, dy}_{w}$$

γ :



$$D = \{ (x,y) \in \mathbb{R}^2 \mid 0 \le x \le 2 , \; 0 \le y \le 7 \}$$

$\partial D = \gamma \cup \gamma_1 \qquad \gamma_1 := \{ (x,y) \in \mathbb{R}^2 \mid y = 0 \;\; 0 \le x \le 2 \}$

$$\underbrace{\int_{\gamma^+} w}_{\shortparallel \atop -I} + \int_{\gamma_1} w = \iint_D dw$$

$\underline{\gamma_1}$ $\qquad y = 0 \qquad dy = 0 \qquad w|_{\gamma_1} = 0$

$$\int_{\gamma_1} w = 0$$

$$dw = f'(x)\, dy \wedge dx + \left(\operatorname{sh} x \lg(2+x) + \operatorname{ch} x \frac{1}{2+x} + f'(x)\right) dx \wedge dy$$

$$= \left(\operatorname{sh} x \lg(2+x) + \operatorname{ch} x \frac{1}{2+x}\right) dx\, dy$$

$$\iint_D dw = \iint_D \left(\operatorname{sh} x \lg(2+x) + \operatorname{ch} x \frac{1}{2+x}\right) dx\, dy$$

$$= \int_0^7 dy \int_0^2 \left(\operatorname{sh} x \lg(2+x) + \operatorname{ch} x \frac{1}{2+x}\right) dx =$$

$$= y\Big|_0^7 \quad \operatorname{ch} x \lg(2+x)\Big|_0^2$$

$$= 7\,(\operatorname{ch} 2 \lg 4 - \lg 2)$$

---

(4) $$\int_\gamma \underbrace{\frac{x-y}{x^2+y^2}\, dx + \frac{x+y}{x^2+y^2}\, dy}_{\omega}$$

POZITIVNO ORIJENTISANA U SMERU KAZALJKE NA SATU

$$\gamma: \left(\frac{x-1}{\alpha}\right)^2 + \left(\frac{y+1}{\beta}\right)^2 = 1 \qquad (0,0) \notin \gamma$$

ELIPSA SA CENTROM U $(1,-1)$

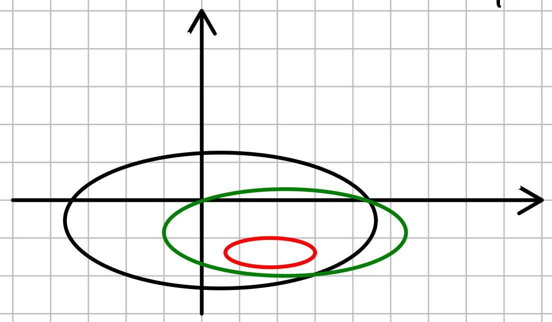
$$dw = \frac{-(x^2+y^2) - (x-y)\cdot 2y}{(x^2+y^2)^2}\, dy \wedge dx + \frac{x^2+y^2 - (x+y)2x}{(x^2+y^2)^2}\, dx \wedge dy$$

$$= \frac{-x^2+y^2-2xy}{(x^2+y^2)^2}\, dy \wedge dx + \frac{-x^2+y^2-2xy}{(x^2+y^2)}\, dx \wedge dy = 0$$

$dw = 0$, ALI ω NIJE DEFINISANA U TAČKI $(0,0)$

$\gamma = \partial D \qquad (0,0) \overset{?}{\in} D$

$(0,0) \in \gamma \qquad \frac{1}{\alpha^2} + \frac{1}{\beta^2} = 1$

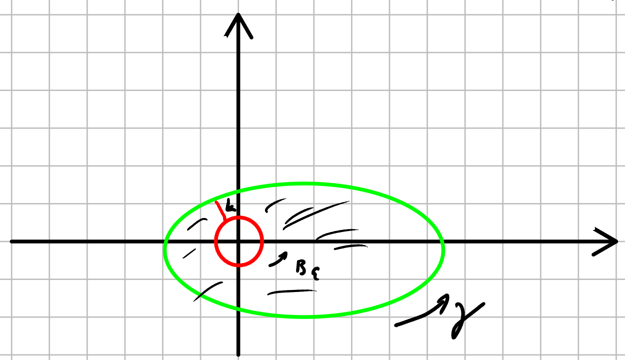


1) $(0,0) \notin D \iff \frac{1}{\alpha^2} + \frac{1}{\beta^2} < 1$

ω DEFINISANA NA D i $\pi_1(D) = 0$

$$\int_\gamma \omega = \iint_D d\omega = \iint_D 0 = 0$$

2) $(0,0) \in D \iff \frac{1}{\alpha^2} + \frac{1}{\beta^2} > 1$



$\partial D_1 = \gamma \cup L^+ \cup B_\varepsilon \cup L^-$

$$\int_\gamma \omega + \cancel{\int_{L^+} \omega} + \int_{B_\varepsilon^-} \omega + \cancel{\int_{L^-} \omega} = \iint_{D_1} d\omega = 0$$

$$\int_\gamma \omega = \int_{B_\varepsilon} \omega \qquad B_\varepsilon \text{ - KUGLA DOVOLJNO MALOG PREČNIKA}$$

$\underline{B_\varepsilon}$:

$x = \varepsilon \cos\varphi \qquad dx = -\varepsilon \sin\varphi \, d\varphi \qquad 0 \le \varphi \le 2\pi$

$y = \varepsilon \sin\varphi \qquad dy = \varepsilon \cos\varphi \, d\varphi$

$$\int_{B_\varepsilon} \omega = \int_0^{2\pi} \frac{\varepsilon\cos\varphi - \varepsilon\sin\varphi}{\underbrace{\varepsilon^2\cos^2\varphi + \varepsilon^2\sin^2\varphi}_{1}} (-\varepsilon\sin\varphi \, d\varphi) + \frac{\varepsilon\cos\varphi - \varepsilon\sin\varphi}{\underbrace{\varepsilon^2\cos^2\varphi + \varepsilon^2\sin^2\varphi}_{1}} (\varepsilon\cos\varphi \, d\varphi)$$

$$= \int_0^{2\pi} \left(-\cos\varphi \sin\varphi + \sin^2\varphi + \cos^2\varphi + \sin\varphi \cos\varphi\right) d\varphi$$

$$= \int_0^{2\pi} d\varphi = 2\pi$$

5

$$\int_l \underbrace{\frac{(2x^3 + 2xy^2 - 6x^2 - 2y^2 + 4x)\, dy - (2y^3 + 2yx^2 - 4xy + 4y)\, dx}{(x^2+y^2)\ \left((x-2)^2 + y^2\right)}}_{w}$$

$(0,0)\ ,\ (2,0) \notin l$

$$w_1 := \frac{x\,dy - y\,dx}{x^2+y^2} \qquad w_2 := \frac{(x-2)\,dy - y\,dx}{(x-2)^2 + y^2}$$

$$w_1 + w_2 = \frac{(x\,dy - y\,dx)(x^2 - 4x + 4 + y^2) + ((x-2)\,dy - y\,dx)(x^2+y^2)}{(x^2+y^2)\ ((x-2)^2+y^2)}$$

$$= \frac{(x^3 - 4x^2 + 4x + xy^2 + x^3 + xy^2 - 2x^2 - 2y^2)\, dy + (-yx^2 + 4xy - 4y - y^3 - yx^2 - y^3)\, dx}{(x^2+y^2)\ ((x-2)^2 + y^2)}$$

$$= \frac{(2x^3 + 2xy^2 - 6x^2 - 2y^2 + 4x)\, dy - (2y^3 + 2yx^2 - 4xy + 4y)\, dx}{(x^2+y^2)\ ((x-2)^2 + y^2)} = w$$

$$\int_l w = \int_l (w_1 + w_2) = \underbrace{\int_l w_1}_{I_1} + \underbrace{\int_l w_2}_{I_2}$$

$\underline{I_1}$

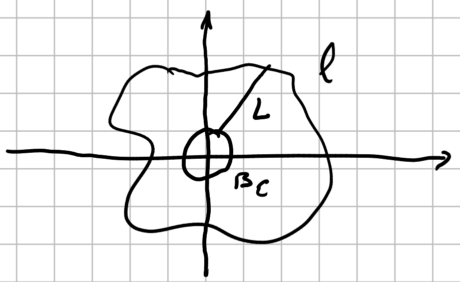
$$\int_l w_1 = ?$$

$$\int_\ell \frac{x\,dy - y\,dx}{x^2+y^2}$$

$d\omega_1 = 0$ $\qquad \ell = \partial D$

$(0,0) \notin D$ $\qquad \int_\ell \omega_1 = 0$

$(0,0) \in D$



$$\int_\ell \omega_1 + \cancel{\int_{L^+} \omega_1} + \int_{B_\varepsilon^-} \omega_1 + \cancel{\int_{L^-}} \omega_1 = \iint_D d\omega = 0$$

$$\int_\ell \omega_1 = \int_{B_\varepsilon} \omega_1$$

$\underline{B_\varepsilon}$ $\qquad x = \varepsilon \cos\varphi$

$y = \varepsilon \sin\varphi$

$\vdots$

$$\int_\ell \omega_1 = 2\pi$$

$\underline{I}_2$ $$\int_\ell \underbrace{\frac{(x-2)\,dy - y\,dx}{(x-2)^2+y^2}}_{\omega_2}$$

$d\omega_2 = 0$

$(2,0) \notin D$ $\qquad \int_\ell \omega_2 = 0$

$(2,0) \in D$

smena $\quad x-2 = X \quad Y = y$

$\rightsquigarrow \int_\ell \omega_1$

$$\int_l w_2 = 2\pi$$

Zaključak:

$$l = \partial D$$

$$\int_l w = \begin{cases} 0, & (0,0),\ (2,0) \notin D \\ 2\pi, & (0,0) \in D \quad (2,0) \notin D \\ 2\pi, & (0,0) \notin D \quad (2,0) \in D \\ 4\pi, & (0,0) \in D \quad (2,0) \in D \end{cases}$$

Domaći:

$|\gamma| \neq 3$

$$\int_l \frac{(y + (x-\gamma)^2 y \sin x + y^3 \sin x)\,dx + ((x-\gamma)^2 e^x + y^2 e^x - x\cdot y)\,dy}{(x-\gamma)^2 + y^2}$$

$l: \ |x| + |y| = 3$ u smeru kazaljke na satu

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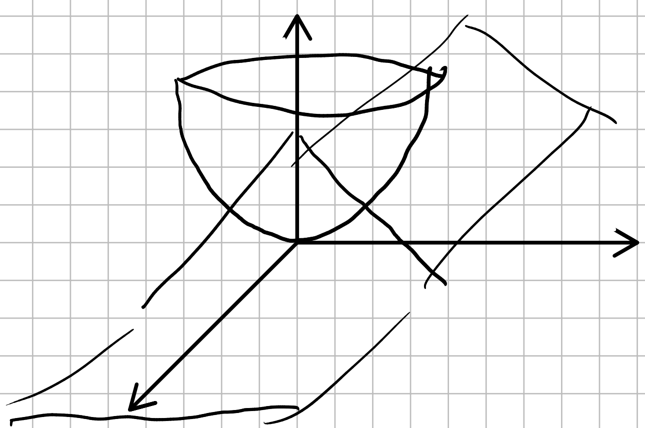
(6) $\gamma: \quad z = x^2 + y^2 \ \wedge \ 2|y| = 3 - z$

pozitivno orijentisana gledano iz tačke $(0, 0, 200)$

$$\int_\gamma z\,dx + x\,dy + y\,dz$$



$\gamma = \gamma_1 \cup \gamma_2 \qquad \gamma_1 = \gamma \cap y \geq 0 \qquad \gamma_2 = \gamma \cap y \leq 0$

$$\int_{\gamma_1} z\,dx + x\,dy + y\,dz$$

$\gamma_1 :$ $\quad y \geq 0$

$$z = x^2 + y^2 \qquad 2y = 3 - z \quad \Rightarrow \quad z = 3 - 2y$$

$$3 - 2y = x^2 + y^2$$

$$x^2 + y^2 + 2y + 1 = 4$$

$$x^2 + (y+1)^2 = 4$$

$$x = 2\cos\varphi \qquad y = -1 + 2\sin\varphi \qquad z = 3 - 2y = 5 - 4\sin\varphi$$

$$dx = -2\sin\varphi\, d\varphi \qquad dy = 2\cos\varphi\, d\varphi \qquad dz = -4\cos\varphi\, d\varphi$$

$$y \geq 0$$

$$-1 + 2\sin\varphi \geq 0$$

$$2\sin\varphi \geq 1$$

$$\sin\varphi \geq \frac{1}{2}$$

$$-\frac{\pi}{6} \leq \varphi \leq \frac{\pi}{6}$$

$$I_1 = \int_{-\pi/6}^{\pi/6} (5 - 4\sin\varphi)(-2\sin\varphi)\,d\varphi + 2\cos\varphi(2\cos\varphi)\,d\varphi + (-1+2\sin\varphi)(-4\cos\varphi)\,d\varphi$$

$$= \ldots \quad \ldots$$

<u>I_2</u> $\quad y \leq 0$

$$z = x^2 + y^2 \qquad -2y = 3 - z \quad \Rightarrow \quad z = 3 + 2y$$

$$3 + 2y = x^2 + y^2$$

$$x^2 + y^2 - 2y = 3$$

$$x^2 + (y-1)^2 = 4$$

$x = 2\cos\varphi$ $\qquad$ $y = 1 + 2\sin\varphi$ $\qquad$ $z = 3+2y = 5+4\sin\varphi$

$dx = -2\sin\varphi \, d\varphi$ $\qquad$ $dy = 2\cos\varphi \, d\varphi$ $\qquad$ $dz = 4\cos\varphi \, d\varphi$

$y < 0$

$1 + 2\sin\varphi \leq 0$

$\sin\varphi \leq -\frac{1}{2}$

$\frac{3\pi}{6} \leq \varphi \leq \frac{7\pi}{6}$

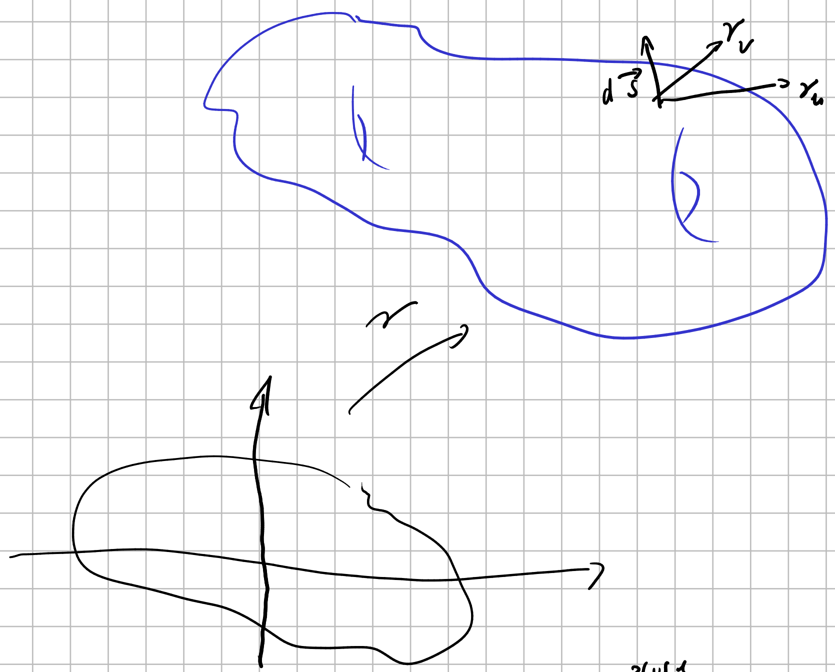
$$I_2 = \int_{\frac{5\pi}{6}}^{\frac{7\pi}{6}} (5+2\sin\varphi)(-2\sin\varphi)\,d\varphi + 2\cos\varphi\,(2\cos\varphi)\,d\varphi + (1+2\sin\varphi)(4\cos\varphi)\,d\varphi$$

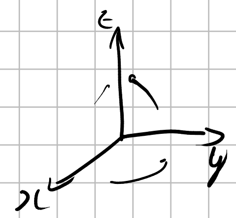
$= - - \_\_\_$

$I = I_1 + I_2$

ANALIZA 2

VEŽBE





$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S P\,\overset{x\cos\alpha}{\overbrace{dy\,dz}} + Q\,\overset{y\cos\beta}{\overbrace{dz\,dx}} + R\,\overset{z\cos\gamma}{\overbrace{dx\,dy}}$$

$$\vec{F} = (P, Q, R)$$

(1)

$$S: \quad x^2 + y^2 + z^2 = a^2$$

$$\vec{F} = (x, y, z)$$

$$\iint_S \vec{F}\, d\vec{S} = \iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$$

$$x = a\cos u \cos v \qquad u \in [0, 2\pi]$$
$$y = a\sin u \cos v \qquad v \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
$$z = a \sin v$$

$$r_u = (-a\sin u \cos v,\ a\cos u \cos v,\ 0)$$

$$r_v = (-a\cos u \sin v,\ -a\sin u \sin v,\ a\cos v)$$

$$d\vec{S} = r_u \times r_v = \begin{vmatrix} i & j & k \\ -a\sin u \cos v & a\cos u\cos v & 0 \\ -a\cos u \sin v & -a\sin u \sin v & a\cos v \end{vmatrix}$$

$$= \left(a^2\cos u \cos^2 v,\ a^2 \sin u \cos^2 v,\ a^2 \sin^2 u \cos v \sin v + a^2 \cos^2 u \cos v \sin v\right)$$

$$= a^2 \left(\cos u \cos^2 v,\ \sin u \cos^2 v,\ \cos v \sin v\right)$$

$$= a^2 \cos v\ (\underbrace{\cos u \cos v}_{x},\ \underbrace{\sin u \cos v}_{y},\ \underbrace{\sin v}_{z})$$

$$\| d\vec{S} \| = a^2 \cos v$$

$$d\vec{S} = a^2 \cos v\ \vec{F}$$

$$\vec{F} \cdot d\vec{S} = a^2 \cos v\ \vec{F}\cdot\vec{F} = a^2 \cos v\ \|\vec{F}\|^2 = a^4 \cos v$$

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} a^4 \cos v \, dv\, du =$$

$$= a^4 \int_0^{2\pi} du \int_{-\pi/2}^{\pi/2} \cos v \, dv$$

$$= a^4\, 2\pi \cdot 2 = \boxed{4\pi a^4}$$

(2) $$\iint_S \vec{F} \cdot d\vec{S}$$

$$\vec{F} = \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right)$$

$$S: \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$x = a \cos u \cos v \qquad u \in [0, 2\pi]$$

$$y = b \sin u \cos v \qquad v \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$z = c \sin v$$

$$r_u = (-a \sin u \cos v, \; b \cos u \cos v, \; 0)$$

$$r_v = (-a \cos u \sin v, \; -b \sin u \sin v, \; c \cos v)$$

$$d\vec{S} = \cos v \,(bc \cos u \cos v, \; ac \sin u \cos v, \; ab \sin v)$$

$$= \cos v \left(\frac{bc}{a} x, \; \frac{ac}{b} y, \; \frac{ab}{c} z \right)$$

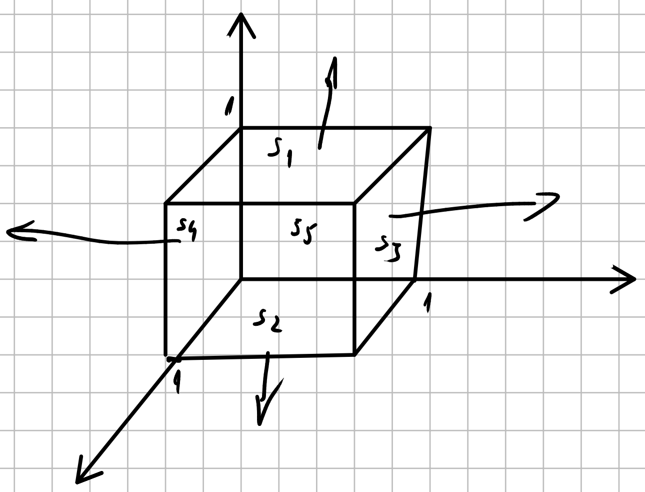
$$\vec{F} \cdot d\vec{S} = \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) \cos v \left(\frac{bc}{a} x, \; \frac{ac}{b} y, \; \frac{ab}{c} z \right)$$

$$= \cos v \underbrace{\left(\frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c} \right)}_{= \xi}$$

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \xi \cos v \, du \, dv = \xi \, 2\pi \cdot 2$$

(3) $\iint\limits_S \underbrace{x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy}_{w}$

S - SPOLJNA STRANA $[0,1]^3$



$$\iint\limits_S \vec{F}\, d\vec{S} = \iint\limits_{S_1} \vec{F} \cdot d\vec{S} + \iint\limits_{S_2} \vec{F}\, d\vec{S} + \ldots + \iint\limits_{S_6} \vec{F}\, d\vec{S}$$

$$w\Big|_{S_1} = dx\, dy$$

$$\iint\limits_{S_1} w = \iint\limits_{\substack{0 \le x \le 1 \\ 0 \le y \le 1}} dx\, dy = 1$$

$$w\Big|_{S_2} = 0 \qquad \iint\limits_{S_2} w = 0$$

$$\iint\limits_S \vec{F}\, d\vec{S} = 3$$

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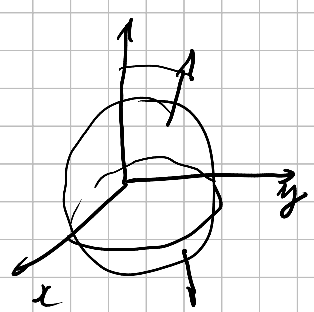
(4) $\iint\limits_{S^2} dx \wedge dy$

SPOLJNA STRANA SFERE

D - PROJEKCIJA NA $z=0$



$$\iint\limits_{S^2} dx \wedge dy = \iint\limits_{S_1} dx \wedge dy + \iint\limits_{S_2} dx \wedge dy$$

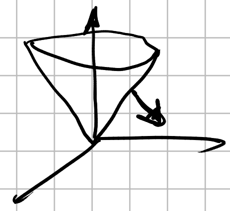
$$= \iint\limits_D dx\, dy - \iint\limits_D dx\, dy = 0$$

"-" ZBOG ORIJENTACIJE

S_1 - GORNJA POLUSFERA

S_2 - DONJA POLUSFERA

(5) $$\iint_S dx \wedge dy = -\pi$$



S - SPOLJNA STRANA KONUSA $\quad z = \sqrt{x^2+y^2} \quad 0 \le z \le 1$

$\boxed{z^2 = x^2+y^2 \quad -1 \le z \le 1}$

$$\iint_S dx \wedge dy = -\iint_P dx\, dy = -P(\text{KRUGA}) = -\pi$$

$$P : 0 \le x^2+y^2 \le 1$$

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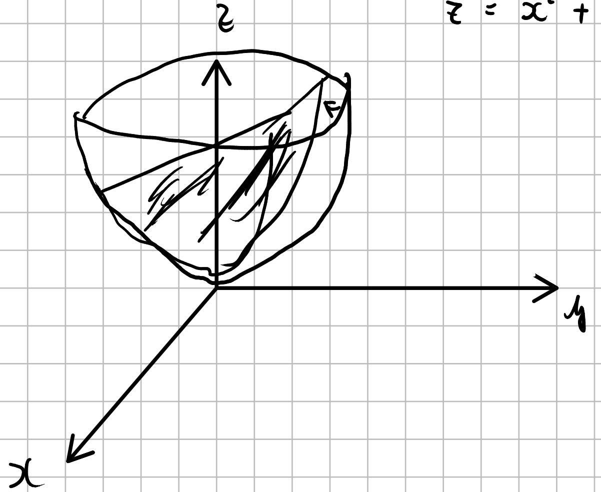
(6) $$\iint_S y\, dz\, dx$$

S - GORNJA STRANA PARABOLOIDA

$z = x^2 + y^2 \quad 0 \le z \le 1$



$S = S_1 \cup S_2$

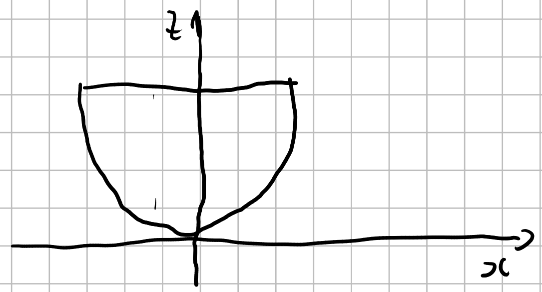
$S_1 = S \cap y \ge 0 \qquad S_2 = S \cap y \le 0$

$$\iint_{S_1} y\, dz\, dx = -\iint_P \sqrt{z-x^2}\, dz\, dx$$

$$\iint_{S_2} y\, dz\, dx = +\iint_P -\sqrt{z-x^2}\, dz\, dx$$

$\int$

$$\iint_S w = \iint_{S_1} w + \iint_{S_2} w = -2 \iint_D \sqrt{z - x^2}\, dz\, dx$$



$y = 0$

$z = x^2$

$$= -2 \int_{-1}^{1} \int_{x^2}^{1} \sqrt{z - x^2}\, dz\, dx =$$

$$= -2 \int_{-1}^{1} \frac{2}{3} (z - x^2)^{\frac{3}{2}} \Big|_{x^2}^{1} dx =$$

$$= -2 \int_{-1}^{1} \frac{2}{3} \left(1 - x^2\right)^{\frac{3}{2}} - \frac{2}{3} \left(x^2 - x^2\right) dx$$

$= - -$

$$S = \{ (x, y, z(x,y)) \mid (x,y) \in D \} \quad \to z \text{ FUNKCIJA}$$

$$\vec{F} = (P, Q, R) \qquad r = (x, y, z)$$

$$r_x = \left(1, 0, \frac{\partial z}{\partial x}\right)$$

$$r_y = \left(0, 1, \frac{\partial z}{\partial y}\right)$$

$$d\vec{S} = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right)$$

$$\iint_S \vec{F}\, d\vec{S} = \iint_{\Pi} \left(-\frac{\partial z}{\partial x} P - \frac{\partial z}{\partial y} Q + R \right) dx\, dy$$

Π - PROJEKCIJA POVRŠI S NA RAVAN $z = 0$

(7)

$$\iint_S \underbrace{(y-z)}_{P}\, dy\, dz + \underbrace{(z-x)}_{Q}\, dz\, dx + \underbrace{(x-y)}_{R}\, dx\, dy$$

$S: \quad x^2+y^2=z^2 \qquad 0 \le z \le 1$

$z = \sqrt{x^2+y^2}$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

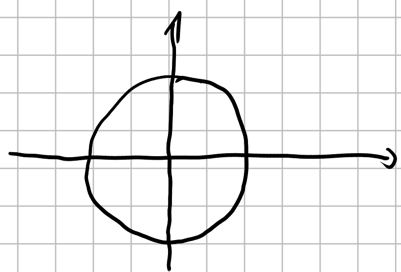
Π - PROJEKCIJA $\quad x^2+y^2 \le 1$

$$I = \iint_\Pi \left(-\frac{x}{\sqrt{x^2+y^2}} \left(y - \sqrt{x^2+y^2}\right) - \frac{y}{\sqrt{x^2+y^2}} \left(\sqrt{x^2+y^2} - x\right) + x - y \right) dx\, dy$$

$$= \iint_\Pi \left(\cancel{\frac{-xy}{\sqrt{x^2+y^2}}} + x - y + \cancel{\frac{yx}{\sqrt{x^2+y^2}}} + x - y \right) dx\, dy$$

NEPARNA FUNKCIJA NA SIMETRIČNOM DOMENU

$$= 2 \iint_\Pi (x-y)\, dx\, dy = 0$$



$$= \iint_{\substack{0 \le \varphi \le 2\pi \\ 0 \le \rho \le 1}} (\rho\cos\varphi - \rho\sin\varphi)\, \rho\, d\rho\, d\varphi$$

$$\int_{\partial S} \omega = \int_S d\omega$$ - UOPŠTENA STOKSOVA FORMULA

$\pi_1(S) = 0$

- GRINOVA FORMULA (U RAVNI)

$$\int_{l=\partial S} P\, dx + Q\, dy = \iint_S \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right) dx\, dy$$

- STOKSOVA U UŽEM SMISLU

$$\int_{\ell = \partial S} P\,dx + Q\,dy + R\,dz = \iint_S \frac{\partial P}{\partial y}\,dy \wedge dx + \frac{\partial P}{\partial z}\,dz \wedge dx$$

$$+ \frac{\partial Q}{\partial x}\,dx \wedge dy + \frac{\partial Q}{\partial z}\,dz \wedge dy + \frac{\partial R}{\partial x}\,dx \wedge dz + \frac{\partial R}{\partial y}\,dy \wedge dz$$

$$= \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

- GAUS - OSTROGRADSKI

$$\iint_{S=\partial T} P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy = \iiint_T \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

(8) $$\iint_S \vec{F} \cdot d\vec{S}$$

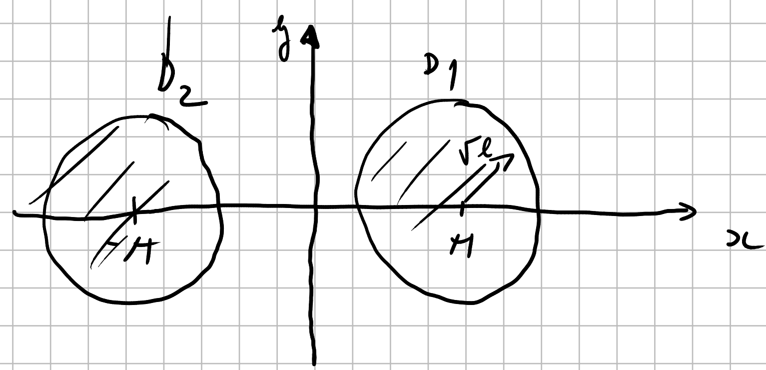
$$\vec{F} = (y e^z,\ z \sin x,\ x + y)$$

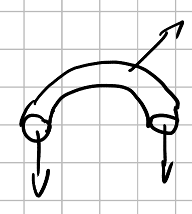
$$S: \left(\sqrt{x^2+z^2} - \pi\right)^2 + y^2 = e \qquad z \geq 0$$

$$\iint_S \underbrace{y e^z\,dy \wedge dz + z \sin x\,dz \wedge dx + (x+y)\,dx \wedge dy}_{\omega}$$

$$d\omega = 0$$

$$S \cap z = 0 \Rightarrow (|x| - \pi)^2 + y^2 = e$$





$$\iint_S w + \iint_{D_1} w + \iint_{D_2} w = \iiint_T dw = \iiint_T 0 = 0$$

$$\iint_S w = -\iint_{D_1} w - \iint_{D_2} w$$

$$w\big|_{D_1} = (x+y)\,dx\,dy \qquad w\big|_{D_2} = (x+y)\,dx\,dy$$

$$\iint_{D_1} w + \iint_{D_2} w = \iint_{D_1} (x+y)\,dx\,dy + \iint_{D_2} (x+y)\,dx\,dy =$$

$$= \iint_{D_1 \cup D_2} x\,dx\,dy + \iint_{D_1 \cup D_2} y\,dx\,dy = 0$$

$$\boxed{I = 0}$$

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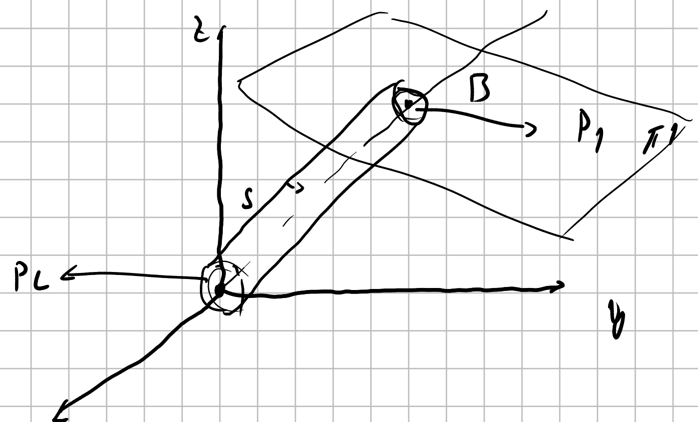
9. $$\iint_S \vec{F} \cdot d\vec{S}$$

$$\vec{F} = \left(x + y e^{zy},\ -(3y + zx + e^{3(y+z)}),\ e^{3(y+z)} + 6z \right)$$

S – UNUTRAŠNJA STRANA CILINDRA POLUPREČNIKA 1 ČIJA OSA PROLAZI KROZ TAČKE A(0,0,0) i B(0,3,3) I OGRANIČENA JE RAVNIMA ORTOGONALNIM NA OSU U TAČKAMA A i B



$$\iint_S \underbrace{(x+y e^{zy})\, dy \wedge dz - (3y+2x+e^{3(y+z)})\, dz \wedge dx + (e^{3(y+z)}+6z)\, dx \wedge dy}_{w}$$

$$dw = dx \wedge dy \wedge dz - 3\, dy \wedge dz \wedge dx + 6\, dz \wedge dx \wedge dy$$

$$dw = 4\, dx \wedge dy \wedge dz$$

$$\iint_{S^-} w + \iint_{P_1} w + \iint_{P_2} w = \iiint_T 4\, dx\, dy\, dz = 4\, V(T)$$

$$= \pi \cdot 1^2 \cdot 3\sqrt{2}$$

π_1 : $y + z + D = 0$

$B \in \pi_1$ $\quad 3+3+D=0 \quad D=-6$

$$y+z-6=0 \qquad z = 6-y$$

$$\sqrt{x^2+(y-3)^2+(z-3)^2} \leq 1$$

$$x^2+(y-3)^2+(z-3)^2 \leq 1$$

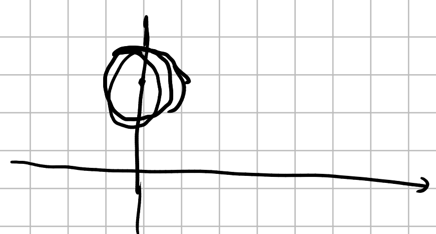
$$x^2+(y-3)^2+(6-y-3)^2 \leq 1$$

$$x^2+2(y-3)^2 \leq 1$$

$w|_{P_1} =$ $\quad z = 6-y \quad \frac{\partial z}{\partial x} = 0 \quad \frac{\partial z}{\partial y} = -1$

$$w|_{P_1} = \left(-3y+2x+e^{3\cdot 6} + e^{3\cdot 6} + 6\cdot(6-y)\right)$$

$$= 2\cdot e^{18} - 9y + 2x + 36$$



$$\iint_{P_1} \left(\underbrace{(2\cdot e^{18}+36)}_{M} - 9y + 2x\right) dx\, dy$$

$$= \iint_{P_1} 4 \, dx\,dy - 9 \iint_{P_1} y \, dx\,dy + 2 \underbrace{\iint_{P_1} x \, dx\,dy}_{=0}$$

$\Pi_2 :$ $y + z + D = 0$

$A \in \Pi_2 \Rightarrow D = 0$

$\Pi_2 :$ $y + z = 0$ $\quad z = -y$ $\quad \frac{\partial z}{\partial x} = 0$ $\quad \frac{\partial z}{\partial y} = -1$

$\phi \quad x^2 + 2y^2 \leq 1$

$$\iint_{P_1} \left(3y + (x+1) + 1 + 6(-y)\right) dx\,dy$$

$$= \underbrace{\iint_{P_1} -3y \, dx\,dy}_{=0} + 2 \underbrace{\iint_{P_1} x \, dx\,dy}_{=0} + 2 \iint_{P_1} dx\,dy = 2 P(e)$$

ANALIZA 2

VEŽBE

① NAĆI FLUKS VEKTORSKOG POLJA

$$F = (x^2, y^2, z^2)$$

KROZ SPOLJNU STRANU SFERE

$$S : (x-a)^2 + (y-b)^2 + (z-c)^2 = R^2 \qquad R > 0$$

$$\iint_{S^+} x^2\, dy\, dz + y^2\, dz\, dx + z^2\, dx\, dy$$

$$X = x - a \qquad Y = y - b \qquad Z = z - c$$

$$S_1 : X^2 + Y^2 + Z^2 = R^2$$

$$= \iint_{S_1} (X+a)^2\, dY\, dZ + (Y+b)^2\, dZ\, dX + (Z+c)^2\, dX\, dY$$

$$= \iint_{S_1} (X+a)^2\, dY\, dZ + \iint_{S_1} (Y+b)^2\, dZ\, dX + \iint_{S_1} (Z+c)^2\, dX\, dY$$

$$\iint_{S_1} (Z+c)^2\, dX\, dY = \iint_{S_1'} (Z+c)^2\, dX\, dY + \iint_{S_1''} (Z+c)^2\, dX\, dY$$

$$S_1' = S_1 \cap Z \geq 0 \qquad S_1'' = S_1 \cap Z \leq 0$$

на S_1' , на S_1'' $z = z(x, y)$

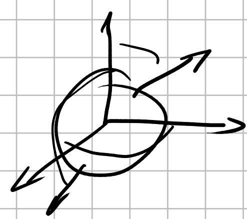
Знамо:

$$\iint\limits_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy =$$

$$= \iint\limits_D \left(-\frac{\partial z}{\partial x} P - \frac{\partial z}{\partial y} Q + R \right) dx\,dy$$

$z = \sqrt{R^2 - x^2 - y^2}$ на S_1'

$z = -\sqrt{R^2 - x^2 - y^2}$ на S_1''



$P = 0$ $Q = 0$ R само имамо

$$= \iint\limits_D \left(c + \sqrt{R^2 - x^2 - y^2}\right)^2 dx\,dy - \iint\limits_D \left(c - \sqrt{R^2 - x^2 - y^2}\right)^2 dx\,dy$$

D : $x^2 + y^2 \le R^2$

$$= \iint\limits_D \left(\cancel{c^2} + 2\sqrt{R^2 - x^2 - y^2}\, c + \cancel{R^2 - x^2 - y^2} - \left(\cancel{c^2} - 2\sqrt{R^2 - x^2 - y^2}\, c + \cancel{R^2 - x^2 - y^2} \right) \right) dx\,dy$$

$$= 4c \iint\limits_D \sqrt{R^2 - x^2 - y^2}\, dx\,dy$$

$= \Big[\ x = \rho\cos\varphi \quad y = \rho\sin\varphi \quad J = \rho$

$0 \le \varphi \le 2\pi \quad 0 \le \rho \le R\ \Big]$

$$= 4c \iint\limits_{\substack{0 < \rho < R \\ 0 < \varphi < 2\pi}} \sqrt{R^2 - \rho^2}\, \rho\, d\rho\, d\varphi$$

$$= 4c \int_0^{2\pi} d\varphi \int_0^R \sqrt{R^2 - \rho^2}\, \rho\, d\rho = \Big[\ \begin{matrix} R^2 - \rho^2 = t \\ -2\rho\, d\rho = dt \end{matrix}$$

ρ	0	R
t	R^2	0

$$= 4c \cdot 2\pi \cdot \int_0^{R^2} \sqrt{t}\,\frac{dt}{2}$$

$$= 4c\pi \,\frac{2}{3}\, R^{\frac{3}{2}} \Big|_0^{R^2} = \frac{8}{3}\, c\pi R^3$$

ИЗ СИМЕТРИЈЕ $\quad I = \frac{8}{3}\pi R^3 (a+b+c)$

II НАЧИН

СТОКСОВА ФОРМУЛА

$$\iint_S x^2\,dy\,dz + y^2\,dz\,dx + z^2\,dx\,dy$$

$$= 2\iiint_T (x+y+z)\,dx\,dy\,dz$$

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(2) $\displaystyle\int_{OA} yz\,dx + 3xz\,dy + 2xy\,dz$

OA: $\quad x = t\cos t$

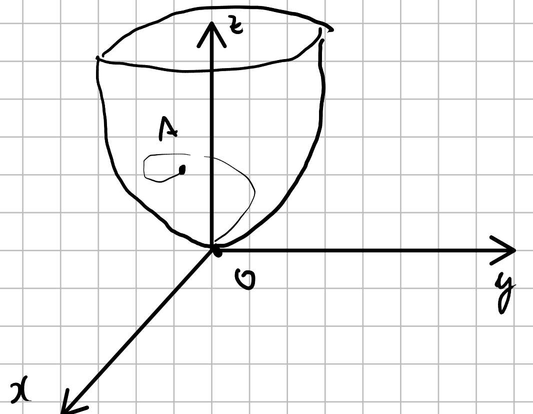
$\quad y = t\sin t \qquad t \in [0, 2\pi]$

$\quad z = t^2$

$$x^2 + y^2 = t^2\cos^2 t + t^2\sin^2 t = t^2 = z$$

КРИВА СЕ НАЛАЗИ НА ПАРАБОЛОИДУ $\quad z = x^2 + y^2$

$O = (0,0,0) \qquad A = (2\pi, 0, 4\pi^2)$



ХОЋЕМО ДА СПОЈИМО

A, B НЕКОМ КРИВОМ

У РАВНИ $y = 0$

МОЖЕМО УЗЕТИ КРИВУ

$\gamma: \quad y = 0 \quad z = x^2$

$$\int_{OA} w + \int_{\gamma} w = \iint_{D} dw \qquad OA \cup \gamma = \partial D$$

$$w|_{\gamma} = 0 \quad \Rightarrow \quad \int_{\gamma} w = 0$$

$$\int_{OA} w = \iint_{D} dw$$

$$w = yz\,dx + 3xz\,dy + 2xy\,dz$$

$$dw = z\,dy \wedge dx + y\,dz \wedge dx + 3z\,dx \wedge dy +$$
$$+ 3x\,dz \wedge dy + 2y\,dx \wedge dz + 2x\,dy \wedge dz$$

$$dw = 2z\,dx \wedge dy - y\,dz \wedge dx - x\,dy \wedge dz$$

$$I = \iint_{D} -x\,dy \wedge dz - y\,dz \wedge dx + 2z\,dx\,dy$$

NA D JE z FUNKCIJA x i y

$$z = z(x,y) \qquad z = x^2 + y^2$$

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y$$

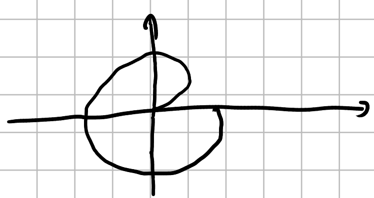
$$P = -x \qquad Q = -y \qquad R = 2z$$

$$I = \iint_{\Pi} \left(2x^2 + 2y + 2(x^2+y^2)\right) dx\,dy$$

Π - PROJEKCIJA NA RAVAN $z = 0$

$$x = t\cos t \qquad x = \rho\cos\varphi$$
$$y = t\sin t \qquad y = \rho\sin\varphi$$

KRIVA $\rho = \varphi$ JE RUB Π



$$0 \le \rho \le \varphi \qquad 0 \le \varphi \le 2\pi$$

$$I = 4 \iint_{\Pi} (x^2+y^2)\, dx\, dy = 4 \iint_{\substack{0 \le \rho \le \varphi \\ 0 \le \varphi \le 2\pi}} \rho^3 \, d\rho \, d\varphi$$

$$= 4 \int_0^{2\pi} \int_0^{\varphi} \rho^3 \, d\rho \, d\varphi = \int_0^{2\pi} \rho^4 \Big|_0^{\varphi} d\varphi =$$

$$= \int_0^{2\pi} \varphi^4 \, d\varphi = \frac{1}{5} \varphi^5 \Big|_0^{2\pi} = \frac{32}{5} \pi^5$$

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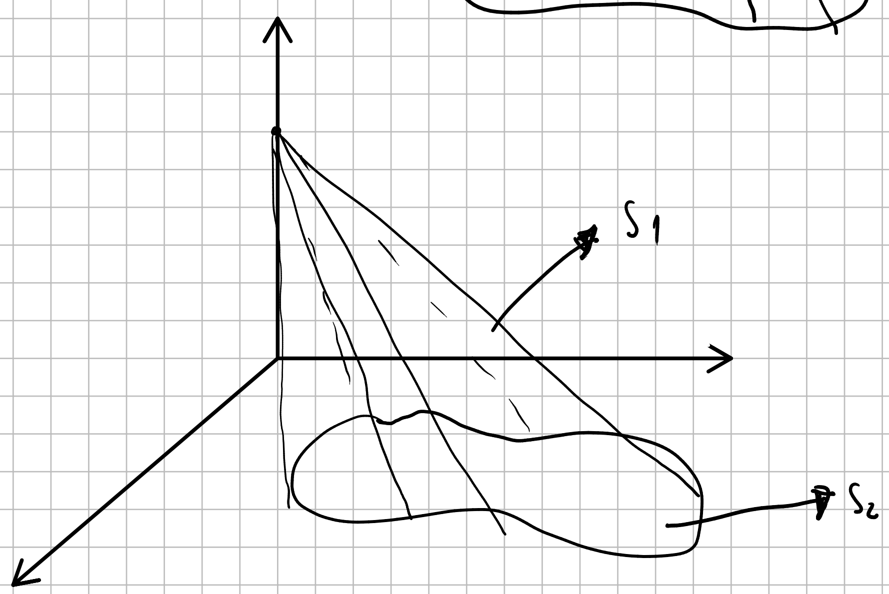
3) NAĆI ZAPREMINU KRIVOLINIJSKE KUPE



S_1 - OMOTAČ

S_2 - BAZA

$$\vec{F} = (x, y, z) \qquad \iint_S \vec{F}\, d\vec{S} = \iint_S x\, dy\, dz + y\, dz\, dx + z\, dx\, dy$$

$$= \iiint_T 3 \, dx\, dy\, dz = 3\, V(T)$$

$S = \partial T$ $\qquad V(T) = \frac{1}{3} \iint_S \vec{F}\, d\vec{S} = \frac{1}{3} \left(\iint_{S_1} \vec{F}\, d\vec{S} + \iint_{S_2} \vec{F}\, d\vec{S} \right)$

$\underline{S_2}$

$$z = 0$$
$$x = x \qquad \vec{F} = (x, y, 0)$$
$$y = y$$
$$d\vec{S} = (0, 0, 1)$$
$$\vec{F} \cdot d\vec{S} = 0$$
$$\iint_{S_2} \vec{F} \, d\vec{S} = 0$$

$$V(T) = \frac{1}{3} \iint_{S_1} \vec{F} \cdot d\vec{S}$$

S_1 - OMOTAČ KUPE, TREBA GA PARAMETRIZOVATI

$$x = -f(u)\, v \qquad v \in [-1, 0]$$
$$y = -g(u)\, v$$
$$z = Hv + H \qquad v = 0 \Rightarrow (0, 0, H)$$
$$v = -1 \Rightarrow (f(u), g(u), 0)$$

$$r_u = (-f' v, -g' v, 0)$$

$$r_v = (-f, -g, H)$$

$$r_u \times r_v = (-g' v H, f' v H, +f' g v - f g' v) =$$
$$= v(-g' H, f' H, f' g - f g')$$

$$\vec{F} = (-f v, -g v, Hv + H)$$

$$\vec{F} \cdot d\vec{S} = v^2 f g' H - f' g v^2 H + (Hv + H) f' g - f g'$$

$$\ldots = vH\,(f'g - fg')$$

$$V(T) = \frac{1}{3}\iint_D vH\,(f'g - fg')\,du\,dv$$

γ - KRIVA

u xy - RAVNI

$$= \frac{1}{3}H\int_{-1}^{0} v\,dv \int_\gamma (f'g - fg')\,du$$

$$= \frac{1}{3}H\,\frac{1}{2}v^2\Big|_{-1}^{0}\int_\gamma y\,dx - x\,dy =$$

$\gamma = \partial D$

$$= \frac{1}{3}H\left(-\frac{1}{2}\right)\iint_D dy\,dx - dx\,dy$$

$$= \frac{-H}{6}(-1)\,2\iint_D dx\,dy = \frac{1}{3}H\,P(D)$$

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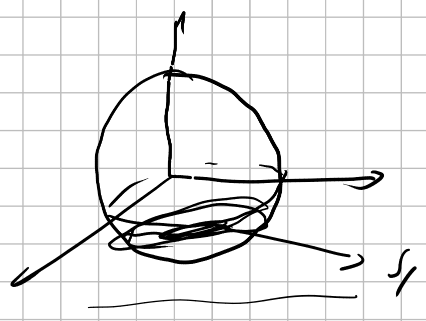
(4)

$$\vec{F} = (x - 3y + 2z,\ -3x + 2y + z,\ 2x + y - 3z)$$

$S:\quad x^2 + y^2 + z^2 = 4,\quad z \geq B,\quad B < 0$

$$\iint_S \vec{F}\cdot d\vec{S} = ?$$



$$\iint_S \underbrace{(x - 3y + 2z)\,dy\,dz + (-3x + 2y + z)\,dz\,dx + (2x + y - 3z)\,dx\,dy}_{w}$$

$$\iint_S w + \iint_{S1} w = \iiint_T 0\,dx\,dy\,dz = 0$$

$B < -2 \qquad \iint_S w = 0$

$-2 < B < 0$

$$\iint_{S_1} w = ? \qquad z = B$$

$$w\big|_{z=B} = 2x + y - 3B$$

$$S_1 : \quad x^2 + y^2 + z^2 = 4$$

$$x^2 + y^2 = 4 - B^2$$

$$\iint_{S_1} w = \iint_{x^2+y^2 \le 4-B^2} (2x + y - 3B)\, dx\, dy$$

$$= 2\cancel{\iint_{x^2+y^2\le 4-B^2} x\, dx\, dy}^{\,0} + \cancel{\iint_{x^2+y^2\le 4-B^2} y\, dx\, dy}^{\,0} - 3B \iint_{x^2+y^2 \le 4-B^2} dx\, dy$$

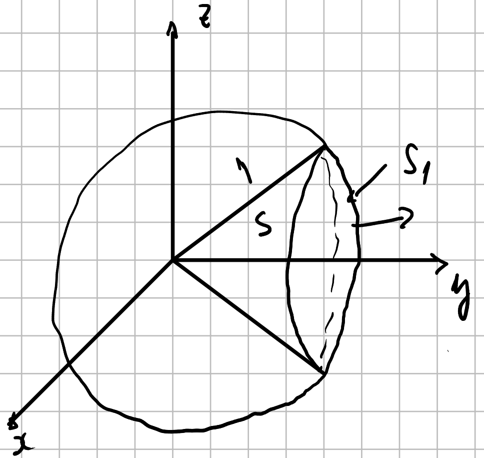
$$= -3B\, \pi\, (4 - B^2)^2$$

5

$$\iint_S \underbrace{4\left(\frac{x^2}{4} + y^2 + z^2\right)^{2025} yz\, dy\, dz + dz\, dx - \left(\frac{x^2}{4} + y^2 + z^2\right)^{2025} xy\, dx\, dy}_{w}$$

S : DEO POVRŠI $y = \sqrt{x^2 + z^2}$ UNUTAR TELA $\frac{x^2}{4} + y^2 + z^2 \le 1$

ORIJENTACIJA SA SLIKE



$$dw = 4 \cdot 2025 \cdot \left(\frac{x^2}{4} + y^2 + z^2\right)^{2024} \frac{2x}{4}\, dx\, dy\, dz + dy\, dz\, dx +$$

$$- 2025 \left(\frac{x^2}{4} + y^2 + z^2\right)^{2024} 2z\, dz\, dx\, dy = dx\, dy\, dz$$

$d\omega$ je dosta jednostavnija, pa idemo na Stoksovu formulu

Moramo imati rub tela. Najjednostavnije je zatvoriti telo sa

S_1: $\frac{x^2}{4}+y^2+z^2=1$ unutar $y \le \sqrt{x^2+z^2}$

jer je ·

$$\omega\big|_{S_1} = 4yz\,dy\,dz + dz\,dx - xy\,dx\,dy$$

$$\iint_{S_1} \omega + \iint_{S} \omega = \iiint_T d\omega = \iiint_T dx\,dy\,dz = V(T)$$

Na S_1 je y funkcija x i z

$$y = \sqrt{1-\frac{x^2}{4}-z^2}$$

$$\frac{\partial y}{\partial x} = \frac{-\frac{x}{2}}{2\sqrt{1-\frac{x^2}{4}-z^2}} = \frac{-x}{4\sqrt{1-\frac{x^2}{4}-z^2}} \qquad \frac{\partial y}{\partial z} = \frac{-z}{\sqrt{1-\frac{x^2}{4}-z^2}}$$

$$r_x = \left(1, \frac{-x}{4\sqrt{1-\frac{x^2}{4}-z^2}}, 0\right)$$

$$r_y = \left(0, \frac{-z}{\sqrt{1-\frac{x^2}{4}-z^2}}, 1\right)$$

$$d\vec{S} = r_x \times r_y = \left(\frac{-x}{4\sqrt{1-\frac{x^2}{4}-z^2}}, -1, \frac{-z}{\sqrt{1-\frac{x^2}{4}-z^2}}\right)$$

$$\vec{F} = (4yz, 1, -xy)$$

$$\vec{F}\cdot d\vec{S} = -xz - 1 + xz = -1$$

$$\iint_{S_1} \vec{F}\,d\vec{S} = \iint_D -dx\,dz = +\iint_D dz\,dx = P(D)$$

D: $\frac{x^2}{4}+y^2+z^2=1$ $\quad y = \sqrt{x^2+z^2}$

$$\frac{5}{4}x^2 + 2y^2 \le 1$$ - elipsa

$$P(D) = \pi \frac{2}{\sqrt{5}}\frac{1}{\sqrt{2}} = \frac{2\pi}{\sqrt{10}}$$

$$V(T) = \iiint_T dx\,dy\,dz = \iint_D \int_{\sqrt{x^2+z^2}}^{\sqrt{1-\frac{x^2}{4}-z^2}} dy\; dz\,dx = \iint_D \left(\sqrt{1-\frac{x^2}{4}-z^2} - \sqrt{x^2+z^2}\right) dz\,dx$$

$$= \iint_D \sqrt{1-\frac{x^2}{4}-z^2}\,dz\,dx - \iint_D \sqrt{x^2+z^2}\,dz\,dx = \dots$$

završiti za domaći

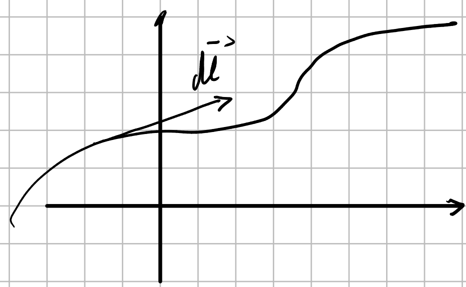
POVRŠINSKI INTEGRAL I VRSTE

PODSEĆANJE:

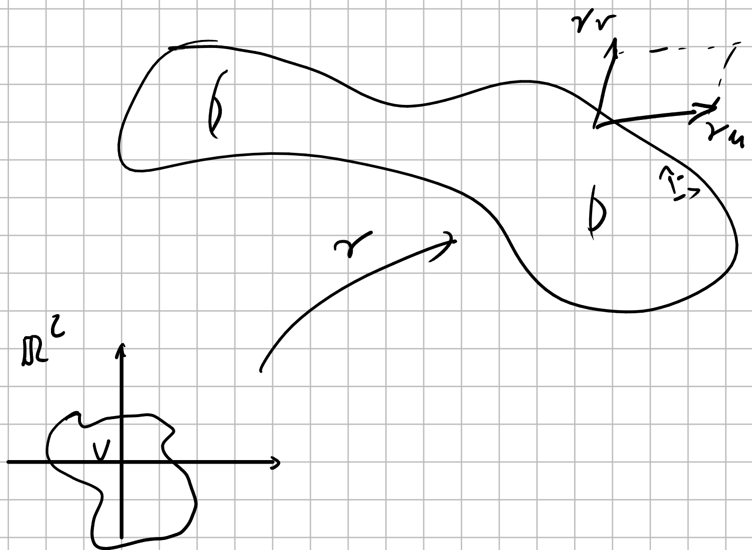
$$\int_l f(x,y,z)\, dl$$

APROKSIMACIJA DUŽINE ELEMENTA KRIVE

dl



POVRŠ:



$$dS = \| r_u \times r_v \| \, du\, dv$$

OSOBINE:

- LINEARNOST
- ADITIVNOST PO SKUPU
- POVRŠINA

$$P(S) = \iint_S dS$$

- VAŽI TEOREMA O SREDNJOJ VREDNOSTI

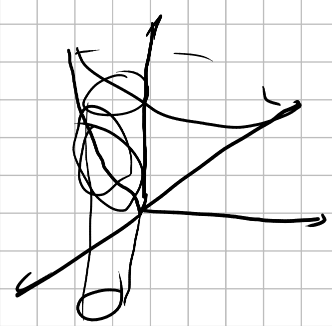
$$z = z(x, y)$$

$$d\vec{S} = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right)$$

$$dS = \| d\vec{S} \| \, du \, dv = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, du \, dv$$

---

(1) $$I = \iint_S (xy + yz + zx) \, dS$$



S: DEO KONUSA $z = \sqrt{x^2 + y^2}$

UNUTAR CILINDRA

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} = \sqrt{\frac{x^2+y^2+x^2+y^2}{x^2+y^2}} = \sqrt{2}$$

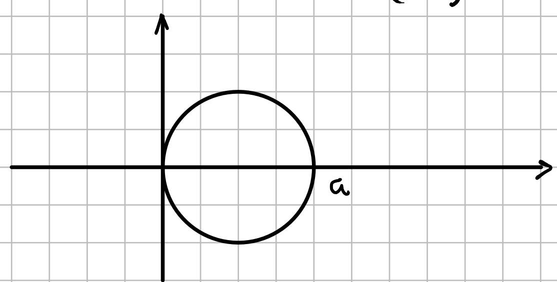
$$I = \iint_D \left(xy + y\sqrt{x^2+y^2} + x\sqrt{x^2+y^2} \right) \sqrt{2} \, dx \, dy$$

D - PROJEKCIJA S NA RAVAN $z = 0$

∂D: $x^2 + y^2 = ax$

$$x^2 - ax + y^2 = 0$$

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$$



$x = \rho \cos\varphi \qquad y = \rho \sin\varphi$

$$\left(\rho\cos\varphi - \frac{a}{2}\right)^2 + \left(\rho\cos\varphi\right)^2 \le \frac{a^2}{4}$$

$$\rho^2\cos^2\varphi - a\rho\cos\varphi + \cancel{\frac{a^2}{4}} + \rho^2\sin^2\varphi \le \cancel{\frac{a^2}{4}}$$

$$\rho^{\cancel{2}}\left(\cos^2\varphi + \sin^2\varphi\right) \le a\cancel{\rho}\cos\varphi$$

$$0 \le \rho \le a\cos\varphi$$

$$\cos\varphi \geq 0 \quad \Rightarrow \quad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$I = \iint\limits_{\substack{0 \le \rho \le a\cos\varphi \\ -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}}} \left(\rho^2\cos\varphi\sin\varphi + \rho^2\sin\varphi + \rho^2\cos\varphi\right)\sqrt{2}\,\rho\,d\rho\,d\varphi$$

$$= \sqrt{2}\int_{-\pi/2}^{\pi/2}\left(\cos\varphi\sin\varphi + \sin\varphi + \cos\varphi\right)\int_0^{a\cos\varphi}\rho^3\,d\rho\,d\varphi$$

$$= \sqrt{2}\int_{-\pi/2}^{\pi/2}\left(\cos\varphi\sin\varphi + \sin\varphi + \cos\varphi\right)\frac{1}{4}\rho^4\Big|_0^{a\cos\varphi}\,d\varphi$$

$$= \frac{\sqrt{2}}{4}a^4\int_{-\pi/2}^{\pi/2}\left(\cos^4\varphi\sin\varphi + \cos\varphi\sin\varphi + \cos^5\varphi\right)d\varphi$$

$$= \frac{\sqrt{2}}{4}a^4\left(\cancelto{0}{\int_{-\pi/2}^{\pi/2}\cos^4\varphi\sin\varphi\,d\varphi} + \cancelto{0}{\int_{-\pi/2}^{\pi/2}\cos\varphi\sin\varphi\,d\varphi} + \int_{-\pi/2}^{\pi/2}\cos^5\varphi\,d\varphi\right)$$

НЕ ПАРНЕ ФУНКЦИЈЕ

$$= \frac{\sqrt{2}}{4}a^4\int_{-\pi/2}^{\pi/2}\cos^5\varphi\,d\varphi = \frac{\sqrt{2}}{2}a^4\int_0^{\pi/2}\cos^5\varphi\,d\varphi$$

$$= \frac{\sqrt{2}}{2}a^4\int_0^{\pi/2}\left(\cos^2\varphi\right)^2\cos\varphi\,d\varphi = \frac{\sqrt{2}}{2}a^4\int_0^{\pi/2}\left(1-\sin^2\varphi\right)^2\cos\varphi\,d\varphi$$

$$= \frac{\sqrt{2}}{2}a^4\int_0^{\pi/2}\left(1 - 2\sin^2\varphi + \sin^4\varphi\right)d(\sin\varphi)$$

$$= \frac{\sqrt{2}}{2}a^4\left(\sin\varphi - \frac{2}{3}\sin^3\varphi + \frac{1}{5}\sin^5\varphi\right)\Big|_0^{\pi/2}$$

$$= \frac{\sqrt{2}}{2}a^4\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{4\sqrt{2}}{15}a^4$$

A N A L I Z A 2

V E Ž B E

$$\iint_S f(x,y,z)\, dS$$

$$dS = \|d\vec{S}\| = \|r_u \times r_v\|$$

1. $$\iint_S (x+y+z)\, dS$$

$$S = \{ (x,y,z) \in \mathbb{R}^3 \mid z \geq 0 \quad x^2+y^2+z^2 = a^2 \} \qquad a>0$$

$$x = u$$
$$y = v$$
$$z = \sqrt{a^2-u^2-v^2}$$

$$r_u = \left(1, \quad 0, \quad \frac{-u}{\sqrt{a^2-u^2-v^2}}\right)$$

$$r_v = \left(0, \quad 1, \quad \frac{-v}{\sqrt{a^2-u^2-v^2}}\right)$$

$$r_u \times r_v = \left(\frac{u}{\sqrt{a^2-u^2-v^2}}, \frac{v}{\sqrt{a^2-u^2-v^2}}, 1\right)$$

$$\| r_u \times r_v \| = \sqrt{\frac{u^2}{a^2-u^2-v^2} + \frac{v^2}{a^2-u^2-v^2} + 1} = \frac{a}{\sqrt{a^2-u^2-v^2}}$$

$$\iint_{u^2+v^2 \le a^2} \left(u+v+\sqrt{a^2-u^2-v^2} \right) \frac{a}{\sqrt{a^2-u^2-v^2}} \, du \, dv$$

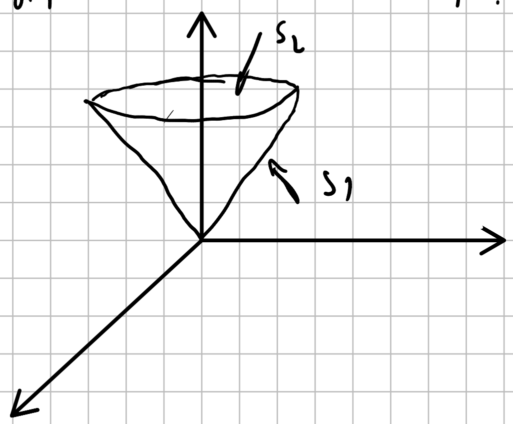
$$= a \iint_{u^2+v^2 \le a^2} \cancelto{0}{\frac{u}{\sqrt{a^2-u^2-v^2}}} \, du \, dv + a \iint_{u^2+v^2 \le a^2} \cancelto{0}{\frac{v}{\sqrt{a^2-u^2-v^2}}} + a \iint_{u^2+v^2 \le a^2} du \, dv$$

$$= \pi a^3$$

2

$$\iint_S (x^2+y^2) \, dS$$

$S = \partial T$ $\qquad$ $T : \sqrt{x^2+y^2} \le z \le 1$



$S = S_1 \cup S_2$

$$\iint_S (x^2+y^2) \, dS = \iint_{S_1} (x^2+y^2) \, dS + \iint_{S_2} (x^2+y^2) \, dS$$

$$I_2 = \iint_{S_2} (x^2+y^2) \, dS = \iint_{x^2+y^2 \le 1} (x^2+y^2) \, dx \, dy$$

$$= \iint_{\substack{0 \le \rho \le 1 \\ 0 \le \varphi \le 2\pi}} \rho^3 \, d\rho \, d\varphi =$$

$$= \int_0^{2\pi} d\varphi \int_0^1 \rho^3 \, d\rho = \varphi \Big|_0^{2\pi} \frac{1}{4} \rho^4 \Big|_0^1 = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$$

$$I_1 = \iint_{S_1} (x^2+y^2) \, dS$$

$$z = \sqrt{x^2 + y^2}$$

$$r_x = \left(1, \ 0, \ \frac{x}{\sqrt{x^2+y^2}}\right)$$

$$r_y = \left(0, \ 1, \ \frac{y}{\sqrt{x^2+y^2}}\right)$$

$$r_x \times r_y = \left(\frac{-x}{\sqrt{x^2+y^2}}, \ \frac{-y}{\sqrt{x^2+y^2}}, \ 1\right)$$

$$\| r_x \times r_y \| = \sqrt{2}$$

$$I_1 = \iint_{S_1} (x^2+y^2) \, dS = \iint_{x^2+y^2 \le 1} (x^2+y^2) \sqrt{2} \, dx \, dy$$

$$= \sqrt{2} \iint_{x^2+y^2 \le 1} (x^2+y^2) \, dx \, dy = \sqrt{2} \, I_1 = \sqrt{2} \, \frac{\pi}{2}$$

$$I = I_1 + I_2 = \frac{\pi}{2} (1 + \sqrt{2})$$

---

(3) $$\iint_S z \, dS$$

$S:$

$$x = u \cos v \qquad u \in [0, a]$$

$$y = u \sin v \qquad v \in [0, 2\pi]$$

$$z = v$$

$$r_u = (\cos v, \ \sin v, \ 0)$$

$$r_v = (-u \sin v, \ u \cos v, \ 1)$$

$$r_u \times r_v = \left(\sin v, \ -\cos v, \ \overset{= u}{u \cos^2 v + u \sin^2 v}\right)$$

$$\| r_u \times r_v \| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}$$

$$\iint_S z \, dS = \int_0^{2\pi} \int_0^a v \sqrt{1+u^2} \, du \, dv$$

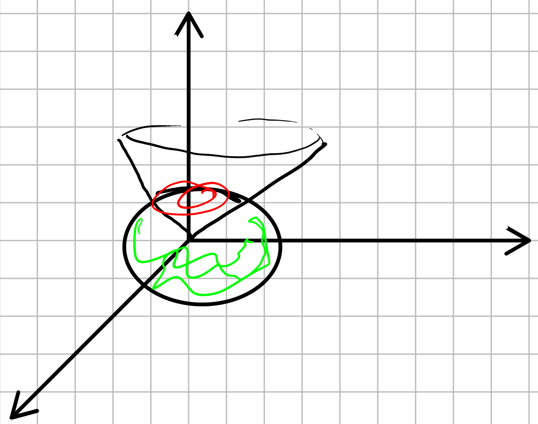
$$= \int_0^{2\pi} v \, dv \int_0^a \sqrt{1+u^2} \, du = \ldots$$

④

$$F(t) = \iint\limits_{S(t)} f(x,y,z)\, dS$$

$$f(x,y,z) = \begin{cases} x^2+y^2, & z \geq \sqrt{x^2+y^2} \\ 0, & z < \sqrt{x^2+y^2} \end{cases}$$

$$S(t): \quad x^2+y^2+z^2 = t^2$$



$$A(t) = S(t) \cap \{ z \geq \sqrt{x^2+y^2} \}$$
$$B(t) = S(t) \cap \{ z < \sqrt{x^2+y^2} \}$$

$$\iint\limits_{S(t)} f(x,y,z)\, dS = \iint\limits_{A(t)} f(x,y,z)\, dS + \iint\limits_{B(t)} \overset{0}{\overset{\shortparallel}{f(x,y,z)}}\, dS$$

$$= \iint\limits_{A(t)} (x^2+y^2)\, dS$$

$$dS = \frac{t}{\sqrt{t^2-x^2-y^2}} \qquad (\text{URAĐENO RANIJE})$$

$$x^2+y^2+z^2 = t^2 \qquad z = \sqrt{x^2+y^2}$$

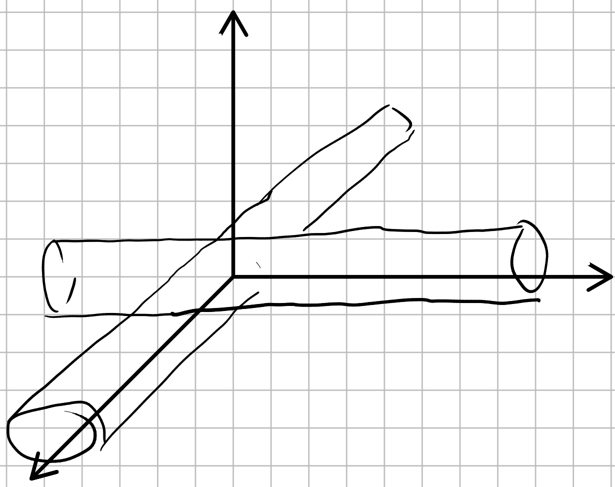
$$2x^2+2y^2 = t^2$$

$$x^2+y^2 = \frac{t^2}{2}$$

$$I = \iint\limits_{x^2+y^2 \leq \frac{t^2}{2}} (x^2+y^2) \frac{t}{\sqrt{t^2-x^2-y^2}}\, dx\, dy = t\int_0^{2\pi} \int_0^{\frac{t}{\sqrt{2}}} \frac{\rho^3}{\sqrt{t^2-\rho^2}}\, d\rho\, d\varphi = \dots$$

⑤ $S: \quad x^2+z^2=a^2 \qquad y^2+z^2=a^2 \qquad a>0$

$P(S)=?$



VOČITI SIMETRIJE

$x \rightsquigarrow -x$

$y \rightsquigarrow -y$

$z \rightsquigarrow -z$

$x \rightsquigarrow y$

$$P(S) = 16\, P(S_1)$$

$$S_1 = S \cap x \geq 0 \cap y \geq 0 \cap z \geq 0 \cap y \geq x$$

$$z = \sqrt{a^2-y^2}$$

$$r_x = (1, 0, 0)$$

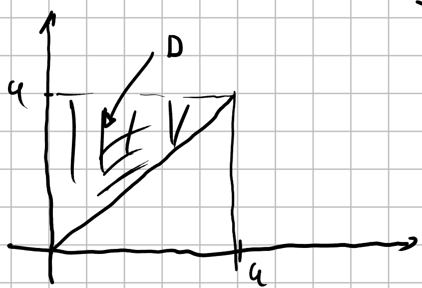
$$r_y = \left(0, 1, \frac{-y}{\sqrt{a^2-y^2}}\right)$$

$$r_x \times r_y = \left(0, \frac{y}{\sqrt{a^2-y^2}}, 1\right)$$

$$\| r_x \times r_y \| = \sqrt{\frac{y^2}{a^2-y^2}+1} = \frac{a}{\sqrt{a^2-y^2}}$$

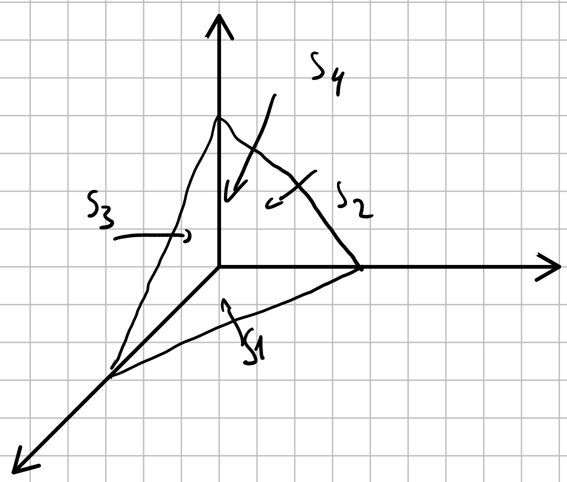
$$P(S) = 16 \iint_D \frac{a}{\sqrt{a^2-y^2}}\, dx\, dy = 16a \int_0^a \int_0^y \frac{dx}{\sqrt{a^2-y^2}}\, dy$$

$$= 16a \int_0^a \frac{y\, dy}{\sqrt{a^2-y^2}} = \ldots$$



⑥ $$\iint_S \frac{ds}{(1+x+y)^2} \qquad S = \partial T \qquad T:$$

$$x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq 1$$



S_1 : x, y - PARAMETRI $z = 0$

$$I_1 = \iint_{D_1} \frac{dx\,dy}{(1+x+y)^2} = \int_0^1 \int_0^{1-x} \frac{dy}{(1+x+y)^2}\,dx = \int_0^1 \left. \frac{-1}{1+x+y} \right|_0^{1-x} dx =$$

$$= \int_0^1 \left(-\frac{1}{2} + \frac{1}{1+x}\right) dx$$

$$= \left(-\frac{1}{2}x + \log(1+x)\right)\Big|_0^1 = -\frac{1}{2} + \log 2$$

S_2 : y i z PARAMETRI $x = 0$

$$I_2 = \int_0^1 \int_0^{1-z} \frac{dy}{(1+y)^2}\,dz = \int_0^1 \left. \frac{-1}{1+y} \right|_0^{1-z} dz$$

$$= \int_0^1 \left(\frac{-1}{2-z} + 1\right) dz = \left(-\log(2-z) + z\right)\Big|_0^1$$

$$= -\log 2 + 1$$

$\underline{\underline{S_3}}$ ZBOG SIMETRIJE $x \rightsquigarrow y$

$$\iint_{S_3} \frac{dS}{(1+x+y)^2} = \iint_{S_2} \frac{dS}{(1+x+y)^2}$$

$\underline{\underline{S_4}}$ $z = 1 - x - y$ $\qquad r_x = (1, 0, -1)$

$$r_y = (0, 1, -1)$$

$$\| r_x \times r_y \| = \sqrt{3}$$

$$I_4 = \iint_{D_1} \frac{\sqrt{3}\, dx\, dy}{(1+x+y)^2} = \sqrt{3}\, I_1$$

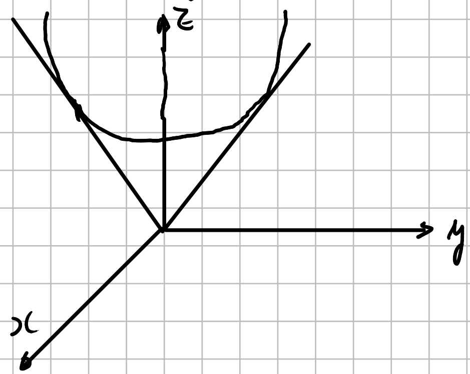
$$I = I_1 + I_2 + I_3 + I_4 = (1+\sqrt{3})(\log 2 - 1) + 2(1 - \log 2)$$

---

⑦ ODREDITI POVRŠINU OMOTAČA TELA T.

T OGRANIČENO POVRŠIMA:

$$z = 1 + x^2 + y^2 \qquad z = \sqrt{4x^2 + 4y^2}$$



$$z = 1 + x^2 + y^2 \quad i \quad z = \sqrt{4x^2 + 4y^2}$$

$$z = 2\sqrt{x^2+y^2}$$

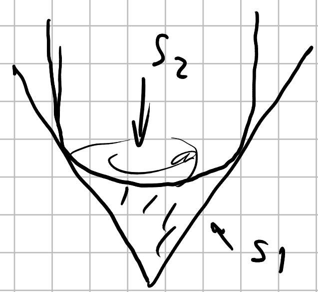
$$z = 1 + \rho^2 \qquad z = 2\rho$$

$$\rho^2 + 1 = 2\rho$$

$$\rho^2 - 2\rho + 1 = 0$$

$$(\rho - 1)^2 = 0$$

$$\rho = 1$$



$$P(S) = P(S_1) + P(S_2)$$

$$P(S_1) = ?$$

$$S_1 : \quad z = 2\sqrt{x^2+y^2}$$

$$r_x = \left(1,\ 0,\ \frac{2x}{\sqrt{x^2+y^2}}\right)$$

$$r_y = \left(0,\ 1,\ \frac{2y}{\sqrt{x^2+y^2}}\right)$$

$$r_x \times r_y = \left(\frac{-2x}{\sqrt{x^2+y^2}},\ \frac{-2y}{\sqrt{x^2+y^2}},\ 1\right)$$

$$\| r_x \times r_y \| = \sqrt{\frac{4x^2}{x^2+y^2} + \frac{4y^2}{x^2+y^2} + 1} = \sqrt{5}$$

$$P(S_1) = \iint\limits_{S_1} ds = \iint\limits_{x^2+y^2\le 1} \sqrt{5}\, dx\, dy = \sqrt{5} \iint\limits_{x^2+y^2\le 1} dx\, dy = \sqrt{5}\,\pi$$

$\underline{\underline{S_2}}$:

$$z = 1 + x^2 + y^2$$

$$r_x = (1,\ 0,\ 2x)$$

$$r_y = (0,\ 1,\ 2y)$$

$$r_x \times r_y = (-2x,\ -2y,\ 1)$$

$$\| r_x \times r_y \| = \sqrt{4x^2 + 4y^2 + 1}$$

$$P(S_2) = \iint\limits_{S_2} ds = \iint\limits_{x^2+y^2\le 1} \sqrt{4x^2+4y^2+1}\, dx\, dy$$

$$= \int_0^{2\pi} d\varphi \int_0^1 \sqrt{1+4\rho^2}\,\rho\, d\rho = \quad \begin{array}{|l} t^2 = 1+4\rho^2 \\ 2t\,dt = 8\rho\, d\rho \end{array} \quad \begin{array}{c|c|c} \rho & 0 & 1 \\ \hline t & 1 & \sqrt{5} \end{array}$$

$$= 2\pi \int_0^{\sqrt{5}} t\,\frac{t\,dt}{4} = \frac{\pi}{2}\,\frac{1}{3}\,t^3\Big|_0^{\sqrt{5}} = \frac{\pi}{6}\, 5\sqrt{5}$$

$$P(S) = P(S_1) + P(S_2) = \pi\sqrt{5} + \frac{5}{6}\pi\sqrt{5} = \frac{11}{6}\pi\sqrt{5}$$

DOMAĆI:

1. $$\iint_S |xyz| \, dS$$

   $S: \quad z = x^2 + y^2 \qquad 0 \le z \le 1$

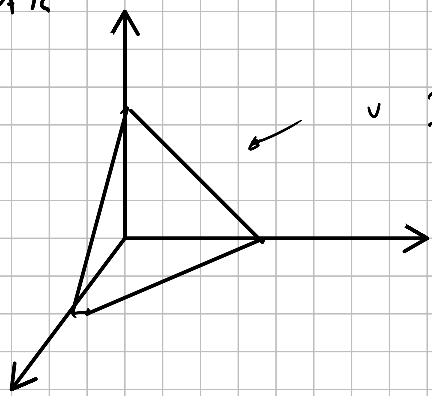
2. ODREDITI POVRŠINU POVRŠI $z = xy$ U CILINDRU $x^2 + y^2 = 4$.

$$\iint_{S^+} \underbrace{\frac{x(y^2 - z^2)\, dy\, dz + y(z^2 - x^2)\, dz\, dx + z(x^2 - y^2)\, dx\, dy}{x^2 + y^2 + z^2}}_{\omega}$$

SINGULARITET U $(0,0,0)$

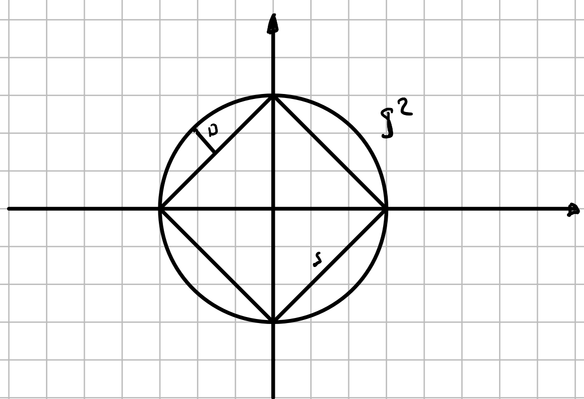
$S^+: \quad |x| + |y| + |z| \le 1$

OKTAEDAR



U I OKTANTU

U OSTALIM SIMETRIČNO



- POPREČNI PRESEK
- KAO KOD FORME $\dfrac{x\,dy - y\,dx}{x^2 + y^2}$

PRAVIMO ZASEK D

$$\iint_{S^+} \omega + \iint_{D^+} \omega + \iint_{S^{2-}} \omega + \iint_{D^-} \omega = \iiint_T d\omega$$

$$d\omega = \frac{(y^2-z^2)(x^2+y^2+z^2) - 2x^2(y^2-z^2)}{(x^2+y^2+z^2)^2}\, dx\, dy\, dz +$$

$$+ \frac{(z^2-x^2)(x^2+y^2+z^2) - 2y^2(z^2-x^2)}{(x^2+y^2+z^2)^2}\, dy\, dz\, dx$$

$$+ \frac{(x^2-y^2)(x^2+y^2+z^2) - 2z^2(x^2-y^2)}{(x^2+y^2+z^2)^2}\, dz\, dx\, dy \quad = 0$$

$$\Rightarrow \quad \iint_{S^+} \omega = \iint_{S^2} \omega$$

$$\omega\Big|_{S^2} = x(y^2-z^2)\, dy\, dz + y(z^2-x^2)\, dz\, dx + z(x^2-y^2)\, dx\, dy$$

$$I = \iint_{S^2} \omega = 3 \iint_{S^2} z(x^2-y^2)\, dx\, dy \qquad \leftarrow \text{ZBOG SIMETRIJE}$$

$$= 3\left(\iint_{S^2 \cap z \geq 0} z(x^2-y^2)\, dx\, dy + \iint_{S^2 \cap z < 0} z(x^2-y^2)\, dx\, dy \right)$$

DELIMO NA DELOVE GDE JE z FUNKCIJA

$$= 3\left(\iint_{x^2+y^2 \leq 1} \sqrt{1-x^2-y^2}\,(x^2-y^2)\, dx\, dy - \iint_{x^2+y^2 \leq 1} \left(-\sqrt{1-x^2-y^2}\right)(x^2-y^2)\, dx\, dy \right)$$

- ZBOG ORIJENTACIJE

$$= 6 \iint_{x^2+y^2 \leq 1} \sqrt{1-x^2-y^2}\,(x^2-y^2)\, dx\, dy =$$

$$= 6 \iint_{\substack{0 \leq \rho \leq 1 \\ 0 \leq \varphi \leq 2\pi}} \sqrt{1-\rho^2}\,(\rho^2\cos^2\varphi - \rho^2\sin^2\varphi)\,\rho\, d\rho\, d\varphi =$$

$$= 6 \int_0^{2\pi} (\cos^2\varphi - \sin^2\varphi)\, d\varphi \int_0^1 \sqrt{1-\rho^2}\,\rho^3\, d\rho$$

АНАЛИЗА 2

ВЕЖБЕ

ФУРИЈЕОВИ РЕДОВИ

ПОСМАТРАМО ПРОСТОР

$C_0\ [a,b]$

УВОДИМО СКАЛАРНИ ПРОИЗВОД

$$f \cdot g = \int_a^b f(x)\, g(x)\, dx$$

ОСОБИНЕ:

$$f \cdot g = g \cdot f$$

$$(\lambda f) \cdot g = \lambda (f \cdot g)$$

$$(f+g)h = fh + gh$$

$$f \cdot f \geq 0$$

$$f \cdot f = 0 \iff f = 0$$

НОРМА

$$\|f\| = \sqrt{f \cdot f} = \sqrt{\int_a^b f^2(x)\, dx}$$

РАСТОЈАЊЕ:

$$d(f,g) = \sqrt{(f-g)(f-g)} = \sqrt{\int_a^b (f(x)-g(x))^2\, dx}$$

KOŠI - ŠVARCOVA NEJEDNAKOST

$$(f\cdot g)^2 \le (f\cdot f)\,(g\cdot g)$$

NEJEDNAKOST MINKOVSKOG

$$\sqrt{(f+g)(f+g)} \le \sqrt{f\cdot f} + \sqrt{g\cdot g}$$

SVE UOBIČAJENE OSOBINE SKALARNOG PROIZVODA

ORTONORMIRANI SISTEM NA $C_0[-\pi,\pi]$

$$\frac{1}{\sqrt{2\pi}},\ \frac{\cos x}{\sqrt{\pi}},\ \frac{\sin x}{\sqrt{\pi}},\ \dots,\ \frac{\cos nx}{\sqrt{\pi}},\ \frac{\sin nx}{\sqrt{\pi}},\ \dots$$

$$m\neq n \qquad \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \int_{-\pi}^{\pi} \frac{1}{2}\left(\cos(m+n)x + \cos(m-n)x\right) dx = 0$$

$$m=n \qquad \int_{-\pi}^{\pi} \cos^2 nx \, dx = \int_{-\pi}^{\pi} \frac{1+\cos 2nx}{2}\, dx = \pi$$

$\vdots$

SLIČNO NA $C_0[-l,l]$ IMAMO ORTONORMIRAN SISTEM

$$\frac{1}{\sqrt{2l}},\ \frac{\cos\frac{\pi x}{l}}{\sqrt{l}},\ \frac{\sin\frac{\pi x}{l}}{\sqrt{l}},\ \dots,\ \frac{\cos\frac{n\pi x}{l}}{\sqrt{l}},\ \frac{\sin\frac{n\pi x}{l}}{\sqrt{l}},\ \dots$$

- FURIJEOVI KOEFICIJENTI

$$\alpha_n = x\cdot e_n \qquad \{e_n\} \text{ - ORTONORMIRAN SISTEM}$$

FURIJEOV RED

$$\sum_{n=1}^{\infty} \alpha_n e_n$$

BESELOVA NEJEDNAKOST

$$\sum_{i=1}^{n} (x e_i)^2 \le \|x\|^2$$

SVE OVO VAŽI U SVAKOM PREDHILBERTOVOM PROSTORU

U HILBERTOVOM PROSTORU EKVIVALENTNO JE

1° $\forall \varepsilon > 0$ i $\forall x \in X$ POSTOJE SKALARI $\lambda_1, \ldots, \lambda_n$ TD.

$$\|x - \sum_{i=1}^{n} \lambda_i l_i \| < \varepsilon$$

2° $\forall x \in X$ $\quad \sum_{n=1}^{\infty} (x \cdot l_n)\, l_n = x$

3° $\forall x \in X$ $\quad \sum_{n=1}^{\infty} (x \cdot l_n)^2 = \|x\|$

PARSERVALOVA JEDNAKOST

JEDNA OD POSLEDICA JE

$\forall n \in \mathbb{N} \quad x \perp l_n \Rightarrow x = 0$

OVAKAV SISTEM SE NAZIVA POTPUNIM, TJ. PREDSTAVLJA ORTONORMIRANU BAZU.

TEOREMA:

$f : \mathbb{R} \to \mathbb{R} \quad f \in C(\mathbb{R}) \quad 2\pi$ - PERIODIČNA

TADA $\forall \varepsilon > 0$ POSTOJI TRIGONOMETRIJSKI POLINOM $t(x)$ TD.

$\forall x \in \mathbb{R} \quad |f(x) - t(x)| < \varepsilon$

SISTEM KOJI SMO DALI JE POTPUN.

FURIJEOVI KOEFICIJENTI SU:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

PARSERVALOVA JEDNAKOST:

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx$$

FURIJEOV RED JE:

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

ČEMU KONVERGIRA FURIJEOV RED?

(FURIJEOV RED MOŽE DA DIVERGIRA)

TEOREMA:

$f: \mathbb{R} \to \mathbb{R}$ $\quad f$ 2π-PERIODIČNA

f APSOLUTNO INTEGRABILNA

AKO:

1° POSTOJE LIMESI $f(x+)$ i $f(x-)$

2° ZA NEKO $\varepsilon > 0$ KONVERGIRAJU

$$\int_0^{\varepsilon} \frac{f(x-u) - f(x-)}{u}\,du \quad ; \quad \int_0^{\varepsilon} \frac{f(x+u) - f(x+)}{u}\,du$$

FURIJEOV RED KONVERGIRA U TAČKI x KA:

$$\frac{f(x+) + f(x-)}{2}$$

SPECIJALNO AKO JE f NEPREKIDNA U x

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

- AKO JE FUNKCIJA PARNA $\Rightarrow \forall n \quad b_n = 0$
- AKO JE FUNKCIJA NEPARNA $\Rightarrow \forall n \quad a_n = 0$
- AKO SU SVI KOEFICIJENTI FUNKCIJA f i g ISTI $\Rightarrow f = g$

① $f(x) = \cos\frac{x}{2}$

a) Razviti funkciju f u Furijeov red

b) Naći sume $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$ i $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

a) f je neprekidna (pa i apsolutno integrabilna na svakom kompaktnom intervalu)

f je 2π periodična

f je parna funkcija

$\Rightarrow \quad \forall n \quad b_n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \left|\cos\frac{x}{2}\right| dx = \frac{2}{\pi}\int_0^{\pi} \cos\frac{x}{2}\,dx$$

$$= \frac{4}{\pi} \sin\frac{x}{2}\Big|_0^{\pi} = \frac{4}{\pi}\left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \left|\cos\frac{x}{2}\right| \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \cos\frac{x}{2}\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left(\cos\left(\frac{x}{2}+nx\right) + \cos\left(\frac{x}{2}-nx\right)\right) dx$$

$$= \frac{1}{\pi}\left(\int_0^{\pi} \cos\frac{2n+1}{2}x\,dx + \int_0^{\pi}\cos\frac{1-2n}{2}x\,dx\right)$$

$$= \frac{1}{\pi}\left(\frac{2}{2n+1}\sin\frac{2n+1}{2}x\Big|_0^{\pi} + \frac{2}{1-2n}\sin\frac{1-2n}{2}x\Big|_0^{\pi}\right)$$

$$= \frac{1}{\pi}\left(\frac{2}{2n+1}(-1)^n + \frac{2}{1-2n}(-1)^n\right)$$

$$= \frac{2(-1)^n}{\pi}\,\frac{1-2n+2n+1}{(2n+1)(1-2n)} = \frac{4(-1)^n}{\pi(1-4n^2)}$$

$$S(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1-4n^2}\cos nx$$

б)

$x=0 \qquad S(0) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$

f је непрекидна у 0 па је

$$S(0) = f(0) = 1$$

$$1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} = \frac{\pi}{4}\left(\frac{2}{\pi}-1\right) = \frac{1}{2} - \frac{\pi}{4}$$

$x=\pi$

$$S(\pi) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\underbrace{\cos n\pi}_{=(-1)^n}$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

f је непрекидна у π

$$S(\pi) = f(\pi) = 0$$

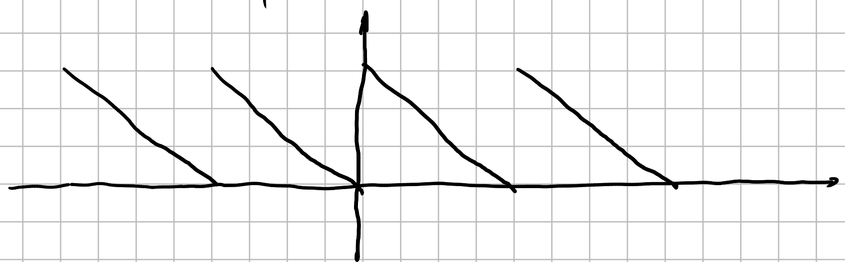
$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}$$

---

(2)

$$f(x) = \begin{cases} 0, & x \le 0 \\ \pi - x, & x > 0 \end{cases} \quad \text{на } (-\pi, \pi)$$

и f је 2π периодична



f је део-по-део непрекидна

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} \overset{0}{\cancel{0\sin nx\,dx}} + \int_0^{\pi}(\pi - x)\sin nx\,dx\right)$$

$$= \int_0^\pi \sin nx - \frac{1}{\pi}\int_0^\pi x \sin nx\, dx = \left| \begin{array}{ll} u = x & dv = \sin nx\, dx \\ du = dx & v = -\frac{1}{n}\cos nx \end{array} \right|$$

$$= -\frac{1}{n}\cos nx \Big|_0^\pi - \frac{1}{\pi}\left(-\frac{1}{n} x \cos nx \Big|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\, dx\right)$$

$$= -\frac{1}{n}\left((-1)^n - 1\right) - \frac{1}{\cancel{\pi}}\left(-\frac{1}{n}\cancel{\pi}(-1)^n + \frac{1}{n^2}\underbrace{\cancel{\sin nx \Big|_0^\pi}}_{0}\right)$$

$$= -\frac{1}{n}(-1)^n + \frac{1}{n} + \frac{1}{n}(-1)^n = \frac{1}{n}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_0^\pi (\pi - x)\, dx$$

$$= \int_0^\pi dx - \frac{1}{\pi}\int_0^\pi x\, dx$$

$$= \pi - \frac{1}{\pi}\,\frac{1}{2}x^2\Big|_0^\pi = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^\pi (\pi - x)\cos nx\, dx =$$

$$= \underbrace{\cancel{\int_0^\pi \cos nx\, dx}}_{0} - \frac{1}{\pi}\int_0^\pi x \cos nx\, dx = \left| \begin{array}{ll} u = x & dv = \cos nx\, dx \\ du = dx & v = \frac{1}{n}\sin nx \end{array} \right|$$

$$= -\frac{1}{\pi}\left(\underbrace{\cancel{\frac{x}{n}\sin nx \Big|_0^\pi}}_{0} - \frac{1}{n}\int_0^\pi \sin nx\, dx\right)$$

$$= \frac{-1}{\pi n^2}\cos nx \Big|_0^\pi = \frac{1 - (-1)^n}{\pi n^2}$$

$$S(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1 - (-1)^n}{\pi n^2}\cos nx + \frac{1}{n}\sin nx\right)$$

---

③

$$f(x) = x \quad \text{na} \quad (0, \pi)$$

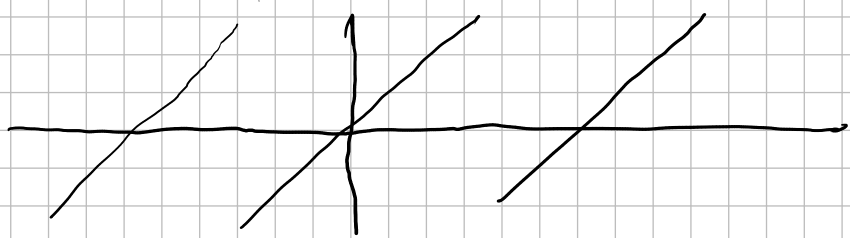
a) Razviti po sinusima

b) Razviti po kosinusima

v) Naći sume $\sum_{n=1}^{\infty} \frac{1}{n^2}$ i $\sum_{n=4}^{\infty} \frac{(-1)^n}{2n+1}$

a) Razvijamo x po sinusima pa funkciju produžujemo neparno i 2π periodično

$f_1(x) = x$ na $(-\pi, \pi)$ i 2π periodična



Funkcija je deo-po-deo neprekidna neparna $\Rightarrow \forall_n \; a_n = 0$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx =$$

$$= \left| \begin{array}{ll} u = x & dv = \sin nx \, dx \\ du = dx & v = -\frac{1}{n} \cos nx \end{array} \right| =$$

$$= \frac{2}{\pi} \left(-\frac{x}{n} \cos nx \Big|_0^{\pi} + \frac{1}{n} \cancel{\int_0^{\pi} \cos nx \, dx}^{\,0} \right)$$

$$= \frac{2}{\pi} \cdot \frac{-\pi}{n} \cos n\pi = \frac{(-1)^{n+1} 2}{n}$$

$$S_1(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

b) po kosinusima

$\Rightarrow$ produžujemo funkciju parno i 2π periodično

$f_2(x) = |x|$ na $(-\pi, \pi)$

f_2 - parna $\Rightarrow$ $\forall n$ $b_n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi} x^2\Big|_0^{\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x \cos nx\,dx = \left| \begin{array}{ll} u = x & dv = \cos nx\,dx \\ du = dx & v = \frac{1}{n}\sin nx \end{array} \right|$$

$$= \frac{2}{\pi}\left(\cancel{\frac{x}{n}\sin nx\Big|_0^{\pi}}^{\,0} - \frac{1}{n}\int_0^{\pi} \sin nx\,dx \right)$$

$$= \frac{+2}{\pi n^2} \cos nx\Big|_0^{\pi} = \frac{2}{\pi n^2}\left((-1)^n - 1\right)$$

$$S_2(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx$$

v) Primenom Parsevalove jednakosti na prvi primer

$$\sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\,\frac{1}{3} x^3\Big|_{-\pi}^{\pi} = \frac{1}{3\pi} 2\pi^3 = \frac{2\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

A zamenom $x = \frac{\pi}{2}$ u prvom primeru dobijamo

$$\frac{\pi}{2} = f\left(\frac{\pi}{2}\right) = S\left(\frac{\pi}{2}\right) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin n\frac{\pi}{2}$$

za parno n ovo je 0

$$= 2\sum_{n=0}^{\infty} \frac{(-1)^{(2n+1)+1}}{2n+1} \sin(2n+1)\frac{\pi}{2} = 2\sum_{n=0} \frac{(-1)^n}{2n+1}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} - 1$$

---

(4) $$f(x) = \sin^2 x$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$f(x) = \frac{1 - \cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$\forall n \quad b_n = 0$

$a_0 = 1 \qquad a_2 = -\frac{1}{2}$

$\forall n \quad n \notin \{0, 2\} \qquad a_n = 0$

---

(5) $$f(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & x \in (1,\pi) \end{cases}$$

f - parna | 2π - periodična

a) Razvij f v Fourierovo vrsto

b) Izračunaj sume $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ $\quad \sum_{n=1}^{\infty} \frac{\sin 2n}{n}$

a) f je parna $\Rightarrow \forall n \quad b_n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\left(\int_0^1 dx + \int_1^{\pi} 0 \cdot dx\right) = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^1 \cos nx\,dx$$

$$= \frac{2}{\pi n} \sin nx \Big|_0^1 = \frac{2\sin n}{\pi n}$$

$$S(x) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin n}{n} \cos nx$$

Za $x=0$ dobijamo

$$1 = f(0) = S(0) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin n}{n}$$

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\pi}{2}\left(1 - \frac{1}{\pi}\right) = \frac{\pi}{2} - \frac{1}{2}$$

Za $x=1$

$$S(1) = \frac{1}{\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{2\sin n \cos n}{n} = \frac{1}{\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2n}{n}$$

Sa druge strane je:

$$S(1) = \frac{f(1^+) + f(1^-)}{2} = \frac{1}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\sin 2n}{n} = \pi\left(\frac{1}{2} - \frac{1}{\pi}\right) = \frac{\pi}{2} - 1$$

---

(6) Data je 2-periodična funkcija

$$f(x) = \begin{cases} \cos \pi x, & x \in [-1, 0) \\ e + \sin \pi x, & x \in [0, 1] \end{cases}$$

a) Razviti funkciju f u Furijeov red

b) Naći sumu $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n + 1}{n^2 - 1}$

v) Da li dobijeni Furijeov red ravnomerno konvergira na intervalu $[-1, 1]$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} \cos\pi x\,dx + \int_{0}^{1} (e+\sin\pi x)\,dx$$

$$= \frac{1}{\pi}\cancelto{0}{\sin\pi x}\Big|_{-1}^{0} + e\int_0^1 dx + \int_0^1 \sin\pi x\,dx$$

$$= e - \frac{1}{\pi}\cos\pi x\Big|_0^1 = e - \frac{-1-1}{\pi} = e + \frac{2}{\pi}$$

$n \neq 1$ $$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{-1}^{0} \cos\pi x \cos n\pi x\,dx +$$

$$+ e\int_0^1 \cancelto{0}{\cos n\pi x\,dx} + \int_0^1 \sin\pi x\cos n\pi x\,dx$$

$$= \int_{-1}^{0} \frac{1}{2}\left(\cos(n+1)\pi x + \cos(n-1)\pi x\right)dx$$

$$+ \int_0^1 \frac{1}{2}\left(\sin(n+1)\pi x + \sin(1-n)\pi x\right)dx$$

$$= \frac{1}{2}\Bigg(\frac{1}{(n+1)\pi}\cancelto{0}{\sin(n+1)\pi x}\Big|_{-1}^{0} + \frac{1}{(n-1)\pi}\cancelto{0}{\sin(n-1)\pi x}\Big|_{-1}^{0}$$

$$- \frac{1}{(n+1)\pi}\cos(n+1)\pi x\Big|_0^1 - \frac{1}{(1-n)\pi}\cos(1-n)\pi x\Big|_0^1\Bigg)$$

$$= \frac{1}{2\pi}\left(\frac{-(-1)^{n+1}+1}{n+1} - \frac{(-1)^{n+1}-1}{(1-n)}\right)$$

$$= \frac{1-(-1)^{n+1}}{2\pi}\cdot\frac{1-\cancel{n}+\cancel{n}+1}{(n+1)(1-n)} = \frac{1+(-1)^n}{\pi(1-n^2)}$$

Za $n=1$ prethodno ne važi, jer bismo imali deljenje nulom.

$$a_1 = \int_{-1}^{1} f(x)\cos\pi x = \int_{-1}^{0}\cos^2\pi x\,dx + e\int_0^1 \cancelto{0}{\cos\pi x} + \int_0^1 \cos\pi x\sin\pi x\,dx$$

$$= \int_{-1}^{0}\frac{1+\cos 2\pi x}{2}\,dx + \frac{1}{2}\int_0^1 \sin 2\pi x\,dx$$

$$= \frac{1}{2} + \frac{1}{2}\int_{-1}^{0} \cos 2\pi x \, dx - \frac{1}{4\pi} \cos 2\pi x \Big|_0^1 = \frac{1}{2}$$

$n \neq 1$

$$b_n = \int_{-1}^{1} f(x) \sin n\pi x \, dx = \int_{-1}^{0} \cos\pi x \sin n\pi x \, dx +$$

$$+ e \int_0^1 \sin n\pi x \, dx + \int_0^1 \sin \pi x \sin n\pi x \, dx$$

$$= \int_{-1}^{0} \frac{1}{2}\left(\sin (n+1)\pi x + \sin (n-1)\pi x \right) dx +$$

$$+ e \frac{-1}{n\pi} \cos n\pi x \Big|_0^1 + \int_0^1 \frac{1}{2}\left(-\cos (n+1)\pi x + \cos (n-1)\pi x\right) dx$$

$$= \frac{1}{2}\left(-\frac{1}{(n+1)\pi} \cos (n+1)\pi x \Big|_0^1 - \frac{1}{(n-1)\pi} \cos (n-1)\pi x \Big|_0^1 \right) +$$

$$- \frac{e}{n\pi}\left((-1)^n - 1\right)$$

$$= \frac{1}{2}\left(-\frac{1}{(n+1)\pi}\left((-1)^{n+1} - 1\right) - \frac{1}{(n-1)\pi}\left((-1)^{n-1} - 1\right)\right) - \frac{e}{n\pi}\left((-1)^n - 1\right)$$

$$= \frac{1 - (-1)^{n+1}}{2\pi}\left(\frac{1}{n+1} + \frac{1}{n-1}\right) - \frac{e\left((-1)^n - 1\right)}{n\pi}$$

$$= \frac{1 - (-1)^{n+1}}{2\pi} \, \frac{2n}{n^2 - 1} - \frac{e}{n\pi}\left((-1)^n - 1\right)$$

$$= \frac{1 - (-1)^{n+1}}{\pi (n^2 - 1)} - \frac{e}{n\pi}\left((-1)^n - 1\right)$$

$n = 1$

$$b_1 = \int_{-1}^{1} f(x) \sin \pi x \, dx =$$

$$= \int_{-1}^{0} \cos\pi x \sin \pi x \, dx + e \int_0^1 \sin \pi x \, dx + \int_0^1 \sin\pi x \sin \pi x \, dx$$

$$= \frac{1}{2}\int_{-1}^{0} \sin 2\pi x \, dx + e \frac{-1}{\pi} \cos \pi x \Big|_0^1 + \int_0^1 \frac{1 - \cos 2\pi x}{2} \, dx$$

$$= \frac{1}{2}\cdot\frac{-1}{2\pi}\cos 2\pi x\Big|_{-1}^{0} - \frac{e}{\pi}\left(\cos\pi - \cos 0\right) + \frac{1}{2}\int_0^1 dx - \frac{1}{2}\int_0^1 \cos 2\pi x\, dx$$

$$= \frac{2e}{\pi} + \frac{1}{2}$$

$$S(x) = \frac{e}{2} + \frac{1}{\pi} + \left(\frac{2e}{\pi} + \frac{1}{2}\right)\sin \pi x + \frac{1}{2}\cos\pi x + \sum_{n=2}^{\infty} \frac{1-(-1)^{n+1}}{\pi(n^2-1)} - \frac{e}{n\pi}\left((-1)^n - 1\right)\sin n\pi x$$

$$+ \sum_{n=2}^{\infty} \frac{1+(-1)^n}{\pi(1-n^2)}\cos n\pi x$$

б)

$$S(0) = \frac{e}{2} + \frac{1}{\pi} + \frac{1}{2} + \sum_{n=2}^{\infty}\frac{1+(-1)^n}{\pi(1-n^2)}$$

$$S(0) = \frac{f(0+) + f(0-)}{2} = \frac{e+1}{2}$$

$$\frac{e+1}{2} = \frac{e+1}{2} + \frac{1}{\pi} + \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{1+(-1)^n}{1-n^2}$$

$$-\frac{1}{\pi} = \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{1+(-1)^n}{1-n^2}$$

$$1 = \sum_{n=2}^{\infty}\frac{1+(-1)^n}{n^2-1}$$

$$1 = \frac{2}{3} + \sum_{n=3}^{\infty}\frac{1+(-1)^n}{n^2-1}$$

$$\sum_{n=3}^{\infty}\frac{1+(-1)^n}{n^2-1} = \frac{1}{3}$$

в) АКО БИ РЕД КОНВЕРГИРАО РАВНОМЕРНО КОНВЕРГИРАО БИ КА НЕПРЕКИДНОЈ ФУНКЦИЈИ. МЕЂУТИМ РЕД КОНВЕРГИРА КА $f(x)$ КОЈА ЈЕ ПРЕКИДНА, ПА КОНВЕРГЕНЦИЈА НИЈЕ РАВНОМЕРНА.

7 FUNKCIJU:

$$f(x) = \begin{cases} \sum_{n=1}^{\infty} \alpha^n \frac{\sin nx}{\sin x}, & x \neq k\pi, \quad |\alpha| < 1 \\ \lim_{x \to k\pi} f(x), & x = k\pi \end{cases}$$

RAZVITI U FURIJEOV RED.

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f je 2π - PERIODIČNA

f je PARNA FUNKCIJA

$\Rightarrow \quad \forall n \quad b_n = 0$

$$\lim_{x \to k\pi} \left| \frac{\sin nx}{\sin x} \right| \overset{L.P.}{=} \lim_{x \to k\pi} \left| \frac{n \cos nx}{\cos x} \right| = n$$

$$x \neq k\pi \qquad \left| \frac{\sin nx}{\sin x} \right| \leq n$$

RED $\sum_{n=1}^{\infty} \alpha^n n$ KONVERGIRA

PA PO VAJERŠTRASOVOM KRITERIJUMU

RED $\sum_{n=1}^{\infty} \alpha^n \frac{\sin nx}{\sin x}$ RAVNOMERNO KONVERGIRA

(INTEGRAL I RED ĆE MOĆI DA ZAMENE MESTA)

$$a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \cos kx \, dx = \frac{2}{\pi} \int_0^{\pi} \left(\sum_{n=1}^{\infty} \alpha^n \frac{\sin nx}{\sin x} \right) \cos kx \, dx$$

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \alpha^n \int_0^{\pi} \frac{\sin nx \cos kx}{\sin x} \, dx$$

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \alpha^n \int_0^{\pi} \frac{\sin (n+k)x + \sin (n-k)x}{\sin x} \, dx$$

$$I_l = \int_0^{\pi} \frac{\sin lx}{\sin x} \, dx$$

$$I_{l+2} - I_l = \int_0^{\pi} \frac{\sin(l+2)x - \sin lx}{\sin x}\,dx =$$

$$= \int_0^{\pi} \frac{2\cos(l+1)x \cancel{\sin x}}{\cancel{\sin x}}\,dx = \frac{2}{l+1} \sin(l+1)x \Big|_0^{\pi} = 0$$

$$I_0 = \int_0^{\pi} \frac{\sin 0x}{\sin x}\,dx = 0 \quad \Rightarrow \forall l \quad I_{2l} = 0$$

$$I_1 = \int_0^{\pi} \frac{\sin x}{\sin x}\,dx = \pi \quad \Rightarrow \forall l \quad I_{2l+1} = \pi$$

$$I_{-l} = -I_l$$ (neparnost sinusa)

Iskoristivši prethodno dobijamo

$$a_k = 2\left(\alpha^{k+1} + \alpha^{k+3} + \dots + \alpha^{k+2l+1} + \dots\right)$$

$$= 2\alpha^{k+1}\left(1 + \alpha^2 + \alpha^4 + \dots\right)$$

$$= 2\alpha^{k+1} \frac{1}{1-\alpha^2}$$

$$S(x) = \frac{\alpha}{1-\alpha^2} + \frac{2}{1-\alpha^2} \sum_{k=1}^{\infty} \alpha^{k+1} \cos kx$$

Domaći:

(1) $f(x) = x$ za $x \in (2\pi, 3\pi)$

f - parna i 2π - periodična

(2) $f(x) = e^{|x|}$ $x \in [-\pi, \pi]$

$S(0) = ?$ $S(2021\pi) = ?$ $S[\mathbb{R}] = ?$

(3) $f(x) = x \sin x$

④ $h(x) = \arcsin \cos x$

⑤ $g(t) = (t) \sin \pi t$

(t) – nastojanje po najbližeg celog broja

⑥ $f(x) = \sin \pi x + (-1)^{[x]}$

PRIMERI DRUGIH ORTOGONALNIH SISTEMA

1) POLINOMI ČEBIŠEVA

$$T_n(x) = \frac{1}{2^{n-1}} \cos(n \arccos x) \qquad (-1, 1)$$

TEŽINSKA FUNKCIJA: $\frac{1}{\sqrt{1-x^2}}$

$$\int_{-1}^{1} \frac{T_m(x)\, T_n(x)}{\sqrt{1-x^2}}\, dx = \frac{\pi}{2^{2n-1}}\, \delta_{mn} \qquad \delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

2) POLINOMI LEŽANDRA

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n\left((x^2-1)^n\right)}{dx^n} \qquad [-1, 1]$$

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \frac{2}{2n+1}\, \delta_{mn}$$

3) POLINOMI ABEL - LAGERA

$$L_n(x) = \frac{e^x}{n!} \frac{d^n\left(x^n e^{-x}\right)}{dx^n} \qquad (0, +\infty)$$

$$\int_0^{+\infty} e^{-x}\, L_n(x)\, L_m(x)\, dx = \delta_{mn}$$

4) POLINOMI ČEBIŠEV - ERMITA

$$H_n(x) = \frac{e^{\frac{x^2}{2}}}{n!} \frac{d^n\left(e^{-\frac{x^2}{2}}\right)}{dx^n}$$

$$\int_0^{+\infty} e^{-\frac{x^2}{2}} H_m(x) H_n(x)\, dx = \frac{\sqrt{2\pi}}{n!} \delta_{mn}$$

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1) FUNKCIJU $f(x) = x^3$ RAZVITI U RED PO POLINOMIMA ČEBIŠEVA NA $[-1, 1]$.

$$x^3 = \sum_{n=1}^{\infty} a_n T_n(x) \qquad a_n = ?$$

$$\int_{-1}^{1} \frac{x^3 T_n(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{2^{2n-1}} a_n$$

$$a_n = \frac{2^{2n-1}}{\pi} \int_{-1}^{1} \frac{x^3 \cos(n \arccos x)}{\sqrt{1-x^2}} dx = \Big/ \arccos x = t$$

$$= \frac{2^{2n-1}}{\pi} \int_0^{\pi} \cos^3 t \cos nt\, dt = \begin{cases} 0, & n \neq 1, n \neq 3 \\ \frac{3}{4}, & n = 1 \\ 1, & n = 3 \end{cases}$$

$$x^3 = \frac{3}{4} T_1(x) + T_3(x) \qquad x \in [-1, 1]$$

---

1) $\{\cos nx\}_{n \in \mathbb{N}_0}$ - NIZ NEPREKIDNIH FUNKCIJA

a) POKAZATI DA SE NIJEDAN ČLAN NIZA NE MOŽE SA PROIZVOLJNOM TAČNOŠĆU RAVNOMERNO APROKSIMIRATI LINEARNOM KOMBINACIJOM IZ OSTALIH NA $[0, \pi]$

b) Važi li isto na $[0, a]$, $0 < a < \pi$

---

a) PPS $\varepsilon = \frac{1}{2}$ $\exists\, n \in \mathbb{N}$

i postoji linearna kombinacija

$$\sum_{\substack{k=0 \\ k\neq n}}^{m} \lambda_k \cos kx$$

tdj.

$$\Big|\cos nx - \sum_{\substack{k=0 \\ k\neq n}}^{m} \lambda_k \cos kx\Big| < \frac{1}{2} \qquad \forall x \in [0, \pi]$$

$$/\ |\cos nx|\ /\ \int_0^\pi$$

$$\Big|\int_0^\pi \Big(\cos^2 nx - \sum_{\substack{k=0 \\ k\neq n}}^{m} \lambda_k \cos kx \cos nx\Big)\,dx\Big| < \frac{1}{2}\int_0^\pi |\cos nx|\,dx$$

$$\int_0^\pi \cos^2 nx\,dx = \begin{cases} \pi, & n = 0 \\ \frac{\pi}{2}, & n > 0 \end{cases} \qquad \int_0^\pi |\cos nx|\,dx = 2$$

$k \neq n$ $\quad \int_0^\pi \cos kx \cos nx\,dx = 0$

Dobijamo:

$$\frac{\pi}{2} < 1 \quad \text{ili} \quad \pi < 1$$

↯

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b) $0 < a < \pi$

Izaberemo h tdj. $a < h < \pi$

i definišemo funkciju

$$f(x) = \begin{cases} 1, & x < h \\ k(x-h)+1, & h \leq x < \pi \end{cases}$$

GDE JE k TAKVO DA JE

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx = 0$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi} \Big(\int_0^{h} \cos nx \, dx +$$

$$+ \int_h^{\pi} (kx + 1 - kh) \cos nx \, dx = \ldots$$

$$\ldots = \frac{2k}{\pi n^2} \left(\cos n\pi - \cos nh \right)$$

$$S(x) = \frac{2k}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - \cos nh}{n^2} \cos nx$$

OVAJ RED RAVNOMERNO KONVERGIRA, PO VAJERŠTRASU. $\left(\leq \frac{1}{n^2} \right)$

SPECIJALNO NA $[0, a]$ RAVNOMERNO KONVERGIRA KA $g(x) = 1$

$\Rightarrow$ $g(x) = 1$ SE MOŽE SA PROIZVOLJNOM TAČNOŠĆU RAVNOMERNO APROKSIMIRATI LINEARNOM KOMBINACIJOM OSTALIH.

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(2) $f : \mathbb{R} \to \mathbb{R}$ $\quad f \in C(\mathbb{R})$

AKO SE f MOŽE SA PROIZVOLJNOM TAČNOŠĆU APROKSIMIRATI RAVNOMERNO ALGEBARSKIM POLINOMIMA TADA JE I f POLINOM.

$$P_n(x) \rightrightarrows f(x), \; n \to \infty \quad \text{na } \mathbb{R}$$

ZA DOVOLJNO VELIKO n I $m > n$

VAŽI

$$\forall x \in \mathbb{R} \qquad |P_n(x) - f(x)| < 1 \qquad |P_m(x) - f(x)| < 1$$

$$\Rightarrow \quad \forall x \in \mathbb{R} \qquad |P_n(x) - P_m(x)| < 2$$

$P_n(x) - P_m(x)$ je ograničen polinom, pa mora biti konstanta.

$$P_m(x) - P_n(x) = C_m \qquad \forall m > n$$

$$/ \lim_{m \to \infty}$$

$$f(x) = P_n(x) + C$$

$\Rightarrow$ f je polinom

---

(3)

a) Dokazati da se proizvoljna neprekidna funkcija na $[0,1]^2$ može sa proizvoljnom tačnošću aproksimirati ravnomerno funkcijama oblika

$$\sum_{i=1}^{n} \varphi_i(x)\, \psi_i(y)$$

$$\varphi_i, \psi_i \in C[0,1]$$

b) polinomima po x i y.

---

a) $\varepsilon > 0$ dato

f je ravnomerno neprekidna na $[0,1]^2$ po Kantoru

$$\exists\, \delta > 0 \quad \text{t.d.} \quad |f(x', y') - f(x'', y'')| < \frac{\varepsilon}{3}$$

$$\text{za} \quad \sqrt{(x'-x'')^2 + (y'-y'')^2} < \delta$$

$n \in \mathbb{N}$ т.д. $\frac{1}{n} < \delta$

$$\varphi_k(x) := \max\{0, 1 - n|x - \frac{k}{n}|\} \quad k = 0, 1, \dots n$$

$$\psi_i(y) = f(x_i, y) \quad x_i = \frac{i}{n} \quad i = 0, \dots, n$$

$f(x,y)$ и $\sum_{i=0}^{n} \varphi_i(x)\psi_i(y)$ су једнаке за $x = x_k$

за $y \in [0,1]$ $\sum_{i=0}^{n} \varphi_i(x)\psi_i(y)$ су линеарне по y

на интервалима $[x_k, x_{k+1}]$

$$|f(x,y) - \sum_{i=0}^{n} \varphi_i(x)\psi_i(y)| \leq$$

$$\leq |f(x,y) - f(x_k,y)| + |f(x_k,y) - \sum_{i=0}^{n} \varphi_i(x_k)\psi_i(y)| +$$

$$+ |\sum_{i=0}^{n} \varphi_i(x_k)\psi_i(y) - \sum_{i=0}^{n} \varphi_i(x_{k+1})\psi_i(y)|$$

$$\leq \frac{\varepsilon}{3} + 0 + |\varphi_k(x_k)\psi_k(y) - \varphi_{k+1}(x_{k+1})\psi_{k+1}(y)|$$

$$= \frac{\varepsilon}{3} + |\psi_k(y) - \psi_{k+1}(y)| =$$

$$= \frac{\varepsilon}{3} + |f(x_k,y) - f(x_{k+1},y)| < \frac{2\varepsilon}{3}$$

b)

$$M = \max_{i,x,y}\{\varphi_i(x), \psi_i(y)\}$$

p_i, q_i полиноми т.д.

$$\left.\begin{aligned} |\varphi_i(x) - p_i(x)| &\leq \frac{\varepsilon}{6(n+1)(M+\varepsilon)} \\ |\psi_i(y) - q_i(x)| &= \frac{\varepsilon}{6(n+1)(M+\varepsilon)} \end{aligned}\right\} \forall x,y \in [0,1]$$

$$|\varphi_i(x)\psi_i(y) - p_i(x)q_i(y)| \leq |\varphi_i(x)\psi_i(y) - \varphi_i(x)q_i(y)| +$$

$+ |\varphi_i(x) q_i(y) - p_i(x) q_i(y)| \leq$

$\leq M |\psi_i(y) - q_i(y)| + |q_i(y)| \, |\varphi_i(x) - p_i(x)| \leq \frac{\varepsilon}{3(n+1)}$

$\Rightarrow$

$$\sum_{i=0}^{n} |\varphi_i(x) \psi_i(y) - p_i(x) q_i(y)| < \frac{\varepsilon}{3}$$

Koristeći tačku a) dobijamo

$$\left| \sum_{i=0}^{n} p_i(x) q_i(y) - f(x,y) \right| \leq$$

$$\left| \sum_{i=0}^{n} p_i(x) q_i(y) - \sum_{i=0}^{n} \varphi_i(x) \psi_i(y) \right| + \left| \sum_{i=0}^{n} \varphi_i(x) \psi_i(y) - f(x,y) \right| <$$

$$< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon$$