ANALIZA 2

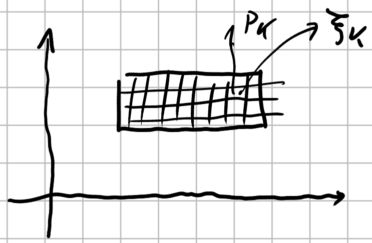
VEŽBE

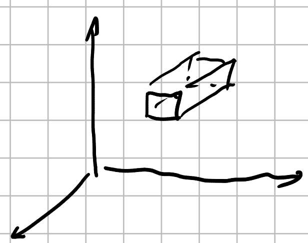
VIŠESTRUKI INTEGRALI

$f : \mathcal{I} \longrightarrow \mathbb{R}$

$\mathcal{I} \subseteq \mathbb{R}^n$

$\mathcal{I}$ - VIŠEDIMENZIONI KVADAR (UOPŠTENJE ZATVORENOG INTERVALA)





$\mathcal{P} = \{ I_j \mid j \in A \}$ - PODELA KVADRA

I_j - PODEONI (MANJI) KVADRI

$\xi_j \in I_j$ $\quad$ ξ_j - ISTAKNUTE TAČKE

$$\sigma(f; \mathcal{P}, \xi) = \sum_{j \in A} f(\xi_j)\, \mu(I_j)$$

$$\mu(I_j) = \prod_{i=1}^{n} |x_i - y_i|$$

$\lambda(\mathcal{P})$ - PARAMETAR PODELE

$\lambda(\mathcal{P}) = \min \mu(I_j)$

$I \in \mathbb{R}$ JE INTEGRAL AKO:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \lambda(P) < \delta \Rightarrow |\sigma(f; P, \xi) - I| < \varepsilon$$

DEFINICIJA:

$A \subseteq \mathbb{R}^n$ IMA LEBEGOVU MERU NULA AKO $\forall \varepsilon > 0$ POSTOJI POKRIVANJE SKUPA A KVADRIMA I_j KOJIH IMA NAJVIŠE PREBROJIVO I VAŽI:

$$\sum_j \mu(I_j) < \varepsilon$$

- TAČKA I KONAČAN SKUP TAČAKA SU MERE NULA
- PODSKUP SKUPA MERE NULA JE MERE NULA
- NAJVIŠE PREBROJIVA UNIJA SKUPOVA MERE NULA JE MERE NULA.

OSOBINE n-INTEGRALA

- LINEARNOST

$$\int_D (\alpha f + \mu g)\, dx = \alpha \int_D f\, dx + \mu \int_D g\, dx$$

- ADITIVNOST PO SKUPU

$$D = D_1 \cup D_2 \qquad \mu(D_1 \cap D_2) = 0$$

$$\int_D f\, dx = \int_{D_1} f\, dx + \int_{D_2} f\, dx$$

- $\forall x \quad f(x) \geq g(x)$

$$\int_D f\, dx \geq \int_D g\, dx$$

SPECIJALNO:

$$f(x) \geq 0 \quad \forall x \qquad \int_D f\, dx \geq 0$$

- $Vol(D) = \int_D 1 \cdot dx \qquad P = \iint_D dx\, dy \qquad V = \iiint_T dx\, dy\, dz$

POSLEDICE FUBINIJEVE TEOREME:

$D: \quad a \leq x \leq b \qquad u(x) \leq y \leq v(x)$

$u, v \in C[a,b]$

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \left(\int_{u(x)}^{v(x)} f(x,y)\,dy \right) dx$$

TEOREMA: (SMENA PROMENLJIVE)

$\varphi : D \to \Delta$

φ JE DIFEOMORFIZAM KLASE C^1

(φ JE BIJEKCIJA i φ, φ^{-1} SU KLASE C^1)

f INTEGRABILNA NA Δ

$J = \det \varphi'(t) \neq 0 \quad \forall t \in D$

TADA JE i

$(f \circ \varphi)|J|$ INTEGRABILNA NA D I VAŽI:

$$\int_\Delta f(x)\,dx = \int_D (f \circ \varphi)(t)\,|J(t)|\,dt$$

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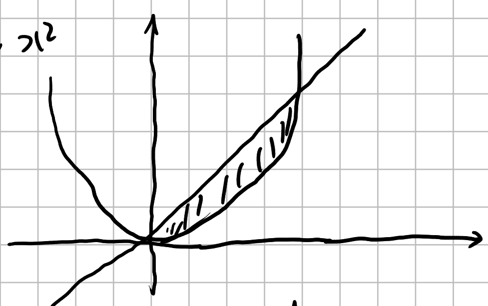
(1) $\iint_D (4x+2)\,dx\,dy$

$D = \{ (x,y) \in \mathbb{R}^2 \mid x \geq y \geq x^2 \}$

$x \geq x^2$



$x = x^2$

$x_1 = 0 \qquad x_2 = 1$

$$\iint_D (4x+2)\,dx\,dy = \int_0^1 \int_{x^2}^{x} (4x+2)\,dy\,dx =$$

$$= \int_0^1 (4x+2) \int_{x^2}^{x} dy\, dx = \int_0^1 (4x+2)\, y\Big|_{x^2}^{x} dx =$$

$$= \int_0^1 (4x+2)(x-x^2)\, dx = \int_0^1 (4x^2 - 4x^3 + 2x - 2x^2)\, dx =$$

$$= \int_0^1 (-4x^3 + 2x^2 + 2x)\, dx = \left(-x^4 + \frac{2}{3}x^3 + x^2\right)\Big|_0^1$$

$$= -1 + \frac{2}{3} + 1 = \boxed{\frac{2}{3}}$$

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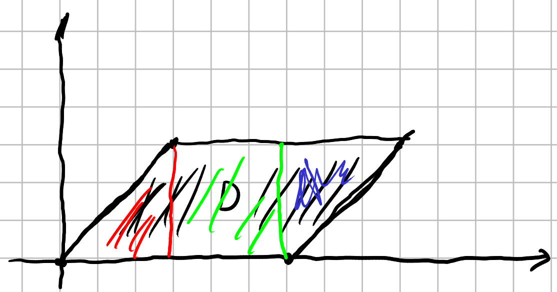
(2)

$$\iint_D \frac{dx\, dy}{1+x+y}$$

D – PARALELOGRAM SA TEMENIMA:

$(0,0)$ $(1,1)$ $(2,0)$ i $(3,1)$



− D_1

− D_2

− D_3

GRANICE ZA y →

D_1: $0 \le y \le x$

D_2: $0 \le y \le 1$

D_3: $x-2 \le y \le 1$

LAKŠE JE DA IDEMO PRVO PO x.

$y \le x \le y+2 \qquad 0 \le y \le 1$

MOŽEMO RADITI: $\iint_D = \iint_{D_1} + \iint_{D_2} + \iint_{D_3} = .$

BOLJE JE:

$$\iint_D \frac{dx\, dy}{1+x+y} = \int_0^1 \left(\int_y^{y+2} \frac{dx}{1+x+y} \right) dy$$

$$= \int_0^1 \ln(1+x+y)\Big|_y^{y+2} dy =$$

$$= \int_0^1 \left(\log(2y+3) - \log(2y+1) \right) dy$$

$$= \int_0^1 \log(2y+3)\, dy - \int_0^1 \log(2y+1)\, dy$$

$$\int \log t \, dt = \left| \begin{array}{ll} u = \lg t & dv = dt \\ du = \frac{dt}{t} & v = t \end{array} \right|$$

$$= t \log t - \int dt = t \log t - t + C$$

$t_1 = 2y+3$ $\qquad$ $t_2 = 2y+1$

$dt_1 = 2dy$ $\qquad$ $dt_2 = 2dy$

y	0	1
t_1	3	5

y	0	1
t_2	1	3

$$= \frac{1}{2} \int_3^5 \log t \, dt - \frac{1}{2} \int_1^3 \log t \, dt =$$

$$= - - -$$

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(3) ODREDITI POVRŠINU OGRANIČENU KRIVOM.

$$D: \quad (\underbrace{a_1 x + b_1 y + c_1}_{u})^2 + (\underbrace{a_2 x + b_2 y + c_2}_{v})^2 = 1$$

AKO SU $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}$ $\qquad a_1 b_2 \neq a_2 b_1$

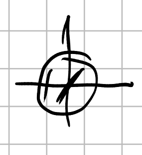
$$P = \iint_D 1 \cdot dx\, dy$$

SMENA:

$u = a_1 x + b_1 y + c_1 \qquad v = a_2 x + b_2 y + c_2$

$$|J^{-1}| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \neq 0 \quad \text{IZ USLOVA}$$

$$\Delta : \quad u^2 + v^2 \leq 1$$



$$P = \iint_D dx\,dy = \iint_\Delta |J|\,du\,dv = \frac{1}{|a_1 b_2 - a_2 b_1|} \iint_{u^2+v^2 \le 1} 1\,du\,dv$$

POVRŠINA JEDINIČNOG KRUGA

$$P = \frac{\pi}{|a_1 b_2 - a_2 b_1|}$$

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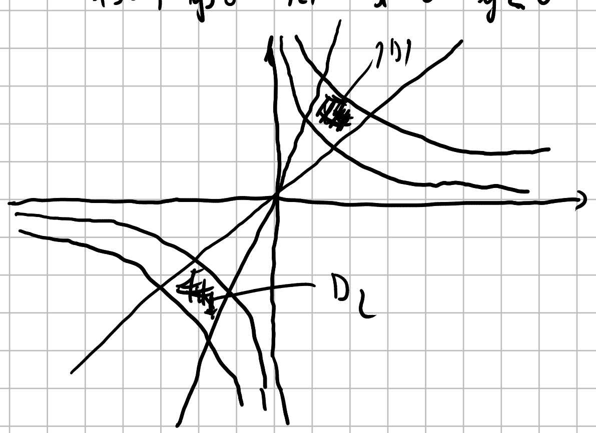
(4)

$$I = \iint_D \underbrace{\frac{x^2+y^2}{xy}}_{f}\,dx\,dy$$

$$D: \quad 1 \le xy \le 2 \qquad y^2 - 5xy + x^2 \le 0$$

$$xy > 0 \qquad (y-x)(y-4x) \le 0$$

$x>0$ i $y>0$ ili $x<0$ $y<0$ $\qquad y = x$ i $y = 4x$



$$D_1 = D \cap \{x>0, y>0\}$$

$$D_2 = D \cap \{x<0, y<0\}$$

$$u = xy$$

$$(y-x)(y-4x) \le 0 \quad \Big/ \frac{1}{x} \qquad x>0$$

$$\left(\frac{y}{x} - 1\right)\left(\frac{y}{x} - 4\right) \le 0$$

$$v = \frac{y}{x}$$

OVAKO UVEDENE SMENE NISU BIJEKCIJA NA D, ALI JESU BIJEKCIJA NA D_1 I D_2

$$f(-x,-y) = f(x,y) \qquad \mu(D_1) = \mu(D_2)$$

$$\iint_D f(x,y)\,dx\,dy = 2 \iint_{D_1} f(x,y)\,dx\,dy$$

NA D_1 UVODIMO SMENU:

$$u = xy \qquad v = \frac{y}{x}$$

$$|J^{-1}| = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{y}{x} + \frac{y}{x} = \underset{\underset{2v}{\parallel}}{2\frac{y}{x}} \neq 0$$

GRANICE:

$$1 \le u \le 2 \qquad (v-1)(v-4) \le 0$$

$$1 \le v \le 4$$

$$f(x,y) = \frac{x^2+y^2}{xy} = \frac{x}{y} + \frac{y}{x} = \frac{1}{v} + v$$

$$\iint_D \frac{x^2+y^2}{xy}\,dx\,dy = \iint_{\substack{1\le u\le 2\\ 1\le v\le 4}} \underbrace{\left(\frac{1}{v}+v\right)\frac{1}{2v}}_{\text{NE ZAVISI OD } u}\,du\,dv =$$

$$= \frac{1}{2}\int_1^2 du \int_1^4 \left(1+\frac{1}{v^2}\right) dv =$$

$$= \frac{1}{2}\; u\Big|_1^2 \;\left(v - \frac{1}{v^2}\right)\Big|_1^4 = \ldots$$

5

$$\iint_D \sin\sqrt{x^2+y^2}\;dx\,dy$$

$$D: \quad \pi^2 \le x^2+y^2 \le 4\pi^2$$

SMENA POLARNIM KOORDINATAMA:

$$x = \rho\cos\varphi \qquad J = \rho$$
$$y = \rho\sin\varphi$$

$$\Delta: \quad \pi^2 \le \rho^2 \le 4\pi^2 \qquad \pi \le \rho \le 2\pi$$

ZA φ NEMAMO USLOV $\quad 0 \le \varphi \le 2\pi$

$$\iint_D \sin\sqrt{x^2+y^2}\,dx\,dy = \iint_\Delta \sin\rho\;\rho\,d\rho\,d\varphi =$$

$$= \int_0^{2\pi} d\varphi \int_\pi^{2\pi} \rho\sin\rho\,d\rho = \left| \begin{matrix} u=\rho & dv=\sin\rho\,d\rho \\ du=d\rho & v=-\cos\rho \end{matrix}\right|$$

$$= \varphi\Big|_0^{2\pi}\left(-\rho\cos\rho\Big|_0^{2\pi} + \int_\pi^{2\pi}\cos\rho\,d\rho\right) = \dots$$

Domaći:

$$\iint_D x^2y^2\,dx\,dy$$

$$D: \quad 1 \le x^2+y^2 \le 4$$

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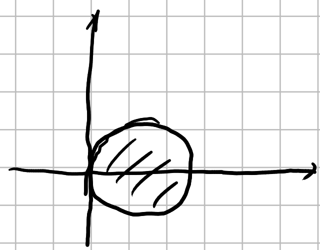
(6) $$\iint_D (x^2+y^2)\,dx\,dy$$



$D:$ $x^2+y^2 = 2x$

$x^2-2x+1+y^2=1$

$(x-1)^2+y^2=1$ - POMERENI KRUG

DA LI GRANICA ILI FUNKCIJA DA BUDE LAKŠE OPISANA?

I NAČIN

$x-1 = \rho\cos\varphi$

$y = \rho\sin\varphi$ $\qquad J=\rho$

$\Delta: \quad 0\le\rho\le 1 \qquad 0\le\varphi\le 2\pi$

$$\iint_D (x^2+y^2)\,dx\,dy = \iint_{\substack{0\le\rho\le 1\\ 0\le\varphi\le 2\pi}} \left((1+\rho\cos\varphi)^2+(\rho\sin\varphi)^2\right)\rho\,d\rho\,d\varphi =$$

$$= \iint_\Delta \left(1+2\rho\cos\varphi + \rho^2\cos^2\varphi + \rho^2\sin^2\varphi\right)\rho\,d\rho\,d\varphi$$

$$= \iint_{\Delta} (1 + 2\rho\cos\varphi + \rho^2)\rho \, d\rho \, d\varphi$$

$$= \iint_{\Delta} \rho \, d\rho \, d\varphi + 2\iint_{\Delta} \rho^2 \cos\varphi + \iint_{\Delta} \rho^3 \, d\rho \, d\varphi$$

$$= \int_0^{2\pi} d\varphi \int_0^1 \rho \, d\rho + 2\underbrace{\int_0^{2\pi} \cos\varphi \, d\varphi}_{0} \int_0^1 \rho^2 d\rho + \int_0^{2\pi} d\varphi \int_0^1 \rho^3 d\rho =$$

$$= 2\pi \frac{1}{2} \rho^2 \Big|_0^1 + 2\pi \frac{1}{4} \rho^4 \Big|_0^1 = 2\pi \left(\frac{1}{2} + \frac{1}{4}\right) = 2\pi \frac{3}{4}$$

$$= \boxed{\frac{3\pi}{2}}$$

II НАЧИН

УВОБИЧАЈЕНЕ ПОЛАРНЕ КООРДИНАТЕ

$$\rho^{\cancel{2}} \le 2\cancel{\rho} \cos\varphi \qquad (x^2 + y^2 \le 2x)$$

$$0 \le \rho \le 2\cos\varphi$$

УСЛОВ: $\cos\varphi \ge 0 \qquad -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$

$$\iint_D (x^2 + y^2) \, dx \, dy = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\varphi} \rho^3 \, d\rho \, d\varphi = \ldots$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{4} \rho^4 \Big|_0^{2\cos\varphi} d\varphi = \frac{1}{4} \int_{-\pi/2}^{\pi/2} 16 \cos^4\varphi \, d\varphi = \ldots$$

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7 ПУАСОНОВ ИНТЕГРАЛ

$\int e^{-x^2} dx$ – НИЈЕ РЕШИВ

$$I = \int_0^{+\infty} e^{-x^2} dx = ?$$

$e^{-x^2} dx$ – ПАРНА ФУНКЦИЈА

$$I = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-x^2} dx \qquad I_1 = \int_{-\infty}^{+\infty} e^{-x^2} dx$$

$$I_1^2 = \int_{-\infty}^{+\infty} e^{-x^2}\,dx \int_{-\infty}^{+\infty} e^{-y^2}\,dy =$$

$$= \iint_{\mathbb{R}^2} e^{-x^2} e^{-y^2}\,dx\,dy = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy$$

ПОЛАРНЕ КООРДИНАТЕ

$$= \iint_{\substack{0 \le \rho < \infty \\ 0 \le \varphi \le 2\pi}} e^{-\rho^2} \rho \, d\rho \, d\varphi = \int_0^{2\pi} d\varphi \int_0^{+\infty} \rho\, e^{-\rho^2}\, d\rho$$

$$\left[\begin{array}{l} t = \rho^2 \\ dt = 2\rho\, d\rho \end{array} \quad \begin{array}{c|c|c} \rho & 0 & +\infty \\ \hline t & 0 & +\infty \end{array} \right]$$

$$= 2\pi \, \frac{1}{2} \int_0^{+\infty} e^{-t}\, dt = \pi \left(-e^{-t} \right) \Big|_0^{+\infty} = \pi$$

$$I_1^2 = \pi$$

$$I_1 = \int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

$$I = \int_0^{+\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

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(7)

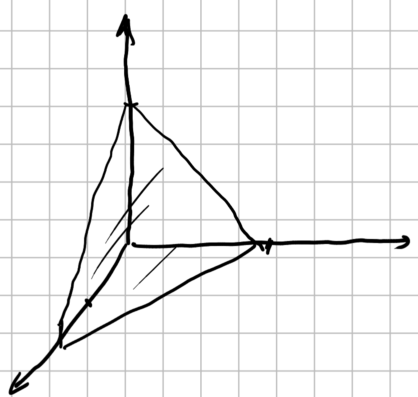
$$\iiint_T \frac{dx\,dy\,dz}{(1+x+y)^3}$$

T : ТЕТРАЕДАР ОГРАНИЧЕН РАВНИМА

$x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$

$$\downarrow$$

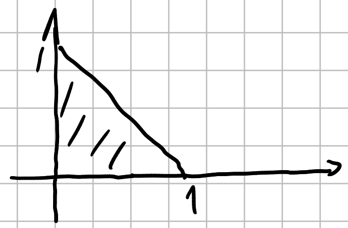
$$z = 1 - x - y$$



D - ПРОЈЕКЦИЈА НА РАВАН $z = 0$

$$\iiint_T \frac{dx\,dy\,dz}{(1+x+y+z)^3} = \iint_D \left(\int \frac{dz}{(1+x+y+z)^3} \right) dx\,dy$$

$D:$



$$= \iint_D -\frac{1}{2} \frac{1}{(1+x+y+z)^2}\Big|_0^{1-x-y} dx\,dy =$$

$$= -\frac{1}{2} \iint_D \left(\frac{1}{(1+x+y+1-x-y)^2} - \frac{1}{(1+x+y+0)^2} \right) dx\,dy =$$

$$= -\frac{1}{2} \iint_D \left(\frac{1}{4} - \frac{1}{(1+x+y)^2} \right) dx\,dy$$

$D: \quad 0 \le y \le 1-x \qquad 0 \le x \le 1$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-x} \left(\frac{1}{4} - \frac{1}{(1+x+y)^2} \right) dy \quad dx$$

$$= -\frac{1}{2} \int_0^1 \left(\frac{1}{4} y + \frac{1}{1+x+y} \right)\Big|_0^{1-x} dx$$

$$= -\frac{1}{2} \int_0^1 \left(\frac{1}{4}(1-x) + \frac{1}{2} - \frac{1}{1+x} \right) dx = \ldots$$

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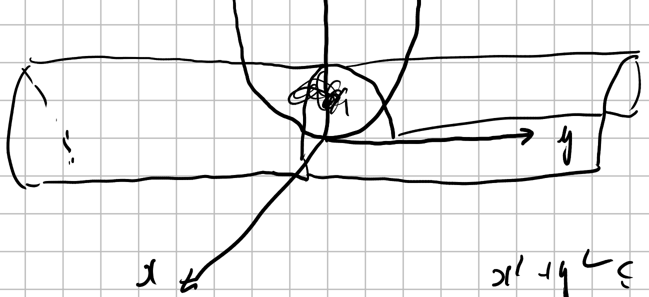
(8) ODREDITI ZAPREMINU TELA OGRANIČENOG

POVRŠIMA

$z = 1 - x^2 \qquad z = x^2 + y^2 \leftarrow$ PARABOLOID



$x^2 + y^2 \le z \le 1 - x^2$

$$V = \iiint_T dx\,dy\,dz = \iint_D \int_{x^2+y^2}^{1-x^2} dz \; dx\,dy =$$

$$= \iint_D (1-2x^2-y^2)\,dx\,dy$$

GRANICE ZA D ?

GORNJA GRANICA VEĆA OD DONJE :

$1 - x^2 \geq x^2 + y^2$

$2x^2 + y^2 \leq 1$

$\sqrt{2}\,x = \rho\cos\varphi \qquad x = \frac{1}{\sqrt{2}}\rho\cos\varphi$

$y = \rho\sin\varphi$

$$J = \begin{vmatrix} \frac{1}{\sqrt{2}}\cos\varphi & -\frac{1}{\sqrt{2}}\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} = \frac{1}{\sqrt{2}}\rho$$

$\rho^2 \leq 1$

$0 \leq \rho \leq 1 \qquad 0 \leq \varphi \leq 2\pi$

$$V = \iint_{\substack{0\leq\rho\leq 1\\ 0\leq\varphi\leq 2\pi}} (1-\rho^2)\frac{1}{\sqrt{2}}\rho\,d\rho\,d\varphi =$$

$$= \frac{1}{\sqrt{2}}\int_0^{2\pi} d\varphi \int_0^1 (\rho-\rho^3)\,d\rho =$$

$$= \frac{1}{\sqrt{2}}\,2\pi\left(\frac{1}{2}\rho^2 - \frac{1}{4}\rho^4\right)\Big|_0^1 = \frac{1}{\sqrt{2}}\,2\pi\,\frac{1}{4} = \boxed{\frac{\pi}{2\sqrt{2}}}$$

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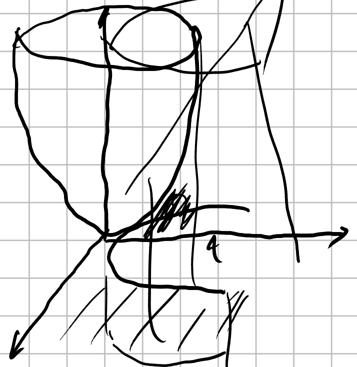
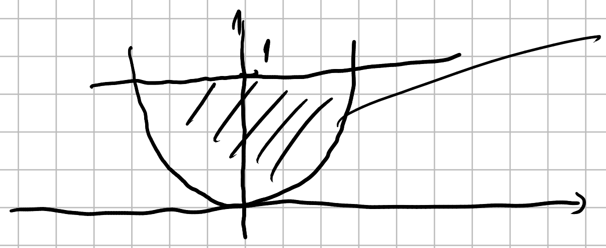
9 ZAPREMINA TELA

$z = 0 \qquad z = x^2 + y^2 \qquad y = x^2 \qquad y = 1$



$$V = \iiint_T dx\,dy\,dz = \iint_D \int_0^{x^2+y^2} dz\,dx\,dy$$

$$= \iint_D (x^2+y^2)\,dx\,dy$$



$D: \quad x^2 \le y \le 1$

$-1 \le x \le 1$

$\rho^2\cos^2\varphi \le \rho\sin\varphi \le 1$

$-1 \le \rho\cos\varphi \le 1$

!

KOMPLIKOVANO

$$= \int_{-1}^{1} \int_{x^2}^{1} (x^2+y^2)\,dy\,dx$$

$$= \int_{-1}^{1} \left(x^2 y + \frac{1}{3}y^3\right)\Big|_{x^2}^{1} dx = \int_{-1}^{1} \left(x^2 + \frac{1}{3} - x^4 - \frac{1}{3}x^6\right) dx = \ldots$$

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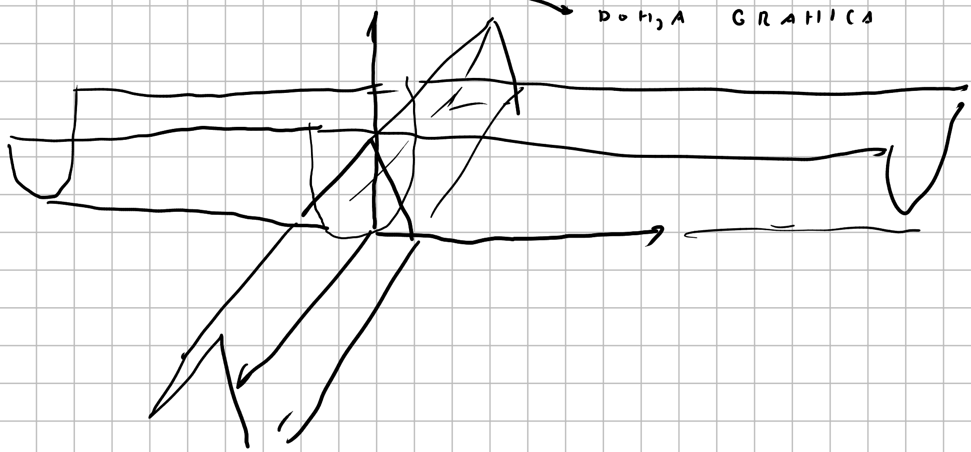
(10) ZAPREMINA

$z = x^2$ $\quad z = 1 - |y|$ → GORNJA GRANICA

DONJA GRANICA



MOŽEMO UPOTREBITI SIMETRIJU U ODNOSU NA RAVAN $y=0$.

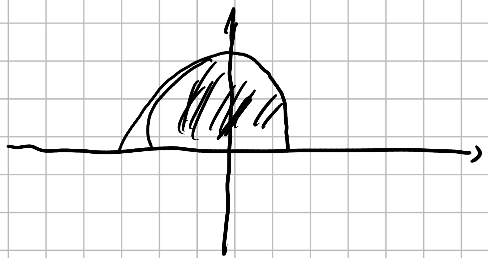
$$V(T) = 2\,V(T')$$

$$T' = T \cap \{y \ge 0\}$$

$$V(T) = 2\,V(T') = 2\iiint_{T'} dx\,dy\,dz =$$

$$= 2 \left(\iint_D \int_{x^2}^{1-y} dz \right) = 2 \iint_D (1-y-x^2)\, dx\, dy$$

$1-y \geq x^2 \geq 0$ $\qquad$ $1-y \geq 0$



$1-y = x^2$

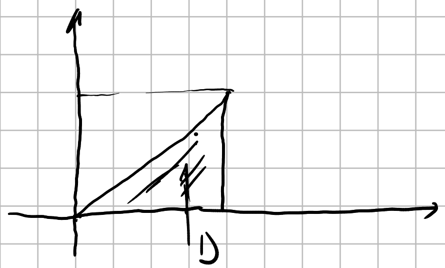
$y = 1-x^2$

$$= 2 \int_{-1}^{1} \int_{0}^{1-x^2} (1-y-x^2)\, dy\, dx =$$

$$= 2 \int_{-1}^{1} \left((1-x^2)y - \frac{1}{2} y^2 \right) \Big|_{0}^{1-x^2} dx =$$

$$= 2 \int_{-1}^{1} \left((1-x^2)^2 - \frac{1}{2}(1-x^2) \right) dx = \ldots$$

① $$\int_0^1 \int_y^1 \frac{xy}{1+x^4}\,dx\,dy = \iint_D \frac{xy}{1+x^4}\,dx\,dy$$



$$= \int_0^1 \int_0^x \frac{xy}{1+x^4}\,dy\,dx$$

$$= \int_0^1 \frac{x}{1+x^4} \int_0^x y\,dy\,dx =$$

$$= \int_0^1 \frac{x}{1+x^4}\, \frac{1}{2} y^2 \Big|_0^x \,dx$$

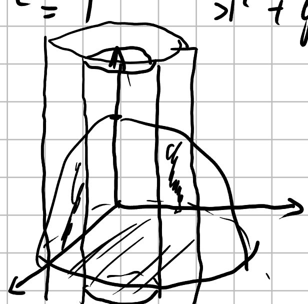
$$= \frac{1}{2} \int_0^1 \frac{x^3}{1+x^4}\,dx = \left| \begin{array}{l} t = 1+x^4 \\ dt = 4x^3\,dx \end{array} \quad \begin{array}{c|c|c} x & 0 & 1 \\ \hline t & 1 & 2 \end{array} \right|$$

$$= \frac{1}{8} \int_1^2 \frac{dt}{t} = \frac{1}{8} \log t \Big|_1^2 = \frac{1}{8} \log 2$$

---

② ODREDITI ZAPREMINU TELA OGRANIČENOG POVRŠIMA

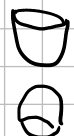
$x^2+y^2=1 \qquad x^2+y^2=4 \qquad z=0 \qquad z = 4-(x^2+y^2)$



$z = x^2+y^2$

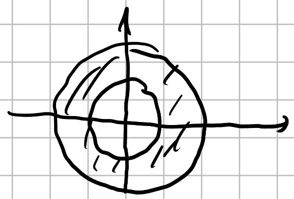
$-z = x^2+y^2$

$4-z$



$$V(T) = \iiint_T dx\,dy\,dz = \iint_D \int_0^{4-(x^2+y^2)} dz \; dx\,dy$$

D - PROJEKCIJA NA RAVAN $z=0$



$$D: \quad 1 \le x^2+y^2 \le 4$$

$$= \iint_D \left(4-(x^2+y^2)\right) dx\,dy =$$

POLARNE KOORDINATE $\quad 1 \le \rho \le 2 \quad 0 \le \varphi \le 2\pi$

$$= \iint_{\substack{1\le\rho\le 2\\ 0\le\varphi\le 2\pi}} (4-\rho^2)\rho \, d\rho\, d\varphi = \int_0^{2\pi} d\varphi \int_1^2 (4\rho-\rho^3)\, d\rho$$

$$= 2\pi \left(2\rho^2 - \frac{1}{4}\rho^4 \right)\Big|_1^2 = \dots$$

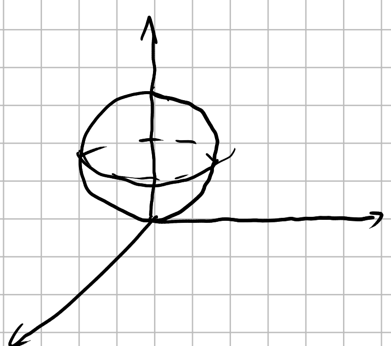
---

(3) $$\iiint_T \sqrt{x^2+y^2+z^2}\; dx\,dy\,dz$$

$$T: \quad x^2+y^2+z^2 = z$$

$$x^2+y^2+z^2 - 2\cdot\frac{1}{2}z + \frac{1}{4} = \frac{1}{4}$$

$$x^2+y^2+\left(z-\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$



$$\left|z-\frac{1}{2}\right| = \sqrt{\frac{1}{4}-x^2-y^2}$$

$$\le z \le \sqrt{\frac{1}{4}-x^2-y^2} + \frac{1}{2}$$

$\vdots$

PAMETNIJE DA PRIMENIMO

SFERNE KOORDINATE

$$x = \rho \cos\varphi \sin\psi \qquad 0 \le \varphi \le 2\pi$$
$$y = \rho \sin\varphi \sin\psi \qquad 0 \le \psi \le \pi$$
$$z = \rho \cos\psi \qquad J = \rho^2 \sin\psi$$

URADENO RANIJE

$$x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\rho^2 \cos^2\varphi \sin^2\psi + \rho^2 \sin^2\varphi \sin^2\psi + \rho^2 \cos^2\psi - \rho\cos\psi + \cancel{\tfrac{1}{4}} = \cancel{\tfrac{1}{4}}$$

$$\rho^2 \sin^2\psi \underbrace{(\cos^2\varphi + \sin^2\varphi)}_{=1} + \rho^2 \cos^2\psi - \rho\cos\psi = 0$$

$$\rho^{\cancel{2}} \underbrace{(\sin^2\psi + \cos^2\psi)}_{=1} - \cancel{\rho}\cos\psi = 0 \qquad \rho = \cos\psi$$

$$0 \le \rho \le \cos\psi \qquad \psi \in \left[0, \frac{\pi}{2}\right]$$

$$\to \cos\psi \ge 0$$

ψ - UGAO SA POZITIVNIM DELOM Z-OSE.

$$\iiint_T \sqrt{x^2 + y^2 + z^2}\, dx\, dy\, dz =$$

$$= \iiint \rho^2 \rho \sin\psi \, d\psi \, d\varphi \, d\rho$$

$$= \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \int_0^{\cos\psi} \rho^3 \sin\psi \, d\rho \, d\psi$$

$$= 2\pi \int_0^{\pi/2} \sin\psi \, \frac{1}{4} \rho^4 \Big|_0^{\cos\psi} d\psi$$

$$= 2\pi \, \frac{1}{4} \int_0^{\pi/2} \sin\psi \cos^4\psi \, d\psi = \left| \begin{array}{l} \cos\psi = t \\ -\sin\psi \, d\psi = dt \end{array} \quad \begin{array}{c|c|c} \psi & 0 & \pi/2 \\ t & 1 & 0 \end{array} \right|$$

$$= \frac{\pi}{2} \int_0^1 t^4 \, dt = \frac{\pi}{2} \, \frac{1}{5} t^5 \Big|_0^1 = \frac{\pi}{10}$$

MOGU SE UVODITI I UOPŠTENE KOORDINATE

SA STEPENIMA

④ ODREDITI POVRŠINU OBLASTI OGRANIČENE KRIVIM

$$\underbrace{\sqrt{x-1}}_{2} + \sqrt{y+1} = \sqrt{2} \qquad x = 1 \qquad y = 1$$

$$x-1 = \rho^2 \cos^4\varphi \qquad x = 1 + \rho^2\cos^4\varphi$$

$$y+1 = \rho^2 \sin^4\varphi \qquad y = -1 + \rho^2 \sin^4\varphi$$

$$J = \begin{vmatrix} 2\rho\cos^4\varphi & -4\rho^2\cos^3\varphi\sin\varphi \\ 2\rho\sin^4\varphi & 4\rho^2\sin^3\varphi\cos\varphi \end{vmatrix} =$$

$$= 8\rho^3\cos^5\varphi\sin^3\varphi + 8\rho^3\cos^3\varphi\sin^5\varphi$$

$$= 8\rho^3\cos^3\varphi\sin^3\varphi\,(\cos^2\varphi + \sin^2\varphi)$$

$$= 8\rho^3\cos^3\varphi\sin^3\varphi$$

$$x = 1 \qquad \rho^4\cos^4\varphi = 0 \quad \Rightarrow \quad \varphi = \frac{\pi}{2}$$

$$y = -1 \qquad \rho^4\sin^4\varphi = 0 \quad \Rightarrow \quad \varphi = 0$$

$$\sqrt{x-1} + \sqrt{y+1} = \sqrt{2} \quad \Rightarrow \quad 0 \le \rho \le \sqrt{2}$$

$$P = \iint_D dx\,dy = \iint_{\substack{0\le\rho\le\sqrt{2}\\ 0\le\varphi\le\frac{\pi}{2}}} 8\rho^3\cos^3\varphi\sin^3\varphi\,d\varphi =$$

$$= 8\int_0^{\sqrt{2}} \rho^3\,d\rho \int_0^{\pi/2} \cos^3\varphi\sin^3\varphi\,d\varphi = \dots$$

$$= 8\,\frac{1}{4}\,\rho^4\Big|_0^{\sqrt{2}} \int_0^1 (1-t^2)\,t^3\,dt = \dots$$

$$\begin{array}{l} t = \sin\varphi \\ dt = \cos\varphi\,d\varphi \end{array} \qquad \begin{array}{c|c|c} \varphi & 0 & \pi/2 \\ \hline t & 0 & 1 \end{array}$$

(5) $V(T) = ?$

$$\sqrt{\frac{x}{2}} + \sqrt{\frac{y}{3}} + \sqrt{\frac{z}{15}} = 1$$

U OPŠTENE SFERNE KOORDINATE:

$$x = 2\rho^2 \sin^4\psi \cos^4\varphi$$
$$\vdots$$

NE DOBIJA SE LAK RAČUN.

II NAČIN

$$u = \sqrt{\frac{x}{2}} \qquad v = \sqrt{\frac{y}{3}} \qquad w = \sqrt{\frac{z}{15}}$$

$$x = 2u^2 \qquad y = 3v^2 \qquad z = 15w^2$$

$$J = \begin{vmatrix} 4u & 0 & 0 \\ 0 & 6v & 0 \\ 0 & 0 & 30w \end{vmatrix} = 720\, u v w \neq 0$$

$$V(T) = \iiint_T dx\,dy\,dz = \iiint_{T'} 720\, u v w \, du\, dv\, dw$$

$$0 \le u + v + w \le 1$$
$$0 \le w \le 1 - u - v$$
$$0 \le v \le 1 - u$$
$$0 \le u \le 1$$

$$= 720 \int_0^1 \int_0^{1-u} \int_0^{1-u-v} u v w \, dw\, dv\, du =$$

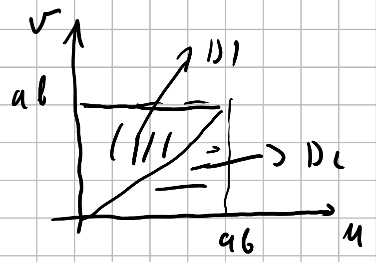
$$= 720 \int_0^1 \int_0^{1-u} uv \frac{1}{2} w^2 \Big|_0^{1-u-v} dv\, du =$$

$$= 720 \frac{1}{2} \int_0^1 \int_0^{1-u} uv (1-u-v)^2 \, dv\, du = \ldots$$

(6) $$\iint\limits_{\substack{0\le x\le a\\ 0\le y\le b}} e^{-\max\{b^2x^2,\, a^2y^2\}}\, dx\, dy \qquad a, b > 0$$

$$u = bx \qquad v = ay$$

$$0 \le u \le ab \qquad 0 \le v \le ab$$



$$J^{-1} = \begin{vmatrix} b & 0 \\ 0 & a \end{vmatrix} = ab$$

$$I = \frac{1}{ab} \iint\limits_{[0,ab]^2} e^{-\max\{u^2, v^2\}}\, du\, dv$$

$$= \frac{1}{ab} \left(\iint\limits_{D_1} e^{-\max\{u^2,v^2\}}\, du\, dv + \iint\limits_{D_2} e^{-\max\{u^2,v^2\}}\, du\, dv \right)$$

$$= \frac{1}{ab} \left(\iint\limits_{D_1} e^{-v^2}\, du\, dv + \iint\limits_{D_2} e^{-u^2}\, du\, dv \right)$$

$$= \frac{1}{ab} \left(\int_0^{ab} \int_0^{v} e^{-v^2}\, du\, dv + \int_0^{ab} \int_0^{u} e^{-u^2}\, dv\, du \right)$$

$$= \frac{2}{ab} \int_0^{ab} \int_0^{v} e^{-v^2}\, du\, dv = \frac{2}{ab} \int_0^{ab} e^{-v^2} \int_0^{v} du\, dv =$$

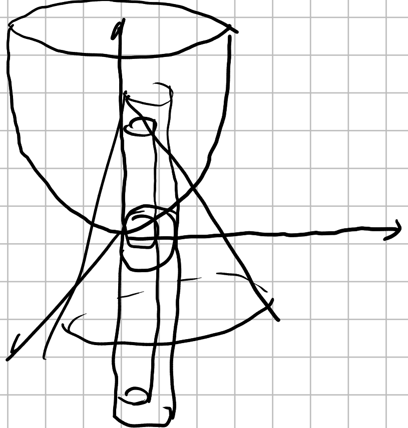
$$= \frac{2}{ab} \int_0^{ab} v\, e^{-v^2}\, dv = \left| \begin{array}{l} v^2 = t \\ 2v\, dv = dt \end{array} \quad \begin{array}{c|c|c} v & 0 & ab \\ \hline t & 0 & a^2b^2 \end{array} \right|$$

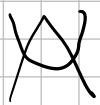
$$= \frac{2}{ab} \int_0^{a^2b^2} e^{-t}\, dt = \frac{2}{ab} \left(-e^{-t} \right) \Big|_0^{a^2b^2}$$

$$= \frac{2}{ab} \left(-e^{-a^2b^2} + 1 \right)$$

$$\iiint_T y\,dx\,dy\,dz \qquad \iiint_T x\,dx\,dy\,dz$$

$T:$ $\quad 10-z=\sqrt{x^2+y^2} \qquad 5z=x^2+y^2 \qquad x^2+y^2=y \qquad x^2+y^2=2y$





$z=\sqrt{x^2+y^2}$

$-z=\sqrt{x^2+y^2}$ ↑ 10.

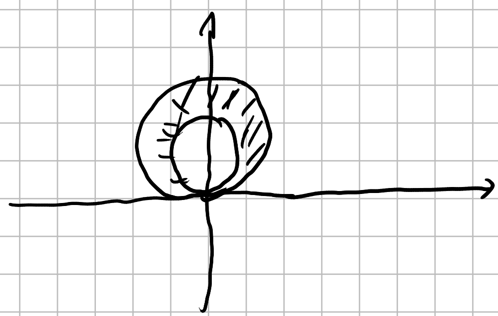
$$x^2+y^2=y \qquad\qquad x^2+y^2=2y$$

$$x^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4} \qquad\qquad x^2+(y-1)^2=1$$

$$I_1=\iiint_T y\,dx\,dy\,dz=\iint_D y\int_{\frac{x^2+y^2}{5}}^{10-\sqrt{x^2+y^2}}dz\;dx\,dy$$

$$=\iint_D y\left(10-\sqrt{x^2+y^2}-\frac{1}{5}(x^2+y^2)\right)dx\,dy$$

$D:$



$$x^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}$$

$$x^2+(y-1)^2=1$$

$$D=D_1\setminus D_2$$

$$I_2=\iint_D \underbrace{x\left(10-\sqrt{x^2+y^2}-\frac{1}{5}(x^2+y^2)\right)}_{\varphi(x,y)}dx\,dy$$

$\varphi(-x,y)=-\varphi(x,y)$ — FUNKCIJA JE NEPARNA PO x

I DOMEN JE SIMETRIČAN PO x.

$$\Rightarrow \boxed{I_2=0}$$

$I_1 \rightsquigarrow$ POLARNE KOORDINATE

$D = D_1 \setminus D_2$

$$\iint_D = \iint_{D_1} - \iint_{D_2}$$

D_1: $x^2 + (y-1)^2 = 1$

$$\rho^2 \cos^2\varphi + (\rho \sin\varphi - 1)^2 = 1$$

$$\rho^2 - 2\rho\sin\varphi = 0$$

$0 \le \rho \le 2\sin\varphi$ $\qquad \sin\varphi > 0$

$0 \le \varphi \le \pi$

D_2: $x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$

$$0 \le \rho \le \sin\varphi$$

ZA CEO D $\quad \sin\varphi \le \rho \le 2\sin\varphi$

$$I_1 = \int_0^{\pi}\int_{\sin\varphi}^{2\sin\varphi} \rho\sin\varphi\left(10 - \rho - \frac{1}{5}\rho^2\right)\rho \, d\rho \, d\varphi$$

ILI DVA ODVOJENA INTEGRALA

$$= \int_0^{\pi} \sin\varphi \int_{\sin\varphi}^{2\sin\varphi} \left(10\rho^2 - \rho^3 - \frac{1}{5}\rho^4\right) d\rho \, d\varphi$$

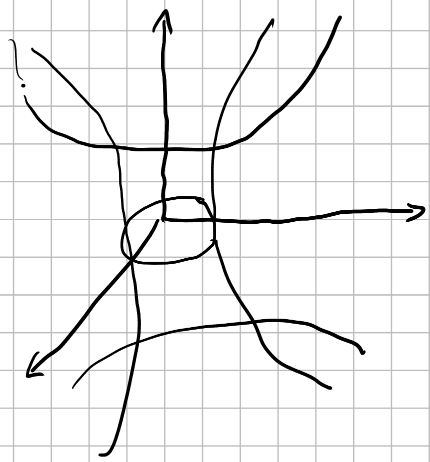
$$= \int_0^{\pi} \sin\varphi \left(\frac{10}{3}\rho^3 - \frac{1}{4}\rho^4 - \rho^5\right)\Bigg|_{\sin\varphi}^{2\sin\varphi} d\varphi$$

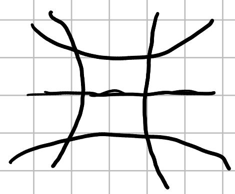
$= \ldots$

ODREDITI ZAPREMINU

$S_1: \quad x^2+y^2-z^2=3$ $\qquad$ $S_2: \quad 5z^2 = x^2+y^2+1$

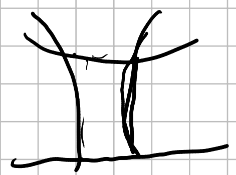
JEDNOGRANI HIPERBOLOID $\qquad$ DVOGRANI HIPERBOLOID





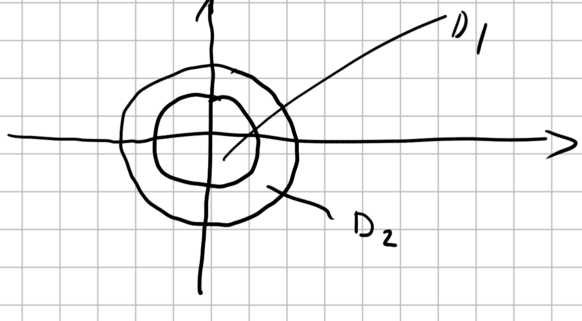
SIMETRIČNO JE U ODNOSU NA RAVAN $z=0$

$V(T) = 2\,V(T') \qquad T' = T \cap \{z \geq 0\}$



$$V(T) = 2\left(\iiint_{T_1} dx\,dy\,dz + \iiint_{T_2} dx\,dy\,dz\right)$$

$$= 2\left(\iint_{D_1}\int_0^{\sqrt{\frac{x^2+y^2+1}{5}}} dz\,dx\,dy + \iint_{D_2}\int_{\sqrt{x^2+y^2-3}}^{\sqrt{\frac{x^2+y^2+1}{5}}} dz\,dx\,dy\right)$$



$D_1: \quad x^2+y^2 \leq 3$

$D_2: \quad 3 \leq x^2+y^2 \leq 4$

TRAŽIMO PRESEK

$$x^2+y^2-z^2=3 \qquad 5z^2 = x^2+y^2+1$$

$$x^2+y^2 = 3+z^2 \nearrow \qquad 5z^2 = 3+z^2+1$$

$$4z^2 = 4 \qquad z^2 = 1 \to 1$$

$$z=1$$

$$x^2+y^2-1=3$$

$$x^2+y^2=4$$

$$= 2\left(\iint_{\substack{0\le\rho\le\sqrt{3}\\ 0\le\varphi\le 2\pi}} \sqrt{\frac{\rho^2+1}{5}}\,\rho\, d\rho\, d\varphi + \iint_{\substack{\sqrt{3}\le\rho\le 2\\ 0\le\varphi\le 2\pi}} \left(\sqrt{\frac{\rho^2+1}{5}} - \sqrt{\rho^2-3}\right)\rho\, d\rho\, d\varphi \right)$$

$$= \ldots$$

---

1. Neka je:

$$\Gamma : \mathbb{R} \to GL(l, \mathbb{K})$$

glatka kriva $\qquad \to$ matrice

$$\gamma(t) = \left(\Gamma(t)\right)^{-1}$$

Dokazati:

$$\gamma''(t) = 2\gamma'\,\Gamma\,\gamma' - \Gamma^{-1}\frac{\partial^2\Gamma}{\partial t^2}\Gamma^{-1}$$

$$\Gamma, \gamma : \mathbb{R} \to GL(l, \mathbb{K})$$

Uradili smo

$$g(L) = L^{-1}$$

$$Dg(L)H = -L^{-1}HL^{-1}$$

$$\gamma'(t) = D(g\circ\Gamma)(t)(h)$$

$$\gamma'(t) = -(\Gamma(t))^{-1}\,\Gamma'(t)\,(\Gamma(t))^{-1}$$

$$\begin{aligned}
\gamma''(t) = &- \overbrace{(\Gamma(t)^{-1})'}^{=\gamma'}\,\Gamma'(t)\,(\Gamma(t))^{-1} \\
&- \Gamma(t)^{-1}\,\Gamma''(t)\,\Gamma(t)^{-1} \\
&- \Gamma(t)^{-1}\,\Gamma'(t)\,\underbrace{(\Gamma(t)^{-1})'}_{=\gamma'} \\
= &- \gamma'(t)\,\Gamma'(t)\,\Gamma(t)^{-1} \\
&- \Gamma(t)\,\Gamma'(t)\,\gamma'(t) \\
&- \Gamma(t)^{-1}\,\Gamma''(t)\,\Gamma(t)^{-1} \\
= &- \gamma'(t)\,\overbrace{\Gamma(t)\,\Gamma(t)^{-1}}^{=I}\,\Gamma'(t)\,\Gamma(t)^{-1} \\
&- \underbrace{\Gamma(t)^{-1}\,\Gamma'(t)\,\Gamma(t)^{-1}}_{\gamma'(t)}\,\Gamma(t)\,\gamma(t) \\
&- \Gamma(t)^{-1}\,\Gamma''(t)\,\Gamma(t)^{-1} \\
= &- 2\,\gamma'(t)\,\Gamma(t)\,\gamma'(t) - \Gamma(t)^{-1}\,\frac{\partial^2\Gamma}{dt^2}\,\Gamma(t)^{-1}
\end{aligned}$$

Teorema: (o inverznoj funkciji)

$A \subseteq \mathbb{R}^m$ - otvoren skup

$a \in A$

$f : A \to \mathbb{R}^m \qquad f \in C^1(A)$

$\det df(a) \neq 0$

Tada postoji U okolina tačke a i V tačke $b = f(a)$ takve da je

$f : U \to V$ bijekcija

$f^{-1} \in C^1(V)$

$$df^{-1}(y) = [df(x)]^{-1}$$

ZA $\forall x \in U$; $y = f(x) \in V$

(1) $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x,y) = (\varphi(x,y), \psi(x,y))$$

$$\varphi, \psi : \mathbb{R}^2 \to \mathbb{R} \qquad \varphi, \psi \in C^1(\mathbb{R}^2)$$

$$\forall x,y \in \mathbb{R}^2 \qquad \varphi'_x \varphi'_y \psi'_x \psi'_y < 0$$

a) f JE „1-1"

b) U SVAKOJ TAČKI SU ISPUNJENI USLOVI TEOREME O INVERZNOJ FUNKCIJI.

v) NAĆI PARCIJALNE IZVODE f^{-1}

---

a) f JE 1-1

$$f(x_1, y_1) = f(x_2, y_2)$$

$$\varphi(x_1, y_1) = \varphi(x_2, y_2) \qquad \psi(x_1, y_1) = \psi(x_2, y_2)$$

$$(\varphi(x_1, y_1) - \varphi(x_1, y_2)) + (\varphi(x_1, y_2) - \varphi(x_2, y_2)) = 0$$

$$(\psi(x_1, y_1) - \psi(x_1, y_2)) + (\psi(x_1, y_2) - \psi(x_2, y_2)) = 0$$

LAGRANŽOVA TEOREMA

$$\varphi'_y(x_1, \eta_1)(y_1 - y_2) + \varphi'_x(\xi_1, y_2)(x_1 - x_2) = 0$$

$$\psi'_y(x_1, \eta_2)(y_1 - y_2) + \psi'_x(\xi_2, y_2)(x_1 - x_2) = 0$$

HOĆEMO DA POKAŽEMO DA SISTEM IMA SAMO TRIVIJALNO REŠENJE

$$\Delta = \begin{vmatrix} \varphi'_y(x_1,\eta_1) & \psi'_x(\xi_1,y_2) \\ \psi'_y(x_1,\eta_2) & \varphi'_x(\xi_2,y_1) \end{vmatrix} =$$

$$= \varphi'_y(x_1,\eta_1)\,\varphi'_x(\xi_2,y_2) - \psi'_y(x_1,\eta_1)\,\psi'_x(\xi_1,y_1)$$

Iz uslova zadatka

$$\varphi'_x\,\varphi'_y\,\psi'_x\,\psi'_y < 0$$

$$\varphi'_x, \varphi'_y, \psi'_x, \psi'_y \in C$$

$\Rightarrow$ svi parcijalni izvodi su stalnog znaka

$$= \varphi'_y\,\psi'_x - \frac{\psi'_y\,\varphi'_x\,\psi'_x\,\varphi'_y}{\psi'_x\,\varphi'_y} \qquad < 0$$

istog znaka $\qquad \Rightarrow \qquad \Delta \neq 0$

$\Rightarrow$ sistem ima jedinstveno rešenje

$$x_1 - x_2 = 0 \quad \text{i} \quad y_1 - y_2 = 0$$

$$\Rightarrow (x_1, y_1) = (x_2, y_2)$$

$\Rightarrow f$ je „1-1"

b) Slično

$$J = \begin{vmatrix} \varphi'_x & \varphi'_y \\ \psi'_x & \psi'_y \end{vmatrix} = \varphi'_x\,\psi'_y - \psi'_x\,\varphi'_y \neq 0$$

v)

$$x'_\varphi \qquad x'_\psi \qquad y'_\varphi \qquad y'_\psi$$

$$\begin{bmatrix} \varphi'_x & \varphi'_y \\ \psi'_x & \psi'_y \end{bmatrix}^{-1} = \frac{1}{\varphi'_x\psi'_y - \psi'_x\varphi'_y} \begin{bmatrix} \psi'_y & -\varphi'_y \\ -\psi'_x & \varphi'_x \end{bmatrix}$$

$$x'_\varphi = \frac{\psi'_y}{\varphi'_x \psi'_y - \psi'_x \varphi'_y} \qquad x'_\psi = \frac{-\varphi'_y}{\varphi'_x \psi'_y - \psi'_x \varphi'_y}$$

$$y'_\varphi = \frac{-\psi'_x}{\varphi'_x \psi'_y - \psi'_x \varphi'_y} \qquad y'_\psi = \frac{\varphi'_x}{\varphi'_x \psi'_y - \psi'_x \varphi'_y}$$

①

$$f(x,y) = \begin{cases} \sqrt[3]{x^4+2x^3+x^2+4y^2} + \dfrac{x^3+4x-2xy}{x^2+(2y-4)^2}, & (x,y) \neq (0,2) \\ \alpha, & (x,y) = (0,2) \end{cases}$$

$\left.\begin{array}{l} \text{НЕПРЕКИДНОСТ} \\ \text{DIFERENCIJABILNOST} \\ \text{IZVOD U PRAVCU} \end{array}\right\}$?

$$h(x,y) = \sqrt[3]{x^4+2x^3+x^2+4y^2} \qquad g(x,y) = \frac{x^3+4x-2xy}{x^2+(2y-4)^2}$$

$$h \in C(\mathbb{R}^2)$$

f је непрекидна $\Leftrightarrow$ g је непрекидна

$$g(x,y) = \frac{x^3+4x-2xy}{x^2+(2y-4)^2} \qquad (x,y) = (0,2) \ ?$$

СМЕНА: $t = 2y-4$ $\qquad y = \frac{1}{2}(t+4)$

$$g(x,t) = \frac{x^3+4x-2x\,\frac{1}{2}(t+4)}{x^2+t^2} = \frac{x^3 - xt}{x^2+t^2}$$

ПОЛАРНЕ КООРДИНАТЕ

$$g(\rho,\varphi) = \frac{\rho^{\cancel{3}}\cos^3\varphi - \cancel{\rho^2}\cos\varphi\sin\varphi}{\cancel{\rho^2}} = \rho\cos^3\varphi - \cos\varphi\sin\varphi$$

$\lim\limits_{\rho\to 0} g(\rho,\varphi)$ НЕ ПОСТОЈИ $\qquad \begin{array}{l}\varphi = 0 \\ \varphi = \frac{\pi}{4}\end{array} \quad \begin{array}{l} 0 \\ \neq \\ -\frac{1}{2}\end{array}$

DIFERENCIJABILNOST ?

U $(0,2)$ NIJE DIF. JER JE PREKIDNA

PROBLEM MOŽE BITI $\sqrt[3]{0}$

$\sqrt[3]{x}$ NIJE DIF U 0.

$\sqrt[3]{x^6} = x^2$ JESTE DIF U 0

$$x^4 - 2x^3 + x^2 + y^2 = x^2(x-1)^2 + y^2 = 0$$

$$\Updownarrow$$

$(x,y) = (0,0)$ i $(x,y) = (1,0)$

U OVIM TAČKAMA TREBA ISPITATI DIF. PO DEFINICIJI

IZVOD U PRAVCU ?

U TAČKAMA U KOJIM FUNKCIJA NIJE DIF. MOŽE POSTOJATI IZVOD U PRAVCU

---

5)

$f(x,y,z) = x^2 + 2y^2 + z$ $\quad S: x^2 + y^2 - z^2 + 2z = 0$

$\nabla f = \lambda \nabla g$

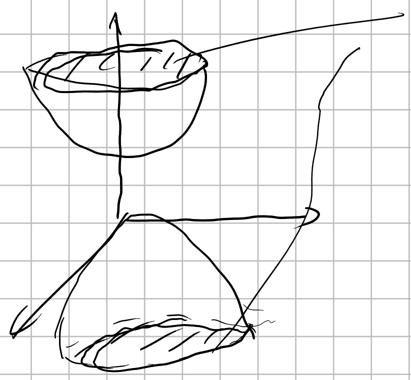
DAJE DVE STACIONARNE TAČKE

$(0,0,0)$ i $(0,0,2)$

ISPITATI PRIRODU ...

NISU APSOLUTNO USLOVNI EKSTREMUMI, ALI JESU USLOVNO ...

d) $z=-2$ $\quad z=4$



NA „POKLOPCIMA" TRAŽITI KANDIDATE ZA EKSTREMUME

DOBIJA SE ISTA PROJEKCIJA NA $z=0$ RAVAN I FUNKCIJA SE RAZLIKUJE ZA KONSTANTU

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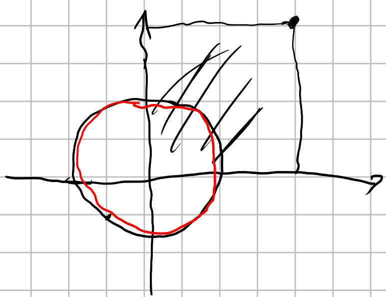
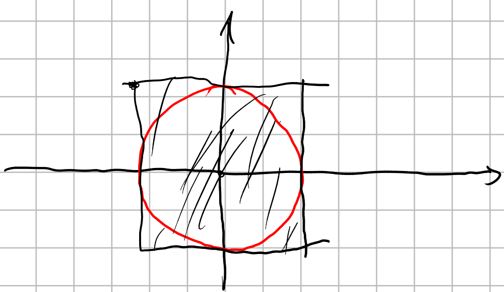
1) $$d(X,Y) = \max\left\{ \sup_{x\in X} e(x,Y), \sup_{y\in Y} e(y,X) \right\}$$

a)



$d([0,1]^2, S^1) = ?$

$d = \max\{1, \underline{2\sqrt{2}-1}\}$



$d(S^1, [-1,1]^2) = ?$

$d = 1$

VIŠESTRUKI INTEGRALI - NASTAVAK

① NAĆI POVRŠINU DELA RAVNI OGRANIČENOG KRIVOM:

$$(x^2+y^2)^2 = a(x^3 - 3xy^2), \quad a>0$$

PROBAMO POLARNE KOORDINATE:

$$x = \rho \cos\varphi \qquad y = \rho \sin\varphi$$

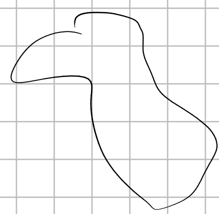
$$\rho^{\cancel{4}} = a(\cancel{\rho^3}\cos^3\varphi - 3\cancel{\rho^3}\cos\varphi \sin^2\varphi)$$

$$\rho = a(\cos^3\varphi - 3\cos\varphi\sin^2\varphi)$$

$$\rho = a(\cos^3\varphi - 3\cos\varphi(1-\cos^2\varphi))$$

$$\rho = a(\underbrace{4\cos^3\varphi - 3\cos\varphi}))$$

$$\rho = a\cos 3\varphi$$



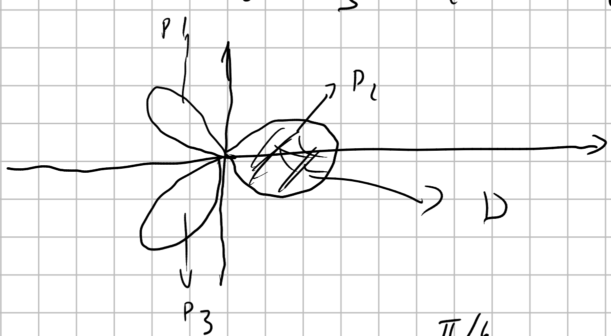
AKO UZMEMO $\rho > a\cos 3\varphi$

DOBIJAMO NEOGRANIČENU OBLAST

$$0 \le \rho \le a\cos 3\varphi$$

USLOV NA φ: $\cos 3\varphi > 0$

$$-\frac{\pi}{6} + \frac{2k\pi}{3} \le \varphi \le \frac{\pi}{6} + \frac{2k\pi}{3} \qquad k \in \{0,1,2\}$$



$$P = 3\iint_D dx\,dy = 3\int_{-\pi/6}^{\pi/6}\int_0^{a\cos 3\varphi} \rho\, d\rho\, d\varphi$$

$$= 3 \int_{-\pi/6}^{\pi/6} \frac{1}{2} \rho^2 \Big|_0^{(a\cos 3\varphi)^{1/2}} d\varphi$$

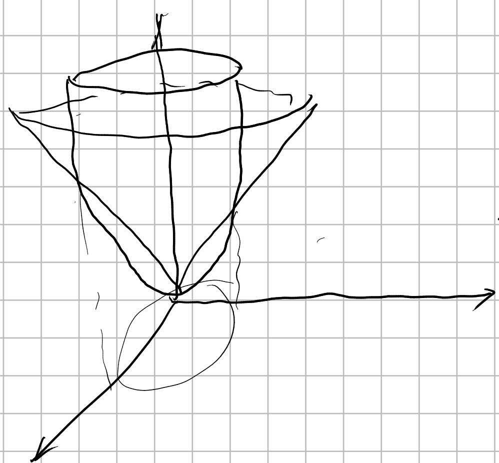
$$= \ldots$$

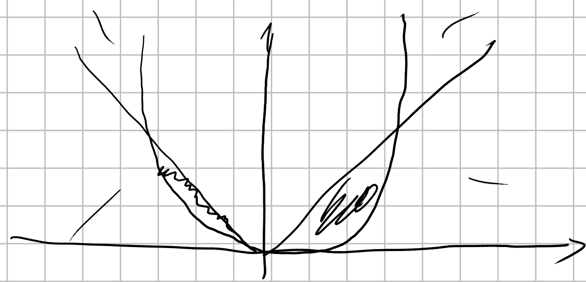
---

(2) ODREDITI ZAPREMINU TELA OGRANIČENOG POVRŠIMA:

$az = x^2 + y^2$ $\qquad a > 0$

$z = \sqrt{x^2+y^2}$





$$V(T) = \iiint_T dx\,dy\,dz = \iint_D \int_{\frac{x^2+y^2}{a}}^{\sqrt{x^2+y^2}} dz\,dx\,dy$$

$$= \iint_D \left(\sqrt{x^2+y^2} - \frac{x^2+y^2}{a}\right) dx\,dy$$

ŠTA JE PRESEK

$$\frac{x^2+y^2}{a} = \sqrt{x^2+y^2}$$

U POLARNIM: $\frac{\rho^2}{a} = \rho \qquad \rho = a$

D JE KRUG POLUPREČNIKA a

$$= \iint_{\substack{0 \le \rho \le a \\ 0 \le \varphi \le 2\pi}} \left(\rho - \frac{\rho^2}{a}\right) \rho \, d\rho\, d\varphi = \ldots$$

3. Odrediti površinu figure ograničene krivom

$$\underbrace{\left(\frac{x}{a}+\frac{y}{b}\right)^4}=\frac{x^2}{h^2}+\frac{y^2}{\ell^2} \quad \text{za } x,y>0$$

Uočimo da ovo zavisi samo od ρ

$$\frac{x}{a}=\rho\cos^2\varphi \qquad \frac{y}{b}=\rho\sin^2\varphi$$

$$x=a\rho\cos^2\varphi \qquad y=b\rho\sin^2\varphi$$

$$(\rho\cos^2\varphi+\rho\sin^2\varphi)^4=\frac{a^2\rho^2\cos^4\varphi}{h^2}+\frac{b^2\rho^2\sin^4\varphi}{\ell^2}$$

$$\rho^{\cancel{4}\,2}=\frac{a^2\cancel{\rho^2}\cos^4\varphi}{h^2}+\frac{b^2\cancel{\rho^2}\sin^4\varphi}{\ell^2}$$

$$0\le\rho\le\sqrt{\frac{a^2}{h^2}\cos^4\varphi+\frac{b^2}{\ell^2}\sin^4\varphi}$$

Moramo izračunati i Jakobijan

$$x=a\rho\cos^2\varphi \qquad y=b\rho\sin^2\varphi$$

$$J=\begin{vmatrix} a\cos^2\varphi & a\rho\,2\cos\varphi(-\sin\varphi) \\ b\sin^2\varphi & b\rho\,2\sin\varphi\cos\varphi \end{vmatrix}=$$

$$=2ab\rho\cos^3\varphi\sin\varphi+2ab\rho\sin^3\varphi\cos\varphi$$

$$=2ab\rho\cos\varphi\sin\varphi(\cos^2\varphi+\sin^2\varphi)$$

$$=2ab\rho\cos\varphi\sin\varphi$$

$$P=\int_0^{\pi/2}\int_0^{\sqrt{\frac{a^2}{h^2}\cos^4\varphi+\frac{b^2}{\ell^2}\sin^4\varphi}}2ab\rho\cos\varphi\sin\varphi\,d\rho\,d\varphi$$

$$=2ab\int_0^{\pi/2}\frac{1}{2}\rho^2\Big|_0^{\sqrt{}}\cos\varphi\sin\varphi\,d\varphi$$

$$= ab \int_0^{\pi/2} \cos\varphi \sin\varphi \left(\frac{a^2}{h^2} \cos^4\varphi + \frac{b^2}{e^2} \sin^4\varphi \right) d\varphi =$$

$$= ab \left(\frac{a^2}{h^2} \int_0^{\pi/2} \cos^5\varphi \sin\varphi + \frac{b^2}{e^2} \int_0^{\pi/2} \cos\varphi \sin^5\varphi \, d\varphi \right)$$

$$= \ldots$$

ČESTO SU ZGODNE UOPŠTENE POLARNE KOORDINATE

$$x = C_1 \rho^{\alpha_1} \cos^{\beta_1}\varphi \qquad y = C_2 \rho^{\alpha_2} \cos^{\beta_2}\varphi$$

---

(4) ODREDITI ZAPREMINU TELA

OGRANIČENOG SA

$$T: \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^4}{c^4} = 1$$
$$\leq 1$$

SVE JE SIMETRIČNO U ODNOSU NA

KOORDINATNE RAVNI

$$x \rightsquigarrow -x$$
$$y \rightsquigarrow -y$$
$$z \rightsquigarrow -z$$

MOŽEMO TRAŽITI ZAPREMINU SAMO

U I OKTANTU

$$T_1 = T \cap \{x \geq 0,\ y \geq 0,\ z \geq 0\}$$

$$V(T) = 8\, V(T_1)$$

PRELAZIMO NA CILINDRIČNE KOORDINATE

$$x = a\rho\cos\varphi$$
$$y = b\rho\sin\varphi$$
$$z = z$$

$$J = \begin{vmatrix} a\cos\varphi & -a\rho\sin\varphi & 0 \\ b\sin\varphi & b\rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= \quad a \quad b \quad \rho$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^4}{c^4} \le 1$$

$$\rho^2 + \frac{z^4}{c^4} \le 1$$

≥ 0

$0 \le \rho \le 1 \qquad z^4 \le c^4(1-\rho^2)$

$0 \le z \le c\sqrt[4]{1-\rho^2}$

$0 \le \varphi \le \frac{\pi}{2}$

$$V(T) = 8 \iiint_{T_1} dx\,dy\,dz = 8\int_0^{\pi/2}\int_0^1\int_0^{c\sqrt[4]{1-\rho^2}} ab\rho\,dz\,d\rho\,d\varphi$$

$$= 8ab\int_0^{\pi/2} d\varphi \int_0^1 \rho\sqrt[4]{1-\rho^2}\,d\rho = \ldots$$

---

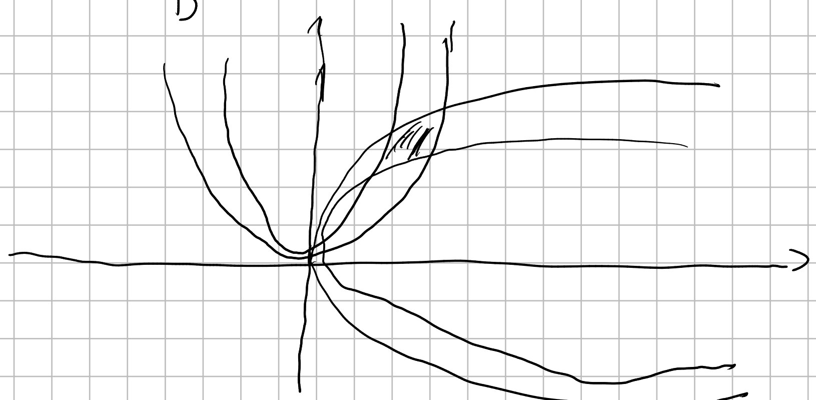
5. ODREDITI ZAPREMINU TELA OGRANIČENOG SA POVRŠINAMA:

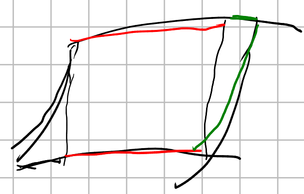
$z = 0 \quad z = xy \quad x^2 = 2y \quad x^2 = y \quad y^2 = x \quad y^2 = 2x$

$\longrightarrow x \ge 0 \quad y \ge 0$

$$V(T) = \iiint_T dx\,dy\,dz = \iint_D \int_0^{xy} dz\,dx\,dy$$

$$= \iint_D xy\,dx\,dy$$





$$y = x^2 \qquad 2y = x^2 \qquad x = y^2 \qquad 2x = y^2$$

$$\frac{x^2}{y} = 1 \qquad \frac{x^2}{y} = 2 \qquad 1 = \frac{y^2}{x} \qquad 2 = \frac{y^2}{x}$$

$$u = \frac{x^2}{y} \qquad v = \frac{y^2}{x}$$

$$1 \le u \le 2 \qquad 1 \le v \le 2$$

$$J^{-1} = \begin{vmatrix} 2\frac{x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & 2\frac{y}{x} \end{vmatrix} = 2\cancel{\frac{x}{y}}\,2\cancel{\frac{y}{x}} - \cancel{\frac{y^2}{x^2}}\,\cancel{\frac{x^2}{y^2}} = 3$$

$J = \frac{1}{3}$ ← ovo je isto funkcija od u i v.

xy — i ovo treba izraziti preko u i v

$$u = \frac{x^2}{y} \quad v = \frac{y^2}{x} \Rightarrow uv = \frac{x^{\cancel{2}}}{\cancel{y}}\,\frac{y^{\cancel{2}}}{\cancel{x}} = xy$$

$$V(T) = \iint_D xy\,dx\,dy = \int_1^2\int_1^2 uv\,\frac{1}{3}\,du\,dv =$$

$$= \frac{1}{3}\int_1^2 u\,du \int_1^2 v\,dv = \frac{1}{3}\,\frac{1}{2}u^2\Big|_1^2\,\frac{1}{2}v^2\Big|_1^2 =$$

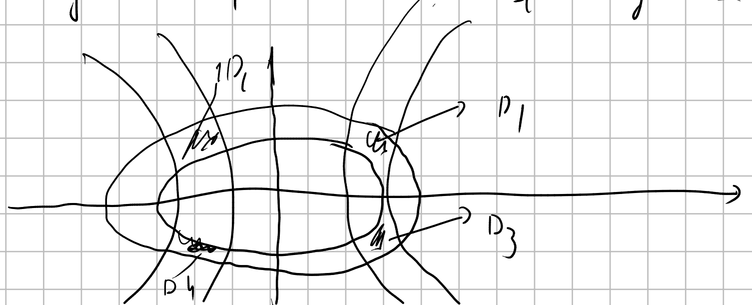
$$= \frac{1}{3}\cdot\frac{1}{2}(4-1)\,\frac{1}{2}(4-1) = \frac{3\cdot\cancel{3}}{\cancel{3}\cdot 2\cdot 2} = \boxed{\frac{3}{4}}$$

---

(6) POVRŠINA FIGURE OGRANIČENE SA

$$\frac{x^2}{4} + y^2 = 1 \qquad x^2 - y^2 = 1$$

$$\frac{x^2}{4} + y^2 = 4 \qquad x^2 - y^2 = 4$$



$$D = D_1 \cup D_2 \cup D_3 \cup D_4$$

МОЖЕМО ДА УВЕДЕМО СМЕНУ

$$u = x^2 - y^2 \qquad v = \frac{x^2}{4} + y^2$$

ОВА СМЕНА НИЈЕ БИЈЕКЦИЈА

НА ЦЕЛОМ D

МОЖЕМО ИСКОРИСТИТИ СИМЕТРИЈЕ

$$P(D) = 4\, P(D_1) \qquad D_1 = D \cap \{x \geq 0,\ y \geq 0\}$$

НА D_1 ЖЕЉЕНА СМЕНА ЈЕ БИЈЕКЦИЈА

$$P(D) = 4 \iint_{D_1} dx\, dy = 4 \iint_{\tilde{D}_1} J\, du\, dv$$

$$u = x^2 - y^2 \qquad v = \frac{x^2}{4} + y^2$$

$$1 \leq u \leq 4 \qquad 1 \leq v \leq 4$$

$$J^{-1} = \begin{vmatrix} 2x & -2y \\ \frac{x}{2} & 2y \end{vmatrix} = 4xy + xy = 5xy$$

$$J = \frac{1}{5xy}$$

xy ТРЕБА ИЗРАЗИТИ ПРЕКО u И v.

$$u + v = \frac{5x^2}{4} \qquad 4v - u = 5y^2$$

$$x = \sqrt{\frac{4(u+v)}{5}} \qquad y = \sqrt{\frac{4v-u}{5}}$$

$$xy = \sqrt{\frac{4(u+v)(4u-v)}{5^2}}$$

$$J = \frac{1}{5\sqrt{\frac{\quad}{5}}}$$

$$P(D) = 4 \int_1^4 \int_1^4 \frac{1}{\sqrt{4(u+v)(4u-v)}}\, du\, dv = \dots$$

ANALIZA 2

VEŽBE

GRUPA 2MHVA

(1) ODREDITI ZAPREMINU TELA OGRANIČENOG POVRŠIMA

$$x^2 - y^2 = 1 \qquad \frac{x^2}{4} + y^2 = 1 \qquad z = 0$$

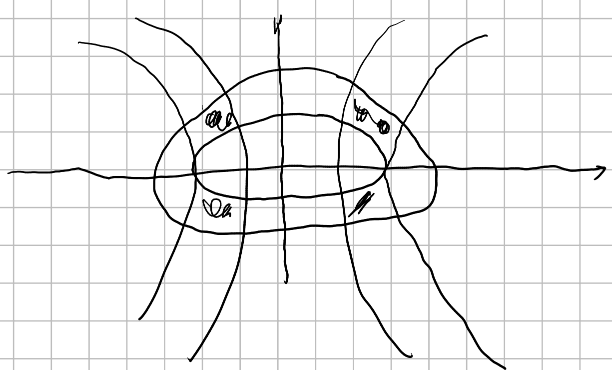
$$x^2 - y^2 = 4 \qquad \frac{x^2}{4} + y^2 = 4 \qquad z = \frac{xy}{x^2+y^2}$$

- ZA GRANICE ZA z NE ZNAMO KOJA JE VEĆA PA STAVL, AKO $\left|\frac{xy}{x^2+y^2}\right| = \frac{|xy|}{x^2+y^2}$

$$V(T) = \iiint_T dx\,dy\,dz = \iint_D \int_0^{\frac{|xy|}{x^2+y^2}} dz\,dx\,dy =$$

$$= \iint_D \frac{|xy|}{x^2+y^2}\,dx\,dy$$

D - PROJEKCIJA NA xy RAVAN



HOĆEMO DA UVEDEMO SMENU

$$u = x^2 - y^2 \qquad v = \frac{x^2}{4} + y^2$$

OVA SMENA NIJE DIFEOMORFIZAM NA D,

ALI DOSTA NA D_1 KOJI DEFINIŠEMO SA

$$D_1 = D \cap \{x \geq 0, \ y \geq 0\}$$

IZ SIMETRIJE DA SE VIDI

$$V(T) = \iint_D \frac{|xy|}{x^2+y^2} \, dx \, dy = 4 \iint_{D_1} \frac{xy}{x^2+y^2} \, dx \, dy$$

UVODIMO SMJENU

$$u = x^2 - y^2 \qquad v = \frac{x^2}{4} + y^2 \qquad 1 \leq u \leq 4 \qquad 1 \leq v \leq 4$$

$$J^{-1} = \begin{vmatrix} 2x & -2y \\ \frac{x}{2} & 2y \end{vmatrix} = 4xy + xy = 5xy$$

$$J = \frac{1}{5xy}$$

$$V(T) = 4 \int_1^4 \int_1^4 \frac{\cancel{xy}}{x^2+y^2} \frac{1}{5\cancel{xy}} \, du \, dv \qquad x, y \text{ SU SAD FUNKCIJE } u \text{ I } v$$

IZRAČUNAMO x I y PREKO u I v

$$x^2 - y^2 = u \quad (+)$$

$$(-4) \quad \frac{x^2}{4} + y^2 = v$$

$$\frac{5}{4} x^2 = u + v \qquad x^2 = \frac{4}{5}(u+v)$$

$$-5y^2 = -4v + u \qquad y^2 = \frac{1}{5}(4v - u)$$

$$x^2 + y^2 = \frac{4}{5}(u+v) + \frac{1}{5}(4v-u) = \frac{1}{5}(3u + 8v)$$

$$V(T) = \frac{4}{\cancel{5}} \int_1^4 \int_1^4 \frac{1}{\frac{1}{\cancel{5}}(3u+8v)} \, du \, dv =$$

$$= 4 \int_1^4 \frac{1}{3} \log(3u+8v) \Big|_1^4 \, dv$$

$$= \frac{4}{3} \int_1^4 \left(\log(12+8v) - \log(3+8v) \right) dv =$$

$$= \ldots \quad (\text{PARCIJALNA INTEGRACIJA})$$

(2) DOKAZATI DA JE OGRANIČENO TELO ODREĐENO SA:

$x \geq y^2$, $x \geq z^2$, $y \geq x^2$, $y \geq z^2$, $z \geq x^2$, $z \geq y^2$

I ZATIM ODREDITI NJEGOVU ZAPREMINU.

$$\left.\begin{array}{l} x \geq y^2 \geq 0 \Rightarrow x \geq 0 \\ y \geq x^2 \geq 0 \Rightarrow y \geq 0 \\ z \geq x^2 \geq 0 \Rightarrow z \geq 0 \end{array}\right\} \quad x, y, z \geq 0$$

$$\left.\begin{array}{l} x \geq y^2 \geq (x^2)^2 = x^4 \Rightarrow x \leq 1 \\ y \geq x^2 \geq y^4 \Rightarrow y \leq 1 \\ z \geq x^2 \geq z^4 \Rightarrow z \leq 1 \end{array}\right\} \quad x, y, z \leq 1$$

IZ PRETHODNOG SLEDI DA JE TELO OGRANIČENO.

IMAMO PREVIŠE NEJEDNAČINA, NIJE JASNO KOJE SU GRANICE.

SIMETRIČNO JE PO PROMENLJIVIM, MOŽEMO DODATI PRETPOSTAVKU:

$$x \geq y \geq z \qquad (1 \geq x \geq y \geq z \geq 0)$$

$$V(T) = 8\,V(T_1) \qquad T_1 = T \cap \{x \geq y \geq z\}$$

OVO NAM OMOGUĆAVA DA NEKE NEJEDNAČINE POSTANU SUVIŠNE (DOBIJAJU SE IZ DRUGIH)

$$\left.\begin{array}{lll} x \geq y \geq y^2 & \Rightarrow & x \geq y^2 \\ x \geq z \geq z^2 & \Rightarrow & x \geq z^2 \\ y \geq z \geq x^2 & \Rightarrow & y \geq x^2 \\ y \geq z \geq z^2 & \Rightarrow & y \geq z^2 \\ x \geq y \Rightarrow x^2 \geq y^2 \text{ i } z \geq x^2 & \Rightarrow & z \geq y^2 \end{array}\right\} \quad \text{SVE OVO NIJE POTREBNO}$$

DOVOLJNO JE:

$$x \geq y \geq z \quad \text{i} \quad z \geq x^2$$

$$V(T_1) = \iiint_{T_1} dx\, dy\, dz = \iint_D \int_z^x dy\, dx\, dz = \iint_D (x-z)\, dx\, dz =$$

D - PROJEKCIJA NA xz RAVAN

$$= \int_0^1 \int_z^{\sqrt{z}} (x-z)\, dx\, dz = \int_0^1 \left(\frac{1}{2}x^2 - xz\right)\Big|_z^{\sqrt{z}} dz =$$

$$= \int_0^1 \left(\frac{1}{2}z - \sqrt{z}\,z - \frac{1}{2}z^2 + z^2\right) dz =$$

$$= \int_0^1 \left(\frac{1}{2}z^2 + \frac{1}{2}z - z^{\frac{3}{2}}\right) dz =$$

$$= \left(\frac{1}{6}z^3 + z^2 - \frac{2}{5}z^{\frac{5}{2}}\right)\Big|_0^1 = \frac{1}{6} + 1 - \frac{2}{5} = \frac{5+30-12}{30} = \boxed{\frac{23}{30}}$$

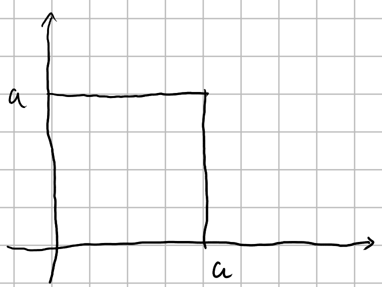
---

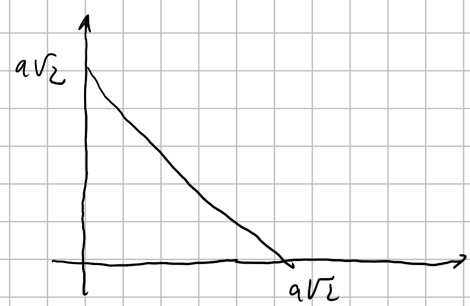
3 DOKAZATI JEDNAKOST:

$$\iint_D \sin(x^2+y^2)\, dx\, dy = \iint_\Delta \sin(x^2+y^2)\, dx\, dy$$

GDE JE:

$a > 0$ $\quad D = [0, a]^2 \quad \Delta = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq a\sqrt{2}\}$



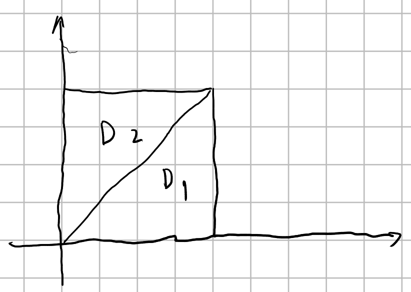


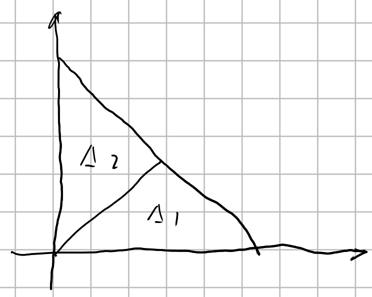
NIJE POTREBNO DA REŠIMO INTEGRALE.

IDEJA JE DA PRESLIKAMO D U Δ DIFEOMORFIZMOM KOJI NE MENJA VREDNOST FUNKCIJE I ČIJI JE JAKOBIJAN 1.

NE MOŽEMO NAĆI TAKAV DIFEOMORFIZAM,

PA DELIMO OBLASTI NA DELOVE:





ROTACIJAMA MOŽEMO PRESLIKATI D_1 NA Δ_2 I D_2 NA Δ_1. ODNOSNO SMENAMA KOORDINATA

I SMENA

$$x = \frac{u}{\sqrt{2}} - \frac{v}{\sqrt{2}}$$

$$y = \frac{u}{\sqrt{2}} + \frac{v}{\sqrt{2}}$$

$J_1 = 1 \qquad D_2 \rightsquigarrow \Delta_1$

II SMENA

$$x = \frac{t}{\sqrt{2}} + \frac{s}{\sqrt{2}}$$

$$y = -\frac{t}{\sqrt{2}} + \frac{s}{\sqrt{2}}$$

$J_2 = 1 \qquad D_1 \rightsquigarrow \Delta_2$

KAKO FUNKCIJA $\sin(x^2+y^2)$ ZAVISI SAMO OD $x^2+y^2=\rho^2$ ROTACIJOM SE NE MENJA

$$\Rightarrow \iint_{D_2} \sin(x^2+y^2)\,dx\,dy = \iint_{\Delta_1} \sin(x^2+y^2)\,dx\,dy$$

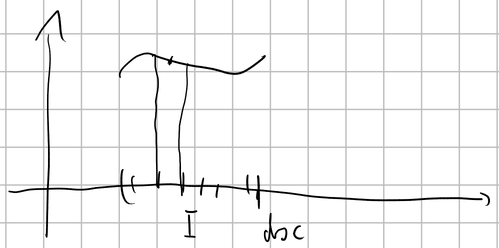
$$\iint_{D_1} \sin(x^2+y^2)\,dx\,dy = \iint_{\Delta_2} \sin(x^2+y^2)\,dx\,dy$$

NA OSNOVU ADITIVNOSTI PO SKUPU

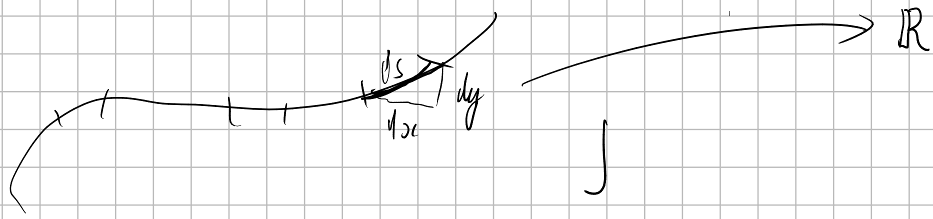
SLEDI ZAKLJUČAK

Rimanov integral

$$f : I \to \mathbb{R}$$



Sada hoćemo po krivoj da integralimo

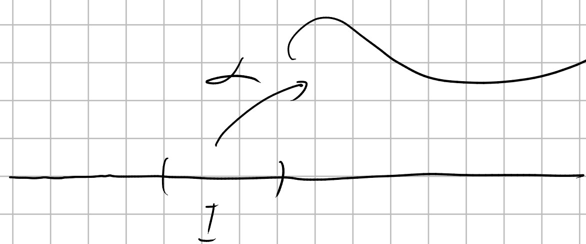


Krive

Definicija:

Kriva u n-dimenzionom afinom prostoru $\mathcal{A}$ je preslikavanje:

$$\alpha : I \to \mathcal{A} \qquad I \text{ - interval}$$



Možemo u $\mathcal{A}$ izabrati koordinate

α definiše vektorsku vrednosnu funkciju

$$r : I \to V_{\mathcal{A}} \quad \to \text{vektorski prostor}$$

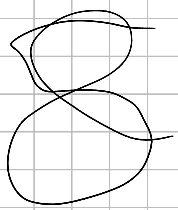
$$x_i = I \to \mathbb{R} \quad \text{koordinatne funkcije}$$

$$x_1 = x_1(t) \quad \ldots \quad x_n = x_n(t)$$

KRIVA JE PRESLIKAVANJE (RAZLIKOVATI OD SLIKE KRIVE)

SLIKA ILI NOSAČ KRIVE JE

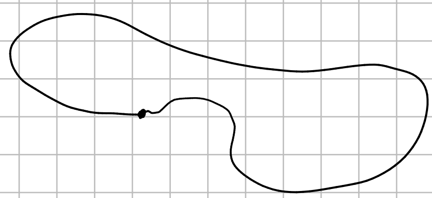
$$\operatorname{supp} \alpha = \{ x \in A \mid \exists t \in I \mid \alpha(t) = x \}$$



PEANOVA KRIVA IMA ZA SLIKU PUN KVADRAT

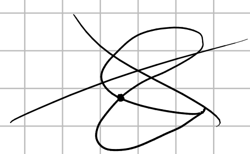
DEFINICIJA:

KRIVA $\alpha : J = [a,b] \to \mathbb{R}$ JE ZATVORENA AKO JE $\alpha(a) = \alpha(b)$



DEFINICIJA:

KRIVA JE PROSTA AKO JE:



1) ZATVORENA I VAŽI

$\alpha(t_1) = \alpha(t_2)$ i $t_1 < t_2 \Rightarrow t_1 = a \quad t_2 = b$

ILI

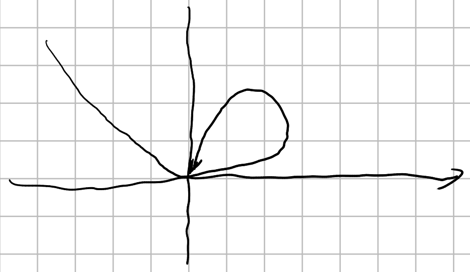
2) NIJE ZATVORENA, A JESTE „1-1" I HOMEOMORFNA

ZA SVAKI NIZ $t_n \in I$ IZ USLOVA DA POSTOJI $\tau \in I$ $\lim_{n \to \infty} \alpha(t_n) = \alpha(\tau)$

$\Rightarrow$ t_n JE KONVERGENTAN

PRIMER:

$$x = \frac{3t}{1+t^3} \qquad y = \frac{3t^2}{1+t^3} \qquad -1 < t < +\infty$$



- OVO NIJE HOMEOMORFNA KRIVA

- ZA ZATVORENE KRIVE NIJE POTREBAN USLOV HOMEOMORFNOSTI

- GLATKOST KRIVE JE GLATKOST FUNKCIJE α.

DEFINICIJA:

$$\alpha : I \to \mathbb{R}^n \quad (\mathbb{C}^n) \qquad \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$$

a) VEKTOR BRZINE KRIVE JE

$$\dot{\alpha}(t) = \left(\frac{\partial \alpha_1}{\partial t}, \frac{\partial \alpha_2}{\partial t}, \dots, \frac{\partial \alpha_n}{\partial t} \right)$$

b) BRZINA KRIVE JE

$$\|\dot{\alpha}(t)\|$$

v) KRIVA JE REGULARNA AKO JE

$$\forall t \in I \quad \|\dot{\alpha}(t)\| \neq 0$$

PRIMERI!

a) PRAVA $\quad \alpha = p + t\vec{q} \qquad \vec{q} \neq 0$

$$\dot{\alpha} = \vec{q}$$

$$\|\dot{\alpha}\| = \|\vec{q}\| \neq 0 \qquad \text{REGULARNA KRIVA}$$

b) KRUG

$$\alpha(t) = (r\cos t, r\sin t) \qquad 0 \le t \le 2\pi$$

$$\dot{\alpha}(t) = (-r\sin t, r\cos t)$$

$$\|\dot{\alpha}(t)\| = \sqrt{r^2\sin^2 t + r^2\cos^2 t} = r \neq 0$$

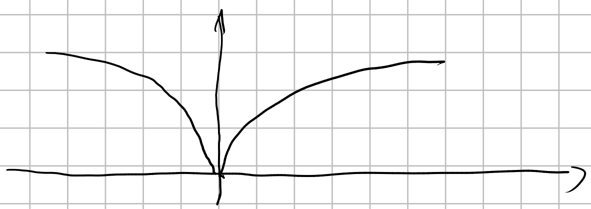
- SPIRALA

$$\alpha(t) = (a\cos t, \ a\sin t, \ bt)$$

$$\dot{\alpha}(t) = (-a\sin t, \ a\cos t, \ b)$$

$$\|\dot{\alpha}(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2} \neq 0$$

- $\alpha(t) = (t^3, \ t^2) \qquad t \in \mathbb{R}$



$$\dot{\alpha}(t) = (3t^2, \ 2t)$$

$$\|\dot{\alpha}(t)\| = \sqrt{9t^4 + 4t^2} \neq 0 \quad \text{ZA} \quad t \neq 0$$

ZA $t=0$ OVA KRIVA NIJE REGULARNA

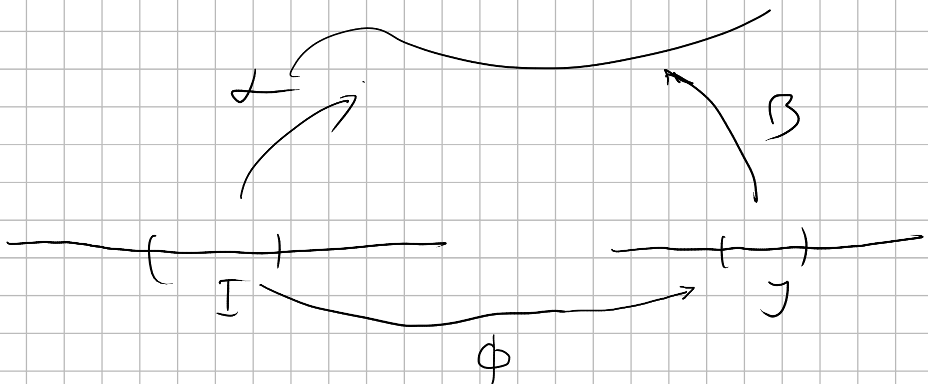
- $f(t) = \begin{cases} e^{-\frac{1}{t}}, & t > 0 \\ 0, & t \leq 0 \end{cases}$

$$\alpha(t) = (f(t), \ f(-t))$$

$\downarrow$ OVA KRIVA JE REGULARNA.

Definicija:

Reparametrizacija krive $\alpha: I \to A$ je preslikavanje $\beta: J \to A$ (I, J - intervali) takvo da postoji difeomorfizam $\phi: I \to J$



$$\alpha = \beta \circ \phi$$

Dužina luka krive je

$$s(t) = \int_{t_0}^{t} \|\dot{\alpha}(u)\| \, du$$

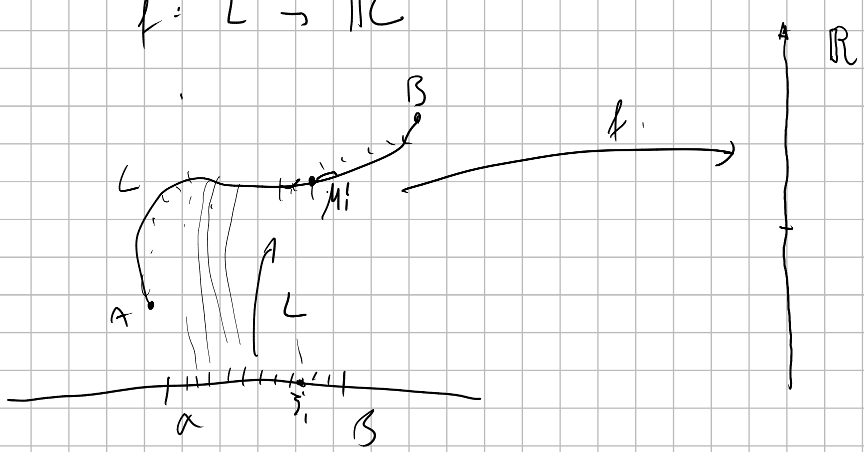
Krivolinijski integral I vrste

Definicija: (u $\mathbb{R}^3$)

Kriva $L = AB$

$$L(t) = (\varphi(t), \psi(t), \chi(t)) \qquad t \in [\alpha, \beta]$$

$$f: L \to \mathbb{R}$$



f definisana i ograničena na L.

$\mathcal{P}: \ \alpha = t_0 < t_1 < \ldots < t_n = \beta$ PODELA

$\xi_i \in [t_{i-1}, t_i]$ - ISTAKNUTA TAČKA

$\xi = \{\xi_1, \ldots, \xi_{n-1}\}$ - SKUP ISTAKNUTIH TAČAKA

$M_i = L(\xi_i)$

$M_i = (\varphi(\xi_i), \psi(\xi_i), \chi(\xi_i))$

ΔS_i - DUŽINA LUKA KRIVE IZMEĐU $L(t_{i-1})$ I $L(t_i)$

PARAMETAR PODELE

$\lambda(\mathcal{P}) = \max \Delta S_i$

$$\sigma(L, f, \mathcal{P}, \xi) = \sum_{i=1}^{n} f(M_i) \Delta S_i$$

$I \in \mathbb{R}$ JE KRIVOLINIJSKI INTEGRAL I VRSTE AKO

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall \mathcal{P} \ \forall \xi \ \lambda(\mathcal{P}) < \delta \Rightarrow |I - \sigma| < \varepsilon$

OZNAKA

$$\int_L f(x, y, z)\, dl \qquad \int_{AB} \qquad \int_A^B$$

dS

OSOBINE:

f, g DEFINISANE NA $L = AB$ I POSTOJE $\int_L f\, dl$ i $\int_L g\, dl$

1) LINEARNOST

$\alpha, \beta \in \mathbb{R}$
$$\int_L (\alpha f + \beta g)\, dl = \alpha \int_L f\, dl + \beta \int_L g\, dl$$

2) ADITIVNOST PO SKUPU

$C \in L$ $\quad$ C IZMEĐU A I B

$$\int_{AB} f\, dl = \int_{AC} f\, dl + \int_{CB} f\, dl$$

3) $\left| \int_{AB} f \, dl \right| \leq \int_{AB} |f| \, dl$

4) $f(M) \leq g(M) \quad \forall M \in L$

$$\int_L f \, dl \leq \int_L g \, dl$$

5) HE ZAVISI OD SMERA

$$\int_{AB} f \, dl = \int_{BA} f \, dl$$

6) TEOREMA O SREDNJOJ VREDNOSTI

$f \in C(L)$

$\Rightarrow \exists M \in L \qquad \int_L f \, dl = f(M) \, s$

DUŽINA LUKA KRIVE

KAKO SE RAČUNA OVAJ INTEGRAL

$L = (\varphi(t), \psi(t), \chi(t)) \qquad t \in [\alpha, \beta]$

$$\int_{AB} f \, dl = \int_{\alpha}^{\beta} f(\varphi(t), \psi(t), \chi(t)) \sqrt{\varphi'^2(t) + \psi'^2(t) + \chi'^2(t)} \, dt$$

Analiza 2

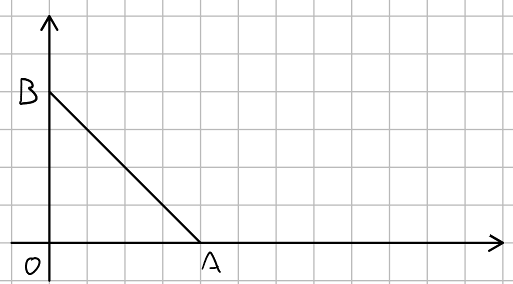
Vežbe

Krivolinijski integrali I vrste

(1) $$I = \int_L (x+y)\, dl$$

L - granica trougla sa temenima

O(0,0) , A(1,0) i B(0,1)



$$I = \int_L (x+y)\, dl = \underbrace{\int_{OA} (x+y)\, dl}_{I_1} + \underbrace{\int_{AB} (x+y)\, dl}_{I_2} + \underbrace{\int_{OB} (x+y)\, dl}_{I_3}$$

$\underline{I_1}$ $\quad y = 0 \,,\quad dl = dx$

$$I_1 = \int_{OA} (x+y)\, dl = \int_0^1 x\, dx = \frac{1}{2} x^2 \Big|_0^1 = \frac{1}{2}$$

$\underline{I_2}$ $\quad y = 1 - x \qquad y' = -1$

$$dl = \sqrt{1+(-1)^2}\, dx = \sqrt{2}\, dx$$

$$I_2 = \int_{OA} (x+y)\, dl = \int_0^1 1 \cdot \sqrt{2}\, dx = \sqrt{2}$$

$I_3 = I_1$ (IZ SIMETRIJE, TJ. ZAMJENE ULOGA x i y)

$I = I_1 + I_2 + I_3 = \frac{1}{2} + \sqrt{2} + \frac{1}{2} = 1 + \sqrt{2}$

---

(2) $\int_L y^2 \, dl$

$L: \quad x = a(t - \sin t)$
$a > 0 \quad y = a(1 - \cos t) \qquad t \in [0, 2\pi] \qquad$ CIKLOIDA

$x'(t) = a(1 - \cos t)$

$y'(t) = a \sin t$

$$dl = \sqrt{a^2(1-\cos t)^2 + a^2 \sin^2 t} \, dt = \sqrt{a^2(1 - \cos 2t + \cos^2 t) + a^2 \sin^2 t} \, dt$$

$$= \sqrt{a^2 - a^2 2\cos t + \underbrace{a^2\cos^2 t + a^2 \sin^2 t}_{= a^2}} \, dt$$

$$= \sqrt{2a^2 - 2a^2 \cos t} \, dt = \sqrt{2} \, a \sqrt{1 - \cos t} \, dt$$

$$= 2a \left|\sin \frac{t}{2}\right| = 2a \sin \frac{t}{2} \qquad \frac{t}{2} \in [0, \pi]$$

$$I = \int_0^{2\pi} a^2 (1 - \cos t)^2 \, 2a \sin \frac{t}{2} \, dt =$$

$$= 8a^3 \int_0^{2\pi} \left(\frac{1 - \cos t}{2}\right)^2 \sin \frac{t}{2} \, dt \qquad u = \frac{t}{2}$$

$$= 8a^3 \int_0^{2\pi} \sin^5 \frac{t}{2} \, dt = 16 a^3 \int_0^{\pi} \sin^5 u \, du =$$

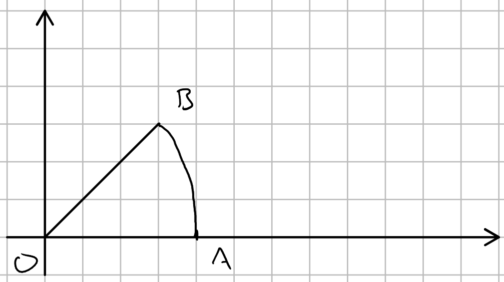
$$= 16 a^3 \int_{-1}^{1} (1 - z^2)^2 \, dz \qquad z = \cos^2 u$$

$$= 32 a^3 \int_0^1 (1 - 2z^2 + z^4) \, dz = \dots = \frac{64 a^3}{15}$$

---

(3) $\int_L e^{\sqrt{x^2+y^2}} \, dl$

L - RUB KRUŽNOG ISEČKA $\quad 0 \le \rho \le a \quad 0 \le \varphi \le \frac{\pi}{4}$



$O\,(0,0)$

$A\,(a,0)$

$B\,\left(\frac{\sqrt{2}}{2}a, \frac{\sqrt{2}}{2}a\right)$

$$I = \underbrace{\int_{OA} e^{\sqrt{x^2+y^2}}\,dl}_{I_1} + \underbrace{\int_{AB} e^{\sqrt{x^2+y^2}}\,dl}_{I_2} + \underbrace{\int_{BO} e^{\sqrt{x^2+y^2}}\,dl}_{I_3}$$

$\underline{I_1}$ $\quad y = 0 \quad dl = dx$

$$I_1 = \int_{OA} e^{\sqrt{x^2+0^2}}\,dx = \int_0^a e^x\,dx = e^a - 1$$

$\underline{I_2}$ $\quad x = a\cos\varphi \quad y = a\sin\varphi \quad 0 \le \varphi \le \frac{\pi}{4}$

$x'(\varphi) = -a\sin\varphi \quad y'(\varphi) = a\cos\varphi$

$$dl = \sqrt{a^2\sin^2\varphi + a^2\cos^2\varphi}\,d\varphi = a\,d\varphi$$

$$I_2 = \int_{AB} e^{\sqrt{x^2+y^2}}\,dl = \int_0^{\pi/4} e^a\,a\,d\varphi = a e^a \frac{\pi}{4}$$

$\underline{I_3}$ $\quad y = x \quad y' = 1$

$$dl = \sqrt{2}\,dx$$

$$I_3 = \int_{BO} e^{\sqrt{x^2+y^2}}\,dx = \int_0^{\frac{\sqrt{2}}{2}a} e^{\sqrt{2}x}\sqrt{2}\,dx = e^a - 1$$

$$I = I_1 + I_2 + I_3 = 2(e^a - 1) + a e^a \frac{\pi}{4}$$

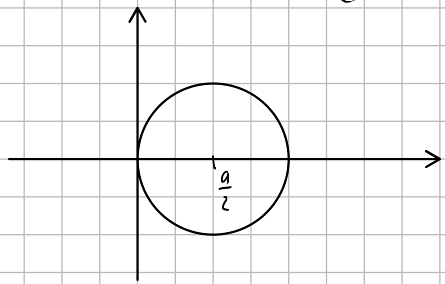
④ $$\int_L \sqrt{x^2+y^2}\, dl$$

L: $x^2+y^2 = ax \qquad a>0$

$$x^2 - ax + y^2 = 0$$

$$x^2 - 2\frac{a}{2}x + \frac{a^2}{4} + y^2 = \frac{a^2}{4}$$

$$\left(x-\frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2 \qquad \text{- KRUG}$$



PARAMETRIZACIJA:

$$x = \frac{a}{2} + \frac{a}{2}\cos\varphi \qquad y = \frac{a}{2}\sin\varphi$$

$$x' = -\frac{a}{2}\sin\varphi \qquad y' = \frac{a}{2}\cos\varphi$$

$$dl = \sqrt{x'^2+y'^2}\, d\varphi = \sqrt{\frac{a^2}{4}\sin^2\varphi + \frac{a^2}{4}\cos^2\varphi}\, d\varphi = \frac{a}{2}\, d\varphi$$

$$I = \int_0^{2\pi} \sqrt{\left(\frac{a}{2}+\frac{a}{2}\cos\varphi\right)^2 + \left(\frac{a}{2}\sin\varphi\right)^2}\, \frac{a}{2}\, d\varphi$$

$$= \frac{a}{2}\int_0^{2\pi} \sqrt{\frac{a^2}{4} + \frac{a^2}{2}\cos\varphi + \frac{a^2}{4}\cos^2\varphi + \frac{a^2}{4}\sin^2\varphi}\, d\varphi$$

$$= \frac{a^2}{2}\int_0^{2\pi} \left|\cos\frac{\varphi}{2}\right| d\varphi = a^2 \int_0^{\pi} |\cos u|\, du = 2a^2$$

---

⑤ $$\int_L (x^2+y^2+z^2)\, dl$$

L: $x^2+y^2+z^2 = a^2 \qquad x+y+z=0$

$$\int_L \underbrace{(x^2+y^2+z^2)}_{a^2}\, dl = a^2\int_L dl = a^2\, 2\pi a = 2\pi a^3$$

OBIM VELIKOG KRUGA NA SFERI

(6) $$\int_L \left(x^{\frac{4}{3}} + y^{\frac{4}{3}}\right) dl$$

L - ASTROIDA

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

PARAMETRIZACIJA

$$x = a\cos^3 t \qquad y = a\sin^3 t$$

$$x' = -a\,3\cos^2 t \sin t \qquad y' = a\,3\sin^2 t \cos t$$

$$dl = \sqrt{a^2\, 9\cos^4 t \sin^2 t + a^2\, 9 \sin^4 t \cos^2 t}\; dt$$

$$= \sqrt{a^2\, 9 \sin^2 t \cos^2 t \,(\cos^2 t + \sin^2 t)}\; dt$$

$$= 3a\, |\sin t \cos t|\; dt$$

$$I = a^{\frac{4}{3}} \int_0^{2\pi} (\cos^4 t + \sin^4 t)\, 3a\, |\sin t \cos t|\; dt$$

$$= 12\, a^{\frac{7}{3}} \int_0^{\pi/2} (\cos^5 t \sin t + \sin^5 t \cos t)\; dt$$

$$= \ldots = 4a^{\frac{7}{3}}$$

---

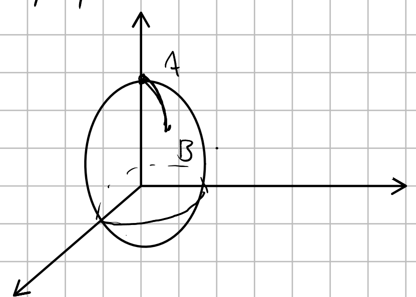
(7) $$I = \int_L (x+y)\; dl$$

L - MALI DEO KRUGA

$$x^2 + y^2 + z^2 = R^2 \qquad x = y$$

IZMEĐU TAČAKA

$$A\,(0, 0, R) \quad \text{i} \quad B\left(\frac{R}{2}, \frac{R}{2}, \frac{R}{\sqrt{2}}\right)$$



$$x = t \qquad y = t \qquad z = \sqrt{R^2 - 2t^2} \qquad (z \geq 0 \text{ NA OVOM DELU KRUGA})$$

$$x' = 1 \qquad y' = 1 \qquad z' = \frac{-2t}{\sqrt{R^2 - 2t^2}}$$

$$dl = \sqrt{1^2 + 1^2 + \frac{4t^2}{R^2 - 2t^2}}\, dt = \frac{\sqrt{2R^2}}{\sqrt{R^2 - 2t^2}}\, dt = \frac{\sqrt{2}\,R}{\sqrt{R^2 - 2t^2}}\, dt$$

$$I = \int_0^{R/2} 2t \frac{\sqrt{2}\,R}{\sqrt{R^2 - 2t^2}}\, dt = \left/ \begin{array}{l} u = R^2 - 2t^2 \\ du = -4t\, dt \end{array} \quad \begin{array}{c|c|c} t & 0 & \frac{R}{2} \\ \hline u & R^2 & \frac{R^2}{2} \end{array} \right/$$

$$= \frac{R}{\sqrt{2}} \int_{R^2/2}^{R^2} \frac{du}{\sqrt{u}} = \ldots = R^2(\sqrt{2} - 1)$$

---

(8)

$$\int_L \sqrt{x^2 + y^2}\, dl$$

$$L: \quad (x^2 + y^2)^{3/2} = a^2 (x^2 - y^2)$$

POLARNE KOORDINATE:

$$\rho^3 = a^2 \rho^2 (\cos^2\varphi - \sin^2\varphi)$$

$$\rho = a^2 \cos 2\varphi$$

$$\cos 2\varphi \geq 0 \quad \Rightarrow \quad \varphi \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$$

AKO PRELAZIMO NA POLARNE KOORDINATE i $\rho = \rho(\varphi)$

$$x(\varphi) = \rho(\varphi) \cos\varphi \qquad x' = \rho' \cos\varphi - \rho \sin\varphi$$

$$y(\varphi) = \rho(\varphi) \sin\varphi \qquad y' = \rho' \sin\varphi + \rho \cos\varphi$$

$$dl = \sqrt{x'^2 + y'^2}\, d\varphi$$

$$= \sqrt{\rho'^2 \cos^2\varphi - \cancel{2\rho\rho' \cos\varphi \sin\varphi} + \rho^2 \sin^2\varphi + \rho'^2 \sin^2\varphi + \cancel{2\rho\rho' \sin\varphi\cos\varphi} + \rho^2 \cos^2\varphi}$$

$$= \sqrt{\rho'^2 + \rho^2}\, d\varphi$$

$\rho^2 = a^2 \cos 2\varphi \qquad \rho' = -2a^2 \sin 2\varphi$

$dl = \sqrt{a^4 \cos^2 2\varphi + 4a^4 \sin^2 2\varphi}\; d\varphi = a^2 \sqrt{1 + 3 \sin^2 2\varphi}\; d\varphi$

$I = \int_{-\pi/4}^{\pi/4} a^2 \cos 2\varphi \; a^2 \sqrt{1 + 3\sin^2 2\varphi}\; d\varphi + \int_{3\pi/4}^{5\pi/4} \text{—}\,||\,\text{—}$

$= 2a^4 \int_{-\pi/4}^{\pi/4} \cos 2\varphi \sqrt{1 + (\sqrt{3}\sin 2\varphi)^2}\; d\varphi$

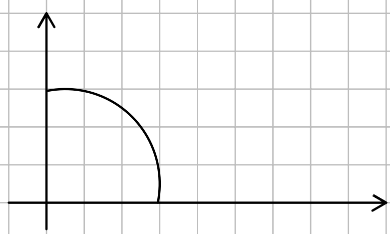
$= \Big|\; \begin{array}{l} u = \sqrt{3}\sin 2\varphi \\ du = 2\sqrt{3}\cos 2\varphi \, d\varphi \end{array} \qquad \begin{array}{c|c|c} \varphi & -\frac{\pi}{4} & \frac{\pi}{4} \\ \hline u & -\sqrt{3} & \sqrt{3} \end{array}$

$= 2a^4 \int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{2\sqrt{3}} \sqrt{1+u^2}\; du = \ldots$

---

(9) $\int_L xy \; dl$

$L: \quad a^2x^2 + b^2y^2 = 1 \qquad a>0,\, b>0, \quad x \geq 0 \quad y \geq 0$



$ax = \cos\varphi \qquad by = \sin\varphi$

$x = \frac{1}{a}\cos\varphi \qquad y = \frac{1}{b}\sin\varphi$

$x' = -\frac{1}{a}\sin\varphi \qquad y' = \frac{1}{b}\cos\varphi$

$dl = \sqrt{x'^2 + y'^2} = \sqrt{\frac{1}{a^2}\sin^2\varphi + \frac{1}{b^2}\cos^2\varphi}\; d\varphi$

$I = \int_L xy \; dl = \int_L \frac{1}{a}\cos\varphi \, \frac{1}{b}\sin\varphi \sqrt{\frac{1}{a^2}\sin^2\varphi + \frac{1}{b^2}\cos^2\varphi}\; d\varphi$

$$= \frac{1}{ab} \int_0^{\pi/2} \cos\varphi \sin\varphi \sqrt{\frac{b^2(1-\cos^2\varphi)+a^2\cos^2\varphi}{a^2 b^2}}\, d\varphi$$

$$= \frac{1}{a^2 b^2} \int_0^{\pi/2} \cos\varphi \sin\varphi \sqrt{b^2 + (a^2-b^2)\cos^2\varphi}\, d\varphi$$

$$= \left| \begin{array}{l} u = \cos\varphi \\ du = -\sin\varphi \, d\varphi \end{array} \quad \begin{array}{c|c|c} \varphi & 0 & \pi/2 \\ \hline u & 1 & 0 \end{array} \right|$$

$$= \frac{1}{a^2 b^2} \int_0^1 u \sqrt{b^2 + (a^2-b^2) u^2}\, du =$$

$$= \left| \begin{array}{l} z = b^2 + (a^2-b^2) u^2 \\ dz = 2(a^2-b^2)\, u\, du \end{array} \quad \begin{array}{c|c|c} u & 0 & 1 \\ \hline z & b^2 & a^2 \end{array} \right|$$

$$= \frac{1}{a^2 b^2} \frac{1}{2(a^2-b^2)} \int_{b^2}^{a^2} \sqrt{z}\, dz = \ldots$$

---

(10) IZRAČUNATI DUŽINU KARDIOIDE

$$(x^2+y^2-8x)^2 = 64(x^2+y^2)$$

POLARNE KOORDINATE:

$$(\rho^2 - 8\rho\cos\varphi)^2 = 64\rho^2$$

$$\cancel{\rho^2}(\rho - 8\cos\varphi)^2 = 64\cancel{\rho^2}$$

$$|\rho - 8\cos\varphi| = 8$$

I

$\rho \geq 8\cos\varphi$

$\rho - 8\cos\varphi = 8$

$\rho = 8 + 8\cos\varphi$

$\rho' = -8\sin\varphi$

$\rho < 8\cos\varphi$

$8\cos\varphi - \rho = 8$

$\rho = 8\cos\varphi - 8 \leq 0$ ↯

$$dl = \sqrt{\rho^2 + \rho'^2}\, d\varphi = \sqrt{(8+8\cos\varphi)^2 + (-8\sin\varphi)^2}\, d\varphi$$

$$= \sqrt{64 + 128\cos\varphi + 64\cos^2\varphi + 64\sin^2\varphi}\, d\varphi$$

$$= \sqrt{128 + 128\cos\varphi}\, d\varphi = \sqrt{256}\sqrt{\frac{1+\cos\varphi}{2}}\, d\varphi$$

$$= 16\left|\cos\frac{\varphi}{2}\right| d\varphi$$

$$\underset{\text{duljina}}{s} = \int_L 1\cdot dl = 16\int_0^{2\pi}\left|\cos\frac{\varphi}{2}\right| d\varphi = \left|\begin{array}{c} u = \frac{\varphi}{2} \\ 2\,du = d\varphi \end{array} \quad \begin{array}{c|c|c} \varphi & 0 & 2\pi \\ \hline u & 0 & \pi \end{array}\right|$$

$$= 32\int_0^{\pi} |\cos u|\, du =$$

$$= 32\left(\int_0^{\pi/2}\cos u\, du - \int_{\pi/2}^{\pi}\cos u\, du\right) =$$

$$= 32\left(\sin u\,\Big|_0^{\pi/2} - \sin u\,\Big|_{\pi/2}^{\pi}\right)$$

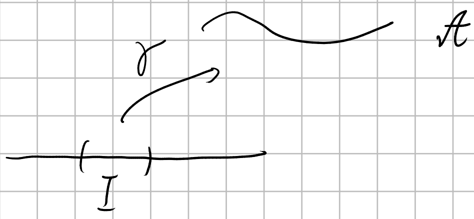
$$= 32\,(1 - 0 - (0 - 1)) = \boxed{64}$$

Analiza 2

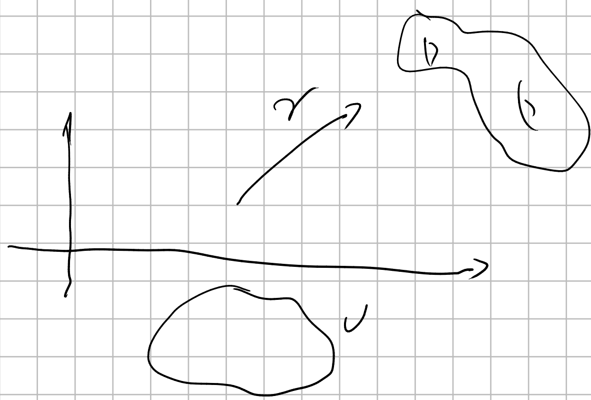
Vežbe

Videli smo jedan način uopštenja integrala na prostore koji nisu Euklidski.

Posmatrali smo integral na krivoj:



Dalje uopštenje mogu biti regularne površi



$r(u,v) = (x, y, z)$

Uslov regularnosti: $\frac{\partial r}{\partial u}$ i $\frac{\partial r}{\partial v}$ linearno nezavisni

- Želimo veće uopštenje a da krive, površi budu specijalni slučaj

Дефиниција:

$$M \subseteq \mathbb{R}^{k+l}$$

M је k-димензиона регуларна подмногострукост класе C^n ако

$\forall p \in M$ има отворену околину $V(p) \subseteq \mathbb{R}^{k+l}$

такву да постоји пресликавање

$$f : V \to \mathbb{R}^l \quad \text{класе } C^n$$

тако да важи:

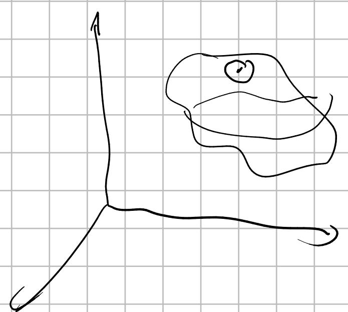
1) $\forall x \in V$ $\quad Df(x)$ је сурјективан

2) $V \cap M = f^{-1}(\{0\})$

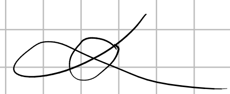
Напомена:

- M може да се зада једначином $f(x) = 0$ за неку субмерзију

- Сваки отворен скуп у $\mathbb{R}^n$ је n-димензиона подмногострукост



$$f(x) = 0$$



Сфера S^2 $\qquad f(x,y,z) = x^2 + y^2 + z^2 - 1$

$$S^2 = f^{-1}(0)$$

Tvrđenje:

$$M \subseteq \mathbb{R}^{k+l}$$

Ekvivalentno je:

a) M je k-dimenzionalna regularna podmnogostrukost klase C^k

b) $\forall p \in M$ postoji okolina $V(p)$ i postoji otvorena okolina nule $U(0) \subseteq \mathbb{R}^{k+l}$ i difeomorfizam:

$$\psi : V \to U$$

$$\psi(V \cap M) = U \cap \{ \{0\} \times \mathbb{R}^k \}$$

$\mathbb{R}^2$ prostor 3 dimenziona podmnogostrukost

$$\psi(V \cap M) = (0, 0, \overbrace{a\ i\ i}^{3})$$

v) $\forall p \in M$ postoji otvorena okolina $V(p)$ i postoji $D \subseteq \mathbb{R}^k$ (k-dimenzion)

$0 \in D$

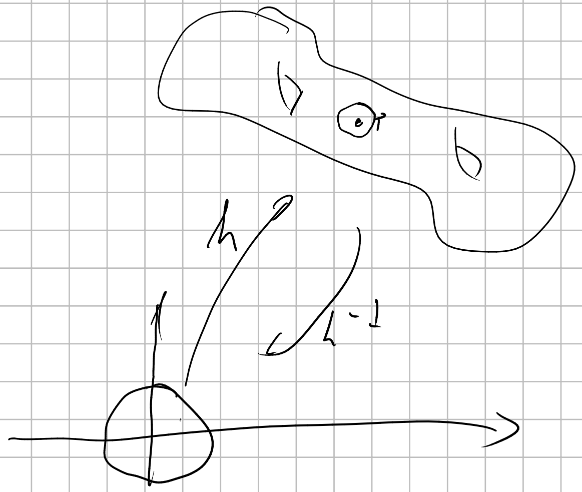
i postoji inkluzija h

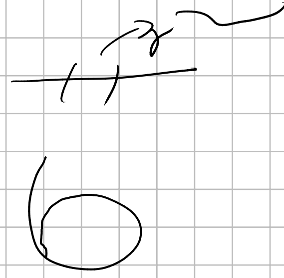
$h : D \to V$ klase C^k

$h : D \to V \cap M$ homeomorfizam

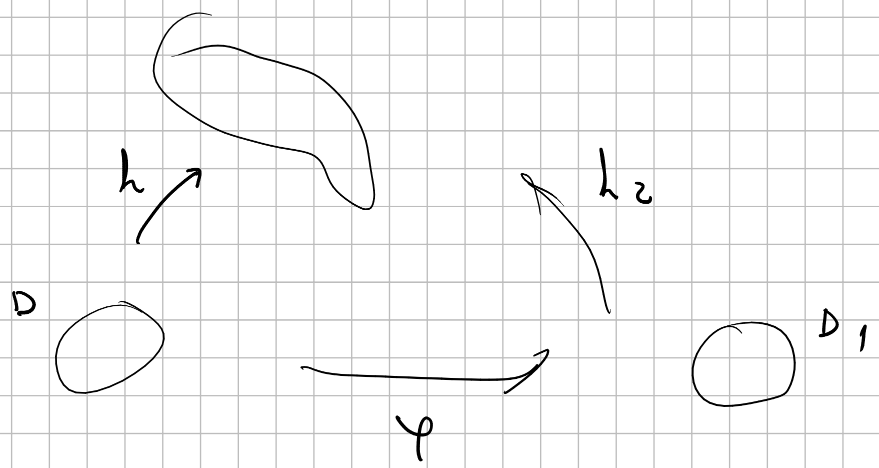
h - parametrizacija

h^{-1} - karta na M





h NIJE JEDINSTVENO



SFERA S^2

$$h(u,t) = (\cos u \sin t, \ \sin u \sin t, \ \cos t)$$

$$dh = \begin{bmatrix} -\sin u \sin t & \cos u \sin t & 0 \\ \cos u \cos t & \sin u \cos t & -\sin t \end{bmatrix}$$

$$dh(0,0) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

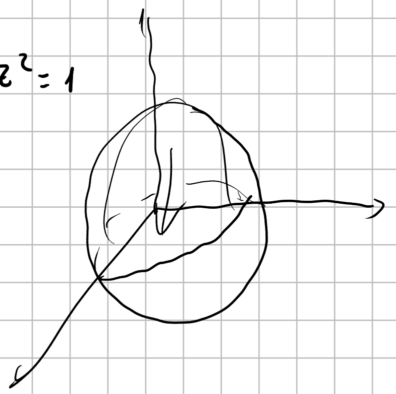
RANG $Dh(0,0) = 1$

$y = x$ $\qquad y^3 - x^3 = 0$

$$f(x,y) = y^3 - x^3$$

$$\nabla f(0,0) = [0,0]$$

$x^2 + y^2 + z^2 = 1$



$z = \sqrt{1 - x^2 - y^2}$

$\boxed{x^2 + y^2 \leq 1}$

ОВО НИЈЕ ОТВОРЕН СКУП

$x^2 + y^2 < 1$

МОРА ДА БУДЕ ОТВОРЕН СКУП

$\varphi_1 : \{x^2 + y^2 < 1\} \to S^2$

ДА ЈЕ ГОРЊУ ПОЛУСФЕРУ БЕЗ ЕКВАТОРА.

ДА ЛИ МОЖЕМО СФЕРУ ПОКРИТИ СА 1 КАРТОМ!

ДА ЛИ ПОСТОЈИ $\varphi : S^2 \to \mathbb{R}^2$

φ - DIFF.

НЕ ПОСТОЈИ ОВАКВО φ

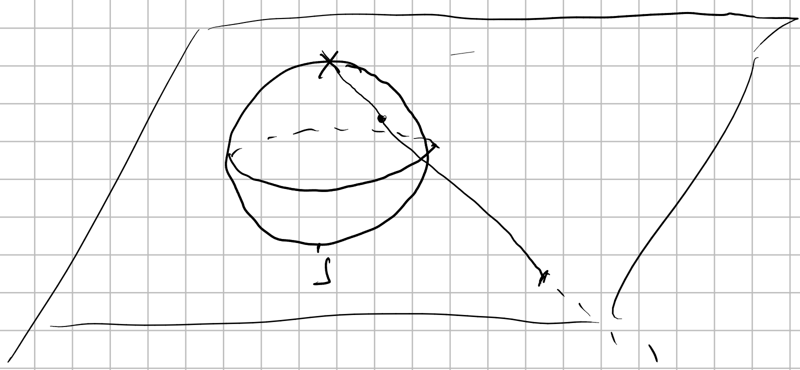
S^2 ЈЕ КОМПАКТАН СКУП И ОТВОРЕН У СЕБИ

$\varphi(S^2)$ - КОМПАКТАН И ОТВОРЕН

НЕ ПОСТОЈИ ОВАКАВ СКУП У $\mathbb{R}^2$ $\lightning$

SFERU BEZ 1 TAČKE MOŽEMO POKRITI JEDNOM KARTOM (STEREOGRAFSKA PROJEKCIJA)

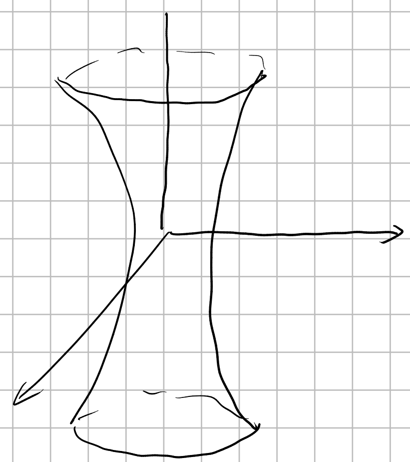
$\mathbb{S}^2 \setminus \{N\}$ N - SEVERNI POL



$\mathbb{S}^2$ "=" $\mathbb{R}^2 \cup \{\infty\}$

SFERU MOŽEMO POKRITI SA 2 KARTE

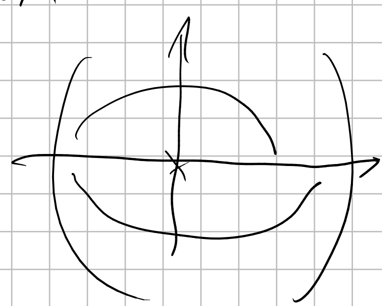
$H: \quad x^2+y^2-z^2=a \qquad a>0$



POSMATRAĆEMO OBLASTI

$x>0$, $x<0$

$y>0$, $y<0$



$x=\pm\sqrt{z^2-y^2+a} \longrightarrow V_1=\{(y,z)\in\mathbb{R}^2 \mid y^2-z^2<a\}$

$y=\pm\sqrt{z^2-x^2+a} \longrightarrow V_2=\{(x,z)\in\mathbb{R}^2 \mid x^2-z^2<a\}$

$\varphi_1 : V_1 \to H \qquad \varphi_1(y,z) = (\sqrt{z^2 - y^2 + 4},\ y,\ z)$

$\varphi_2 : V_1 \to H \qquad \varphi_2(y,z) = (-\sqrt{z^2 - y^2 + 4},\ y,\ z)$

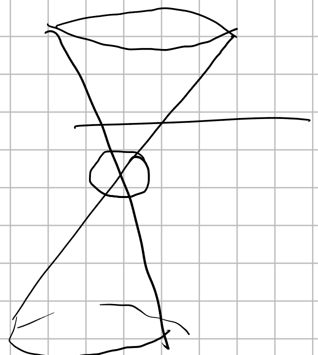
$\varphi_3 : V_2 \to H \qquad \varphi_3(x,z) = (x,\ \sqrt{z^2 - x^2 + 4},\ z)$

$\varphi_4 : V_2 \to H \qquad \varphi_4(x,z) = (x,\ -\sqrt{z^2 - x^2 + 4},\ z)$

POKRILI SMO KARTAMA CEO HIPERBOLOID.

$$M = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 0 \}$$

NIJE MNOGOSTRUKOST



P.P.S. M JESTE PODMNOGOSTRUKOST

POSTOJI KARTA U i φ

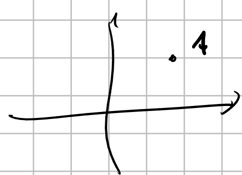
$(0,0,0) \in U$

$\varphi((0,0,0)) = A \in \mathbb{R}^2$

φ JE HOMEOMORFIZAM

$\hat{\varphi} : M \setminus \{(0,0,0)\} \to \mathbb{R}^2 \setminus \{A\}$ I OVO JE HOMEOMORFIZAM

$M \setminus \{(0,0,0)\}$ NIJE POVEZAN SKUP



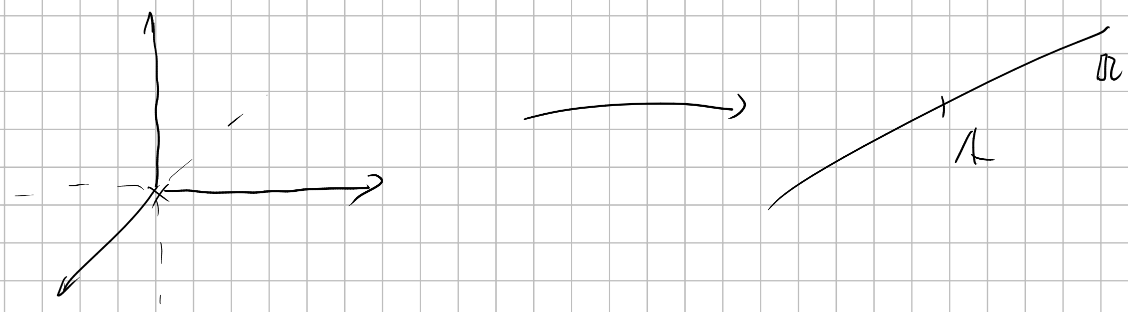
$\mathbb{R}^2 \setminus \{A\}$ JESTE POVEZAN

$\lightning$

$H = O_x \cup O_y \cup O_z$

H - NIJE PODMNOGOSTRUKOST

P.P.S. JESTE



PROBLEM JE TAČKA $(0,0,0)$ $(\varphi(0,0,0) = A)$

OPET DOBIJAMO KONTRADIKCIJU

$H \setminus \{(0,0,0)\}$ IMA 6 KOMPONENTI POVEZANOSTI

$\mathbb{R} \setminus \{A\}$ IMA 2 —— || ——

$M \subseteq \mathbb{R}^n$

M JE n-DIMENZIONA PODMNOGOSTRUKOST

$\Leftrightarrow$ M JE OTVOREN

$\Leftarrow$ M OTVOREN

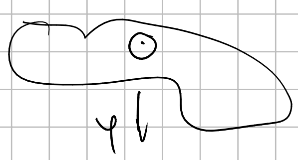
$x \in M$ $\quad B(x, \varepsilon_x) \subseteq M$

POSTOJI KARTA NA $B(x)$ $\quad h = 1$

$\Rightarrow$ M PODMNOGOSTRUKOST

$x \in M$

POSTOJI KARTA (U, φ) $\quad x \in U$

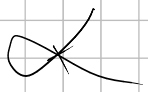


$\varphi : U \cap M \xrightarrow[\text{DIFF}]{} V \in \tau_{\mathbb{R}^n}$

$x \in S = \varphi^{-1}(u)$

OTVOREN

НАПОМЕНА: БИТНО ЈЕ ДА ЈЕ ИСТА ДИМЕНЗИЈА



$$GL(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\}$$

$$f : \mathbb{R}^{n\times n} \to \mathbb{R}$$

$$f(A) = \det A$$

$$GL(n,\mathbb{R}) = f^{-1}\big((-\infty,0) \cup (0,+\infty)\big)$$

ОТВОРЕНИ

ИНВЕРЗНА СЛИКА ОТВОРЕНОГ ЈЕ ОТВОРЕН.

ВАЖИ СЛЕДЕЋЕ:

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$M = f^{-1}(\{0\}) \subseteq \mathbb{R}^n$$

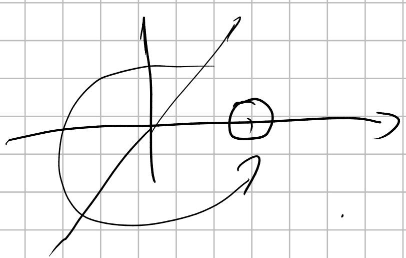
M ЈЕ ПОДМНОГОСТРУКОСТ АКО ЈЕ

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) \neq 0$$

$$\dim M = n-1$$

$0 < r < R$ $\qquad T : r^2 = \underbrace{z^2 + \left(\sqrt{x^2+y^2} - R\right)^2}$

$$f : \mathbb{R}^3 \to \mathbb{R}$$



$$f(x,y,z) = z^2 + \left(\sqrt{x^2+y^2} - R\right)^2$$

$$\frac{\partial f}{\partial x} = \frac{2(\sqrt{x^2+y^2}-R)}{\cancel{2}\sqrt{x^2+y^2}}\cancel{2}x$$

$$T = f^{-1}(\{r^2\})$$

$$\frac{\partial f}{\partial y} = \frac{2(\sqrt{x^2+y^2}-R)}{\sqrt{x^2+y^2}}\, y$$

$$\frac{\partial f}{\partial z} = 2z$$

$$\nabla f(x,y,z) = 0 \rightarrow (x,y,z) = (0,0,0)$$

$$\rightarrow z = 0 \quad \sqrt{x^2+y^2} = R$$

$$f(0,0,0) = R^2 > r^2 \qquad T = f^{-1}(\{r^2\})$$

$$(0,0,0) \notin T$$

$$\sqrt{x^2+y^2} = R \qquad f(x,y,0) = 0 \qquad (x,y,z) \notin T$$

OVIM SMO POKAZALI DA T JESTE PODMNOGO-STRUKOST T.

$$\dim T = 2$$

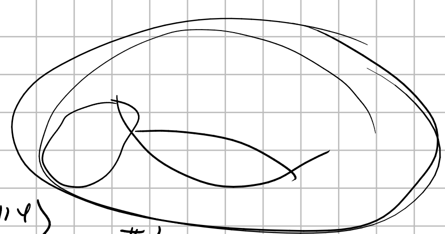
$$x = \cos\theta\,(R + r\cos\varphi)$$
$$y = \sin\theta\,(R + r\cos\varphi)$$
$$z = r\sin\varphi$$

$$f(x,y,z) = r^2\sin^2\varphi + \left(\sqrt{\cos^2\theta\,(R+r\cos\varphi)^2 + \sin^2\theta\,(R+r\cos\varphi)^2} - R\right)^2$$
$$= r^2$$

OVO JESTE PARAMETRIZACIJA M

(ODAVDE SE LAKŠE VIDI DA JE TORUS)

$\mathbb{T}^2 = S^1 \times S^1 \subseteq \mathbb{R}^4$



$((x_1, x_2), (x_3, x_4)) = (\cos\theta, \sin\theta, \cos\varphi, \sin\varphi) = \mathbb{T}^2$

$\varphi : \mathbb{T}^2 \to T$

$\varphi((x_1,x_2),(x_3,x_4)) = (x_1(R + r x_3), \; x_2(R + r x_3), \; r x_4)$

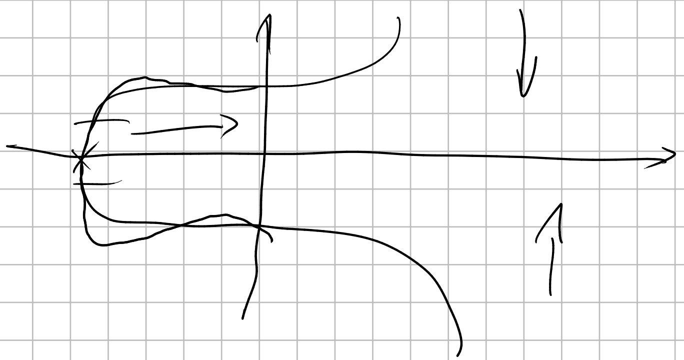
$\varphi^{-1} : T \to \mathbb{T}^2$

$\varphi^{-1}(x, y, z) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, \frac{R - \sqrt{x^2+y^2}}{r}, \frac{z}{r} \right)$

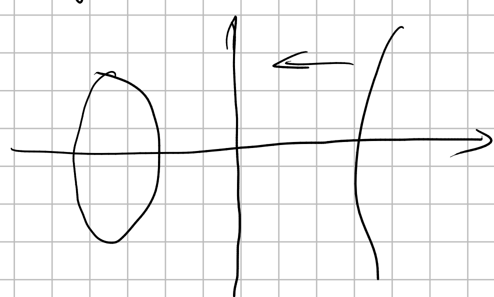
φ, φ^{-1} su DIFEOMORFIZMI

$\varphi \circ \varphi^{-1} = \mathbb{1}$

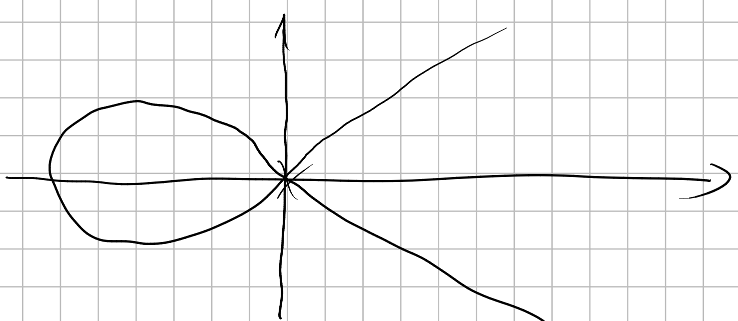
$y^2 = (x+1)(x^2+4)$



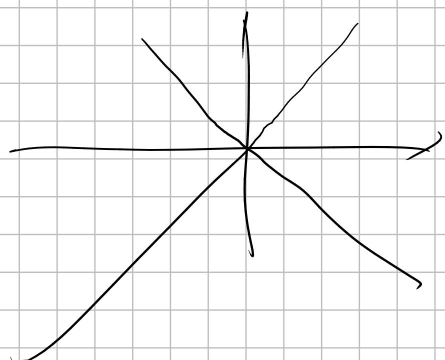
$y^2 = (x+1)(x^2-4)$

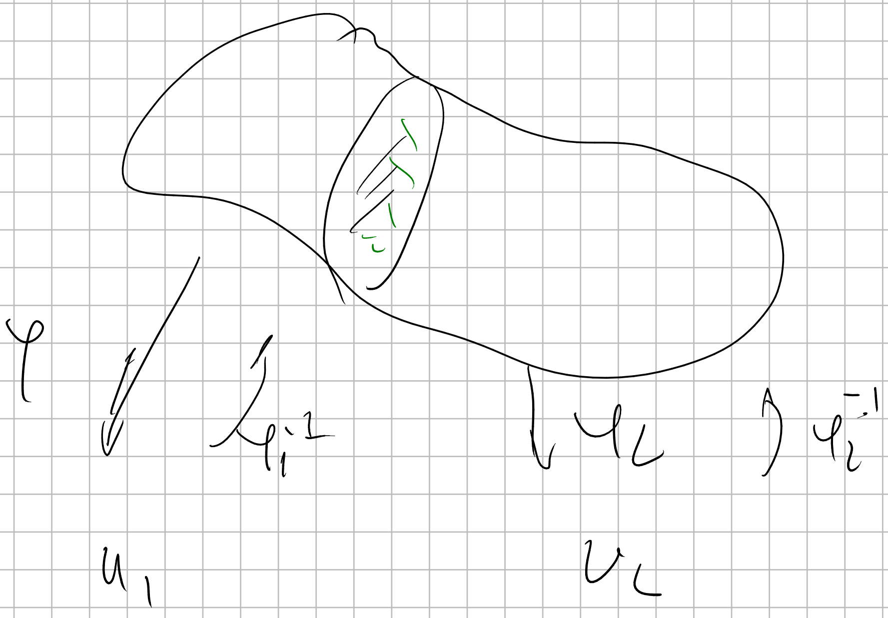


$y^2 = (x+1)x^2$



$x^2 - y^2 = 0$





U_1, V_2 — открыты

ANALIZA 2

VEŽBE

- DEFINISALI SMO PODMNOGOSTRUKOSTI U $\mathbb{R}^n$

k-DIMENZIONA REGULARNA PODMNOGOSTRUKOST KLASE C^n JE SKUP M AKO VAŽI:

$\forall p \in M \quad \exists V(p)$ - OTVORENA OKOLINA

$$f : V \to \mathbb{R}^l$$

1) $\forall x \in V \quad Df(x) : \mathbb{R}^{k+l} \to \mathbb{R}^l$ JE SURJEKTIVNA

2) $V \cap M = f^{-1}(0)$

II OPIS

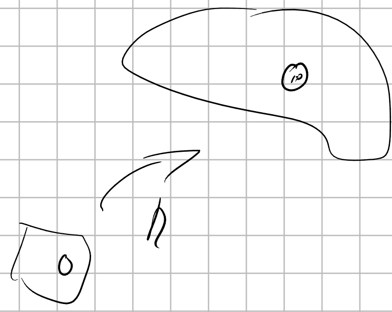
$\forall p \in M \quad \exists V(p)$

$\exists D \quad 0 \in D \subseteq \mathbb{R}^k$

I IMERZIJA

$h : D \to V \qquad h : D \to V \cap M$ JE HOMEOMORFIZAM

DEFINICIJA!



$$h : D \to \mathbb{R}^n, \quad D \subset \mathbb{R}^k$$

h - lokalna parametrizacija k-dimenzione regularne podmnogostrukosti M u okolini tačke $h(0) = p$ (klase bar C^1)

Tangentni prostor na M u tački p je afini prostor

$$x - p = Dh(0)\, t$$

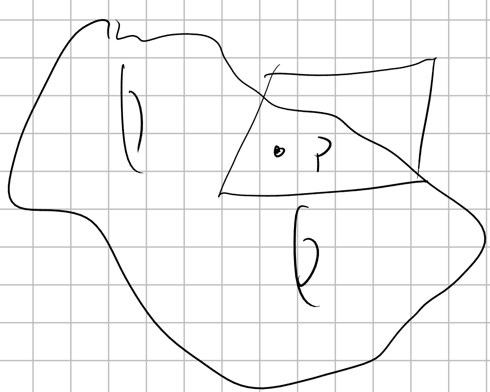
ili u koordinatama

$$x_1 - p_1 = \frac{\partial h_1}{\partial t_1}(0)\, t_1 + \dots + \frac{\partial h_1}{\partial t_k}(0)\, t_k$$

$$\vdots$$

$$x_n - p_n = \frac{\partial h_n}{\partial t_1}(0)\, t_1 + \dots + \frac{\partial h_n}{\partial t_k}(0)\, t_k$$

- Tangentni prostor možemo videti i kao prostor tangenti na sve krive koje su u M i prolaze kroz p



Vektorizaciju dobijamo tako što uzmemo p za nula vektor.

T_pM - OZNAKA

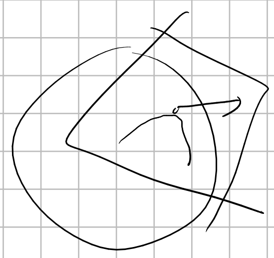
$TM = \bigcup_{p\in M} T_pM$ - TANGENTNO RASLOJENJE

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$M = f^{-1}(\{0\})$$

M JE PODMNOGOSTRUKOST $\iff$ $\nabla f \neq 0$

$$\dim M = n-1$$



$p \in M$ - PROIZVOLJNO

$$\nabla f(p) \perp T_pM$$

$X_p \in T_pM$ - PROIZVOLJAN VEKTOR

$\iff$ POSTOJI KRIVULJA $\gamma : (-\varepsilon, \varepsilon) \to M$

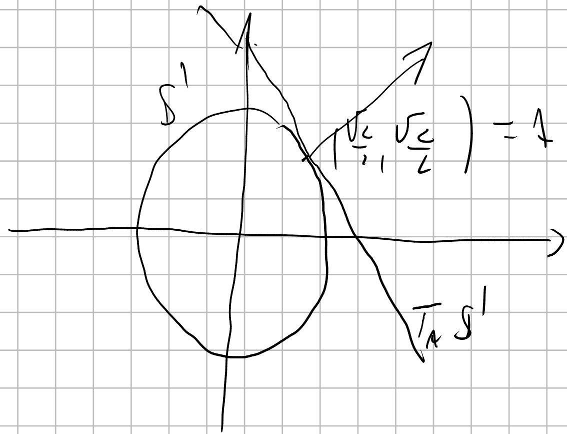
$$\gamma(0) = p \qquad \frac{d\gamma}{dt}(0) = X_p$$

$$\forall t \in (-\varepsilon,\varepsilon) \qquad f(\gamma(t)) \overset{\gamma \in M}{=} 0 \quad / \frac{d}{dt}$$

$$\nabla f(\gamma(t)) \overbrace{\frac{d\gamma}{dt}}^{= X_p} = 0 \Big|_{t=0}$$

KAKO JE X_p BIO PROIZVOLJAN $\Rightarrow$ $\nabla f \perp T_pM$

Krug $\quad x^2+y^2=1 \qquad f: x^2+y^2-1$



$$S^1 = f^{-1}(\{0\})$$

$$\nabla f = (2x, 2y)$$

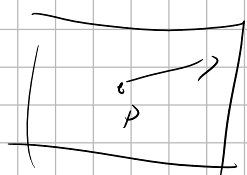
$$\nabla f(A) = (\sqrt{2}, \sqrt{2}) \qquad (1,1)$$

$T_A S^1 \qquad x + y + c = 0$

$A \qquad \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + c = 0 \qquad c = -\sqrt{2}$

$$x + y - \sqrt{2} = 0$$

$$y = \sqrt{1-x^2}$$



$$\gamma = \left(x, \sqrt{1-x^2}\right)$$

$$\frac{\partial \gamma}{\partial x} = \left(1, \frac{-x}{\sqrt{1-x^2}}\right)$$

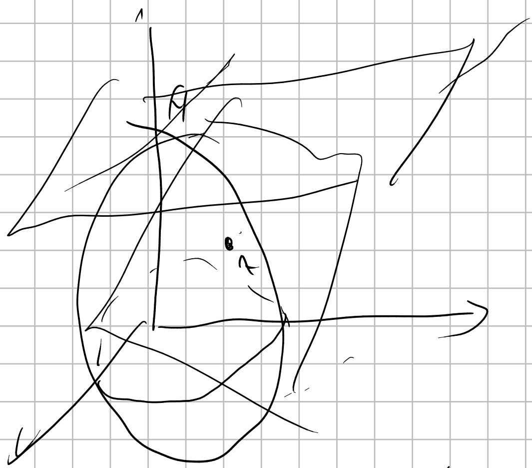
$T_A S^1 \qquad (x,y) = A + \alpha\left(1, \frac{-x}{\sqrt{1-x^2}}\right)$

$$\left.\begin{aligned} x &= \frac{\sqrt{2}}{2} + \alpha \\ y &= \frac{\sqrt{2}}{2} - \alpha \end{aligned}\right\} + \qquad x + y = \sqrt{2}$$

$S^2: \quad x^2+y^2+z^2=1 \qquad f(x,y,z) = x^2+y^2+z^2-1$

$S^2 = f^{-1}(0)$

$\nabla f = (2x, 2y, 2z)$



$T_N S^2: \quad z = 1$

$A = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$

$\nabla f(A) = (1, 1, \sqrt{2})$

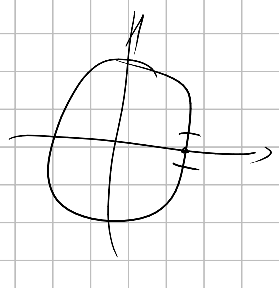
$T_A S^2: x + y + \sqrt{2}\, z + D = 0$

$\in$

A

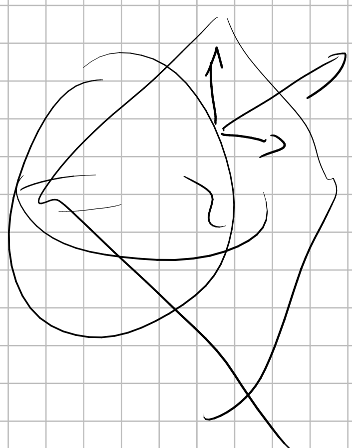
$\frac{1}{2} + \frac{1}{2} + 1 + D = 0 \qquad D = -2$

$x + y + \sqrt{2}\, z - 2 = 0$



$z = \sqrt{1-x^2-y^2}$

$r = \left(x, y, \sqrt{1-x^2-y^2}\right)$



$\frac{\partial r}{\partial x} = \left(1, 0, \frac{-x}{\sqrt{1-x^2-y^2}}\right)$

$\frac{\partial r}{\partial y} = \left(0, 1, \frac{-y}{\sqrt{1-x^2-y^2}}\right)$

$$\frac{\partial r}{\partial x}(A) = \left(1, 0, \frac{-\frac{1}{2}}{\frac{1}{\sqrt{2}}}\right) = \left(1, 0, -\frac{1}{\sqrt{2}}\right)$$

$$\frac{\partial r}{\partial y}(A) = \left(0, 1, -\frac{1}{\sqrt{2}}\right)$$

$$(x, y, z) = A + \alpha \frac{\partial r}{\partial x} + \beta \frac{\partial r}{\partial y}$$

$$\left.\begin{aligned} x &= \frac{1}{2} + \alpha \\ y &= \frac{1}{2} + \beta \\ z &= \frac{1}{\sqrt{2}} - \frac{\alpha}{\sqrt{2}} - \frac{\alpha}{\sqrt{2}} \end{aligned}\right\}$$

DOMAĆI:

ODREDITI TANGENTNU RAVAN NA TORUS I JEDNOGRANI HIPERBOLOID.

$S_1: \quad (x-1)^2 + y^2 + z^2 = 1$

$S_2: \quad x^2 + y^2 + (z-\lambda)^2 = 3$

ODREDITI λ TAKO DA SU TANGENTNE RAVNI U TAČKAMA PRESEKA ORTOGONALNE

$$f_1 : \mathbb{R}^3 \to \mathbb{R}$$

$$f_1(x, y, z) = (x-1)^2 + y^2 + z^2 - 1$$

$$S_1 = f_1^{-1}(\{0\})$$

$f_2 : \mathbb{R}^3 \to \mathbb{R}$

$f_2(x,y,z) = x^2 + y^2 + (z-\lambda)^2 - 3$

$S_2 = f_2^{-1}(3)$

$\nabla f_1 = (2(x-1),\ 2y,\ 2z)$

$\nabla f_2 = (2x,\ 2y,\ 2(z-\lambda))$

IZ ORTOGONALNOSTI $\quad \nabla f_1 \cdot \nabla f_2 = 0$

$\nabla f_1 \cdot \nabla f_2 = 4x^2 - 4x + 4y^2 + 4z^2 - 4\lambda z = 0$

$x^2 - 2x + \cancel{1} + y^2 + z^2 = \cancel{1} \qquad x^2 + y^2 + z^2 = 2x$

$x^2 + y^2 + z^2 - 2\lambda z + \lambda^2 = 3$

$8x - 4x - 4\lambda z = 0$

$x - \lambda z = 0 \qquad x = \lambda z$

$2x - 2\lambda z + \lambda^2 = 3$

$2\lambda z - 2\lambda z + \lambda^2 = 3 \qquad \boxed{\lambda = \pm\sqrt{3}}$

$GL(n,\mathbb{R}) = \{ A \in \mathbb{R}^{n\times n} \mid \det A \neq 0 \}$

$f(A) = \det A$

$GL(n,\mathbb{R}) = f^{-1}\big((-\infty, 0) \cup (0, +\infty) \big)$

$T_{\mathbb{1}} \, GL(n,\mathbb{R}) = \mathbb{R}^{n\times n}$ $\rightsquigarrow$ SKUP SVIH MATRICA $n \times n$

$\subseteq$: OČIGLEDNO

$\supseteq$:

$X \in \mathbb{R}^{n\times n}$ – PROIZVOLJNA

TREBA NAM $A(t)$ KRIVA U $GL(n,\mathbb{R})$

$A(0) = \mathbb{1}$

$\frac{dA}{dt}(0) = X$

$A(t) = \mathbb{1} + tX$

$A(0) = \mathbb{1}$

$\frac{dA}{dt}\Big|_{t=0} = X$

$A(t) \in GL(n,\mathbb{R})$ ZA DOVOLJNO MALO t.

$A(t) = \mathbb{1} + tX$

$t \in \left(-\frac{1}{\|X\|}, \frac{1}{\|X\|} \right)$

$\supseteq$:

$$f = \det A : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$$

$$df(A)\,\hat{A} = \det A \ \mathrm{Tr}\ A^{-1}\cdot\hat{A}$$

$$A \in GL(n,\mathbb{R})$$

$$df(A) : T_A \mathbb{R}^{n\times n} \to T_{f(x)}\mathbb{R} = \mathbb{R}$$

$$df(A)\,\hat{A} = \frac{d}{d\varepsilon} \underbrace{f(A+\varepsilon\hat{A})}\Big|_{\varepsilon=0}$$

$$f(A+\varepsilon\hat{A}) = \det(A+\varepsilon\hat{A}) = \det A\,(\mathbb{1}+\varepsilon A^{-1}\hat{A})$$

$$= \underbrace{\det A}\ \det(\mathbb{1}+\varepsilon \underbrace{A^{-1}\hat{A}}_{B})$$

$$\mathbb{1}+\varepsilon B = \begin{bmatrix} 1+\varepsilon b_{11} & \varepsilon b_{12} & \cdots & \varepsilon b_{1n} \\ \varepsilon b_{21} & 1+\varepsilon b_{22} & & \\ & & \ddots & \\ & & & 1+\varepsilon b_{nn} \end{bmatrix}$$

$$\det(\mathbb{1}+\varepsilon B) = \sum_{\sigma} (-1)^{\mathrm{sgn}\,\sigma}\, c_{1\sigma_1} \cdots c_{n\sigma_n}$$

σ_i – PROIZVOLJNA PERMUTACIJA

AKO NIJE $\sigma_i = i$ POJAVLJUJE SE ε^2

OSTAJE SAMO KADA JE $\sigma_i = i$

$$\det(\mathbb{1}+\varepsilon B) = \frac{d}{d\varepsilon}\Big((1+\varepsilon b_{11})(1+\varepsilon b_{22})\cdots(1+\varepsilon b_{nn})\Big)\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}\Big(1+\varepsilon(b_{11}+b_{22}+\cdots+b_{nn})+\varepsilon^2 D\Big)\Big|_{\varepsilon=0} = \mathrm{Tr}\,B$$

$$B = A^{-1}\hat{A}$$

$$\boxed{\det(A)\hat{A} = \det \operatorname{Tr} A^{-1}\hat{A}}$$

$$SL(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det A = 1\}$$

$$SL(n,\mathbb{R}) = f^{-1}(\{1\}) \qquad f = \det$$

DA LI JE 1 REGULARNA VREDNOST?

$A \in f^{-1}(\{1\})$ PROIZVOLJNO

$$df(A) : T_A\mathbb{R}^{n\times n} \to \mathbb{R}$$

$$\det A = 1$$

$$df(A)\hat{A} = \operatorname{Tr} A^{-1}\hat{A}$$

$ny \in \mathbb{R}$ – PROIZVOLJNO

$$\hat{A} = A\begin{bmatrix} y & & 0 \\ & \ddots & \\ 0 & & y \end{bmatrix} \qquad \operatorname{Tr} A^{-1}\hat{A} = A^{-1}A\begin{bmatrix} y & & \\ & \ddots & \\ & & y \end{bmatrix} = ny$$

$df(A)$ JE RANGA 1

$$\dim SL(n,\mathbb{R}) = n^2 - 1$$

$$f|_{SL(n,\mathbb{R})} = 1 \qquad df|_{SL(n,\mathbb{R})} = 0$$

$$\det I = 1$$

$$df(I)\,A = 0$$

$\det D \; Tr(D^{-1} A) = 0$

$Tr = 0$

$T_I \, SL(n, \mathbb{R}) = \{ A \in \mathbb{R}^{n \times n} \mid Tr\, A = 0 \}$

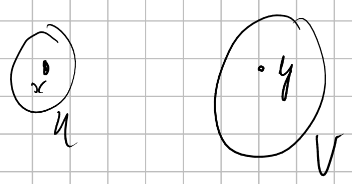
Topološka mnogostrukost M je prostor koji je

- Hauzdorfov (T_2)
- lokalno Euklidski
- prebrojiva baza topologije

$\Rightarrow$ Hauzdorfov prostor je onaj u kome se svake dve tačke mogu razdvojiti otvorenim skupovima

$\forall x, y \quad \exists U, V \qquad U, V$ - otvoreni

$x \in U \qquad y \in V \qquad U \cap V = \emptyset$



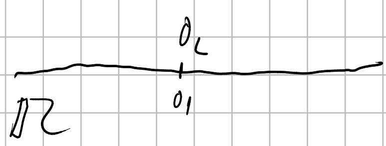
- Primer da nije Hauzdorfov:

antidiskretna topologija

X - skup $\quad \# A \geq 2$

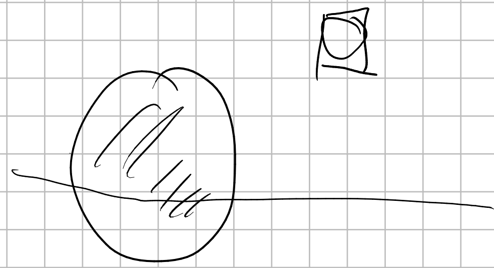
Otvoreni skupovi su samo $\emptyset$ i A.

-



- Lokalno euklidski

Za svaku tačku x postoji otvorena okolina $U(x)$ t.d. U homeomorfno $\mathbb{R}^n$.



- Prebrojiva baza topologije

Glatka mnogostrukost M

$\dim M = n$

Karte (U, h)

$U \subset M$

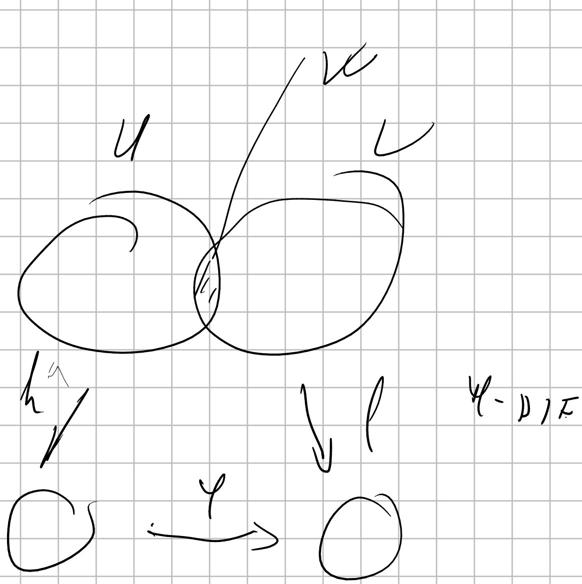
$h : U \to U' \subset \mathbb{R}^n$ bijekcija

Karte su saglasne

(U, h) i (V, ℓ)

1) $U \cap V = \emptyset$

2) $\emptyset \neq W = U \cap V$



$M_1, \dots, M_k$ – MNOGOSTRUKOSTI,

$M_1 \rightsquigarrow$ ATLAS $\{(U_{\alpha_1}, \varphi_{\alpha_1}) \mid \alpha_1 \in A_1\}$

$\vdots$

$M_n \rightsquigarrow$ ATLAS $\{(U_{\alpha_n}, \varphi_{\alpha_n}) \mid \alpha_n \in A_n\}$

$M_1 \times M_2 \times \dots \times M_n$

NA PROIZVODU IMAMO ATLAS

$\{(U_{\alpha_1} \times U_{\alpha_2} \times \dots \times U_{\alpha_n}, \varphi_{\alpha_1} \times \dots \times \varphi_{\alpha_n}) \mid \alpha_i \in A_i\}$

$\Rightarrow$ MNOGOSTRUKOSTI SU:

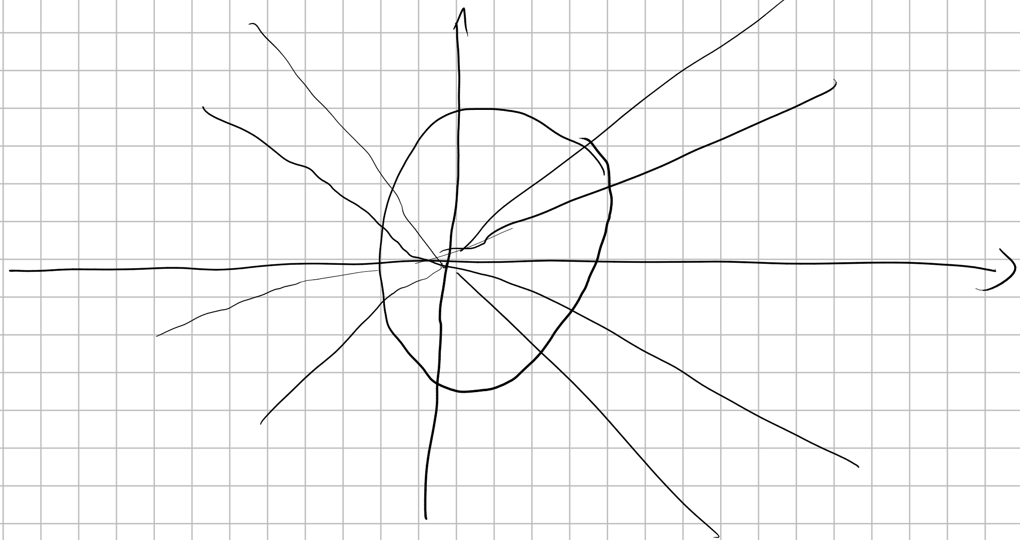
$\mathbb{T}^2, \quad \mathbb{T}^n = \underbrace{\mathbb{S}^1 \times \dots \times \mathbb{S}^1}_{n}$

PROJEKTIVNI PROSTORI

$(\mathbb{R}P^n, \ \mathbb{C}P^n)$

$\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim \qquad x \sim y \Leftrightarrow x = \lambda y \quad \lambda \in \mathbb{R} \setminus \{0\}$

$\searrow$ MOŽEMO SHVATITI TAČKE KAO PRAVE KROZ KOORDINATNI POČETAK.



IDENTIFIKUJEMO x i $-x$ NA SFERI

$$p \mapsto [p]$$

$$p = (p_1, p_2, \dots, p_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}$$

$$U_k = \{ [p] \in \mathbb{R}P^n \mid p_k \neq 0 \}$$

$$\varphi_k : U_k \to \mathbb{R}^n$$

$$\varphi_k(p) = \left(\frac{p_1}{p_k}, \frac{p_2}{p_k}, \dots, \frac{p_{k-1}}{p_k}, \frac{p_{k+1}}{p_k}, \dots, \frac{p_{n+1}}{p_k} \right)$$

↳ n KOORDINATA

ATLAS $\{ (U_k, \varphi_k) \mid k \in \{1, \dots, n\} \}$

ANALIZA 2

VEŽBE

GLATKA MNOGOSTRUKOST KLASE C^r JE TOPOLOŠKA MNOGOSTRUKOST M DIMENZIJE n ZA KOJU POSTOJI FAMILIJA

$$\mathcal{U} = \{ (U_\alpha, \varphi_\alpha) \mid \alpha \in A \}$$

TAKO DA JE $\{U_\alpha\}_{\alpha \in A}$ OTVORENO POKRIVANJE M;

$$\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$$

FAMILIJA HOMEOMORFIZAMA (DIFEOMORFIZAMA)

I MORA DA VAŽI

1) $U_\alpha \cap U_\beta = \emptyset$

ILI

2) $W = U_\alpha \cap U_\beta$

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(W) \longrightarrow \varphi_\beta(W)$$

JE DIFEOMORFIZAM KLASE C^r

$(U_\alpha, \varphi_\alpha)$ - KARTE

φ_α^{-1} - LOKALNA PARAMETRIZACIJA

- UNIJA KARATA SE NAZIVA ATLASOM

DEFINICIJA:

$(X, \mathcal{T})$ – TOPOLOŠKI PROSTOR

$\sim$ – RELACIJA EKVIVALENCIJE

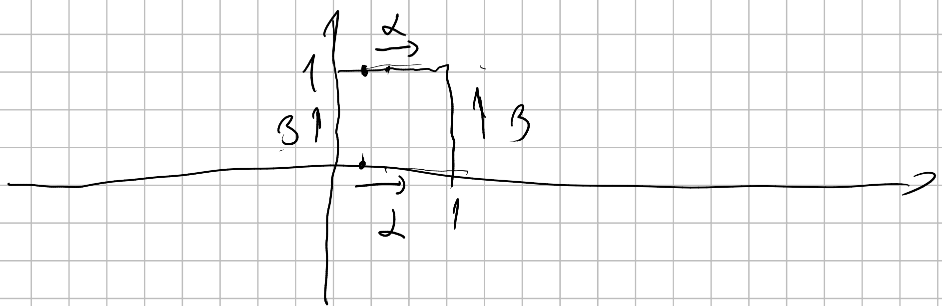
$X/\sim$ – KOLIČNIČKI PROSTOR

$V \subset X/\sim$ JE OTVOREN $\Leftrightarrow$ $\pi^{-1}(V)$ JE OTVOREN U X

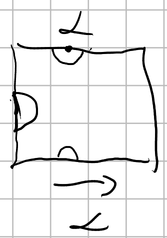
π – PROJEKCIJA (NEPREKIDNA JE)

PRIMERI:

1) TORUS $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$



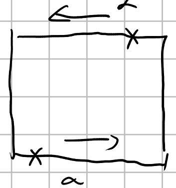
2) CILINDAR



$C = [0,1]^2 \setminus \sim$

$(0, x) \sim (1, x)$

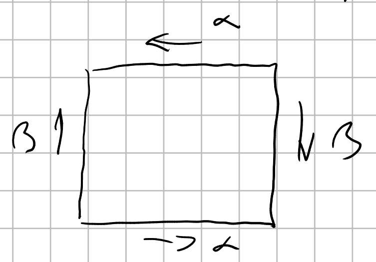
3) MEBIJUSOVA TRAKA



$M = [0,1]^2 / \sim$

$(0, x) \sim (1, 1-x)$

4) KLAJNOVA BOCA



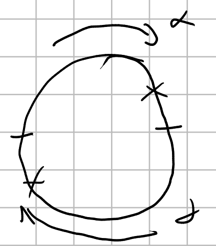
$K = [0,1]^2 / \sim$

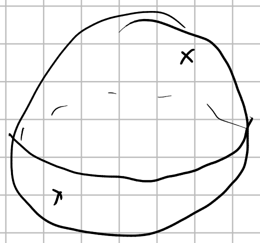
$(0,x) \sim (1, 1-x)$

$(y,0) \sim (1-y, 1)$

5) $\mathbb{RP}^n = S^n / \sim$

$x \sim -x$





GRAFIK GLATKOG PRESLIKAVANJA JE GLATKA MNOGOSTRUKOST

$f : M \to N$

DEFINICIJA:

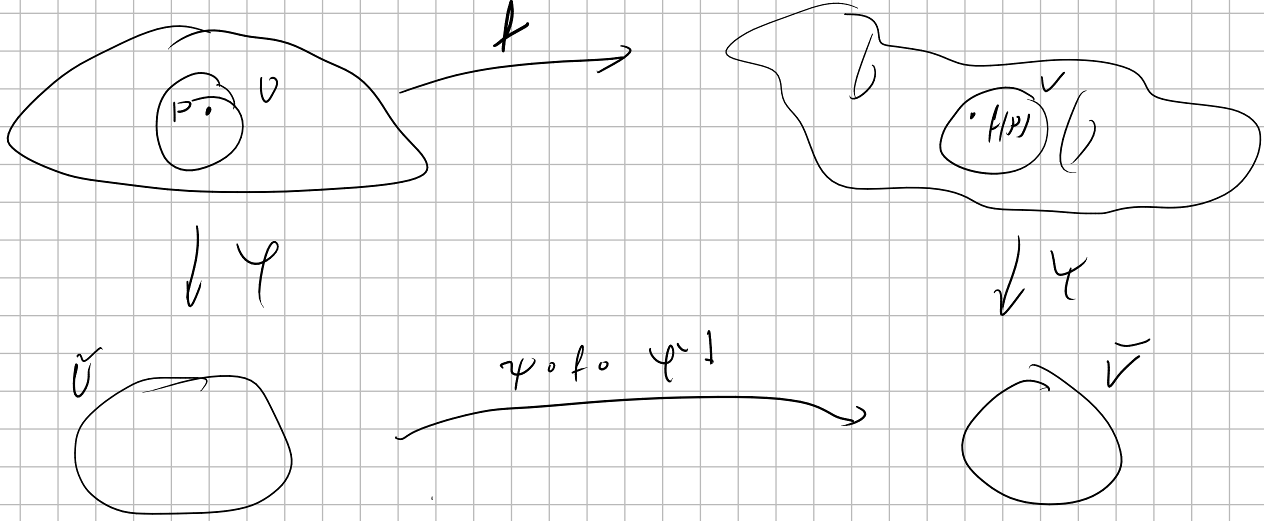
$f : M \to N$ M, N MNOGOSTRUKOST.

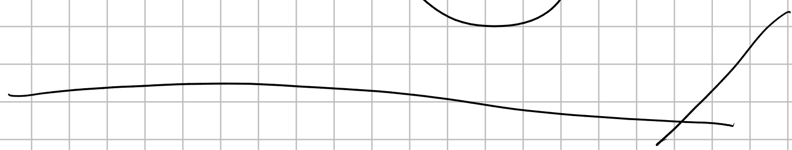
f JE GLATKO PRESLIKAVANJE

$\iff \forall p \in M$ POSTOJE KARTE (U, φ) i (V, ψ)

$p \in U$ $f(p) \in V$

$\psi \circ f \circ \varphi^{-1}$ JE GLATKO





$f : M \to N$ GLATKO

$\Gamma(f) = \{ (x, f(x)) \mid x \in M \} \subseteq M \times N$

$\pi : M \times N \to M$ PROJEKCIJA

$\tilde{\pi} = \pi \big|_{\Gamma(f)}$ - NEPREKIDNO

$(x_0, y_0) \in \Gamma(f)$

$x_0 \in M$ POSTOJI OKOLINA $O(x_0)$ I NA KOJ

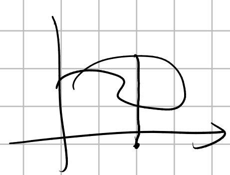
ZADATA KARTA OTVOREN

$\varphi : O(x_0) \to U_0 \subseteq \mathbb{R}^m$

DIREKTNA SLIKA

$\tilde{O} = O \times f(O)$

$\tilde{O}$ JE OKOLINA (x_0, y_0)



$\varphi \circ \tilde{\pi}$

- NEPREKIDNO KAO KOMPOZICIJA

$(\varphi \circ \tilde{\pi})^{-1} = \tilde{\pi}^{-1} \circ \varphi^{-1}$

$$\tilde{\pi}^{-1}(x) = (x, f(x))$$

$f : M \to N$ GLATKO

RANG GLATKOG PRESLIKAVANJA U $p \in M$ JE

$$A = D(\psi \circ f \circ \varphi^{-1})(\varphi(p))$$

- RANG $A = \dim M$ - IMERZIJA
- RANG $A = \dim N$ - SUBMERZIJA

<u>ULAGANJE:</u>

1) f JE 1-1
2) f JE IMERZIJA
3) $f : M \to f(M)$ JE HOMEOMORFIZAM

- AKO JE f IMERZIJA $\Rightarrow$ POSTOJE KARTE TDJ:

$m \leq n$

$$(\psi \circ f \circ \varphi^{-1})(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, 0, \dots, 0)$$

- AKO JE f SUBMERZIJA $\Rightarrow$ POSTOJE KARTE TDJ

$$(\psi \circ f \circ \varphi^{-1})(x_1, \dots, x_m) = (x_1, \dots, x_n)$$

$m \geq n$

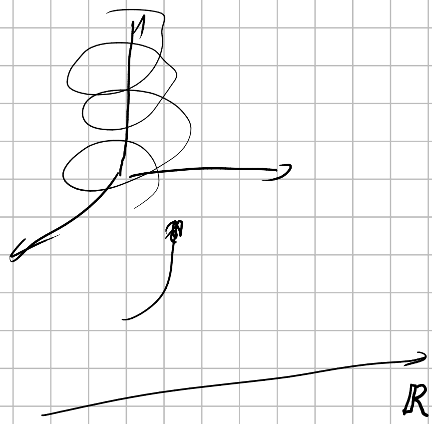
- f DIFEOMORFIZAM

$\psi \circ f \circ \varphi^{-1} = \mathbb{1}$ $\quad m = n$

<u>PRIMERI:</u>

1) $f : \mathbb{R} \to \mathbb{R}^3$

$f(t) = (\cos t, \sin t, t)$



$f'(t) = (-\sin t, \cos t, 1) \neq (0,0,0)$

$\Rightarrow$ f JE IMERZIJA

RANG $f'(t) = 1$

1-1 ZBOG 3. KOORDINATE

$f(t_1) = f(t_2)$ $\quad (\cos t_1, \sin t_1, t_1) = (\cos t_2, \sin t_2, t_2)$

$\Rightarrow t_1 = t_2$

f JE NEPREKIDNO PO KOORDINATAMA

ŠTA JE INVERZ f ?

π_3 INVERZ $\quad \pi_3(x, y, z) = z$

$\pi_3 \circ f = \mathbb{1}$

JESTE ULAGANJE

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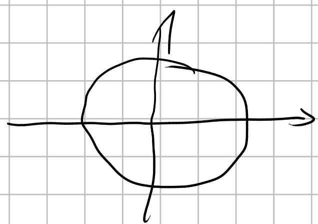
2)

$f : \mathbb{R} \to \mathbb{R}^2$



$f(t) = (\cos t, \sin t)$

DA LI JE IMERZIJA

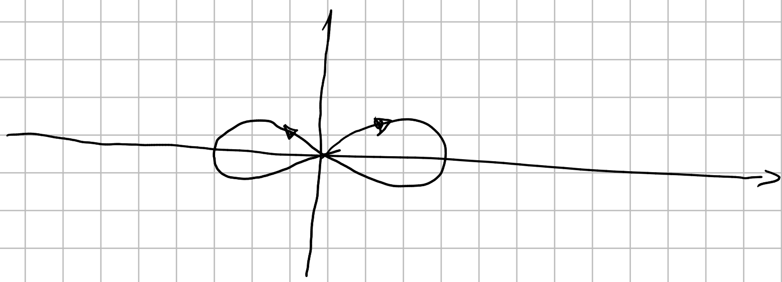
$f'(t) = (-\sin t, \cos t) \neq (0,0)$ JESTE IMERZIJA

NIJE 1-1 PA NIJE ULAGANJE.

3) Имерзија и 1-1, а није улагање

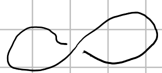
$f : \mathbb{R} \to \mathbb{R}^2$

$f(t) = \left(2\cos\left(t-\frac{\pi}{2}\right),\ \sin\left(2\left(t-\frac{\pi}{2}\right)\right)\right)$



$f'(t) \neq (0,0)$ ЈЕСТЕ ИМЕРЗИЈА

А НИЈЕ 1-1 У $(0,0)$ ПА НИЈЕ УЛАГАЊЕ



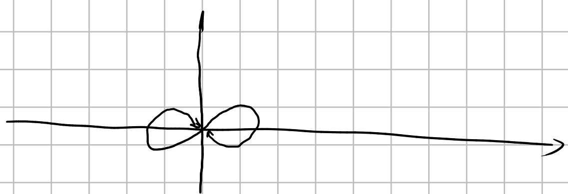
$h : \mathbb{R} \to (0, 2\pi)$

h - СТРОГО РАСТУЋА

$\lim\limits_{t\to-\infty} h(t) = 0 \qquad \lim\limits_{t\to+\infty} h(t) = 2\pi$

$h(t) = \pi + 2 \operatorname{arctg} t$

$g = f \circ h$

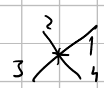


g ЈЕСТЕ ИМЕРЗИЈА

g ЈЕСТЕ 1-1

g НИЈЕ УЛАГАЊЕ ЈЕР НИЈЕ ХОМЕОМОРФИЗАМ

POSMATRAMO $(0,0)$



P.P.S. JESTE HOMEOMORFIZAM

PA JESTE I U OKOLINI $(0,0)$

$g(t_0) = (0,0)$

$U \ni t_0 \qquad V \ni (0,0)$

$f(U) = V$

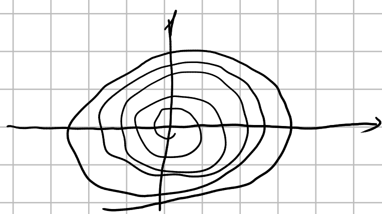
$f : U \setminus \{t_0\} \longrightarrow V \setminus \{(0,0)\}$ ↯

2 KOMPONENTE   4 KOMPONENTE

---

(5) $g : (1, +\infty) \longrightarrow \mathbb{R}^2$

$g(t) = \left(\frac{\cos t}{t}, \frac{\sin t}{t} \right)$



$g'(t) = \left(\frac{-\sin t\, t - \cos t}{t^2}, \frac{\cos t\, t - \sin t}{t^2} \right) \overset{?}{=} (0,0)$

$t = \frac{-\cos t}{\sin t} \qquad t = \frac{\sin t}{\cos t}$ ↯ $\tan t = -\operatorname{ctan} t$

NE MOŽE BITI $(0,0)$ $\Rightarrow$ JESTE IMERZIJA

$\frac{\cos t_1}{t_1} = \frac{\cos t_2}{t_2} \qquad \frac{\sin t_1}{t_1} = \frac{\sin t_2}{t_2}$

$\tan t_1 = \tan t_2$

$t_2 = t_1 + k\pi$

JESTE 1-1

g JE NEPREKIDNO PO KOORDINATAMA

ŠTA JE INVERZ OD g?

$g^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$g^{-1}(x,y) = \frac{1}{\sqrt{x^2+y^2}}$$

PROVERITI $g \cdot g^{-1} = \mathbb{1}$

g^{-1} JE NEPREKIDNO

JESTE ULAGANJE.

DOMAĆI:

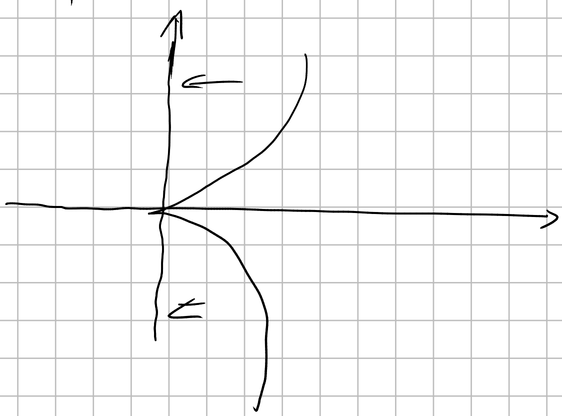
$$g : (1, +\infty) \to \mathbb{R}$$

$$g(t) = \left(\frac{t+1}{2t} \cos t \; , \; \frac{t+1}{2t} \sin t \right)$$

7

$$f : \mathbb{R} \to \mathbb{R}^2$$

$$f(t) = (t^2, t^3)$$



$f'(t) = (2t, 3t^2)$ $\qquad f'(0) = (0,0)$

NIJE IMERZIJA

$\Gamma(f)$ JE MNOGOSTRUKOST DIMENZIJE 1

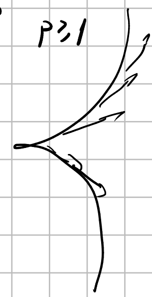
JEDNA KARTA POKRIVA

$$\pi_2(x,y) = y$$

DA LI POSTOJI IMERZIJA KLASE C^p $p \geq 1$

$$h : \mathbb{R} \to \mathbb{R}^2$$

$$h(\mathbb{R}) = \Gamma$$



P.P.S. POSTOJI

$h'(t)$ $\vec{t}$ NEPREKIDNO POLJE TANGENTI

IZ NEPREKIDNOSTI U TAČKI t_0 TDJ. $h(t_0) = (0,0)$

MORA BITI $h'(t_0) = (0,0)$

$\Rightarrow$ h NIJE IMERZIJA

Γ NIJE PODMNOGOSTRUKOST U $\mathbb{R}^2$

A JESTE 1-DIMENZIONA MNOGOSTRUKOST.

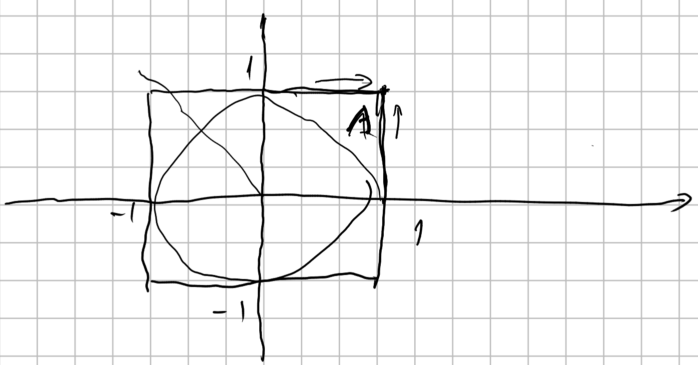
---

(8)

$$K = \{ (x, y) \mid \max\{|x|, |y|\} = 1 \}$$

K NIJE PODMNOGOSTRUKOST U $\mathbb{R}^2$

A JESTE MNOGOSTRUKOST DIMENZIJE 1.



IZ NEPREKIDNOSTI PROIZVOLJNE PARAMETRIZACIJE

$h'(t_0) = (0,0)$

$h(t_0) = A$

ZAŠTO NIJE PODMNOGOSTRUKOST?

ZBOG TEMENA

RADIMO PROJEKCIJU NA S^1.

$S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$

$K \subseteq \mathbb{C}$

$h : K \to S^1$

$h(z) = \dfrac{z}{\|z\|}$ $\quad$ h НЕПРЕКИДНО

h ЈЕСТЕ 1-1

ШТА ЈЕ h^{-1}?

ДОКАЗАТИ ХОМЕОМОРФИЗАМ

ДОМАЋИ:

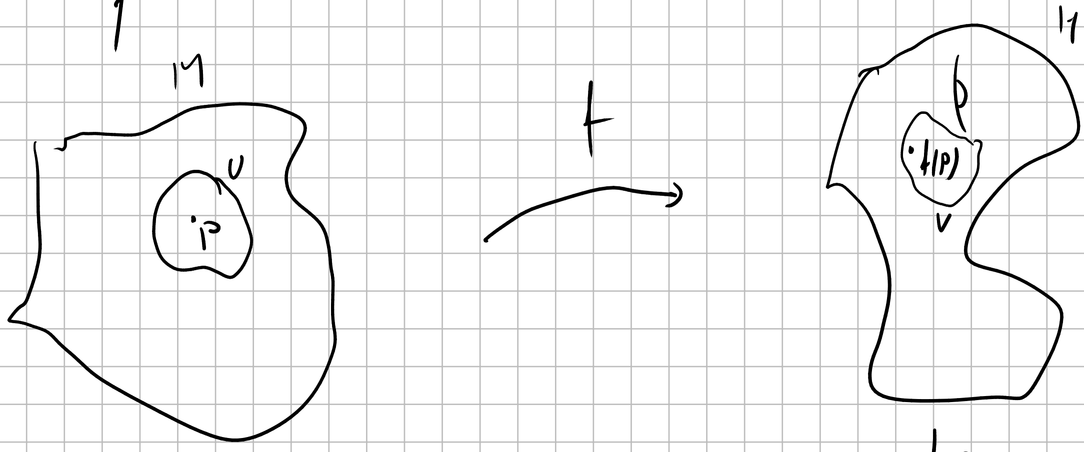
* ТАНГЕНТНИ ПРОСТОР НА $\Gamma(f)$
* СВАКА ИМЕРЗИЈА ЈЕСТЕ ЛОКАЛНО УЛАГАЊЕ

ANALIZA 2

VEŽBE

M, N - GLATKE MNOGOSTRUKOSTI

$f: M \to N$ GLATKO



$$\varphi \downarrow \qquad\qquad\qquad\qquad \downarrow \psi$$

$$\varphi(U) \xrightarrow{\psi \circ f \circ \varphi^{-1}} \psi(V)$$

RANG GLATKOG PRESLIKAVANJA JE RANG LINEARNOG PRESLIKAVANJA U p

$$L = D(\psi \circ f \circ \varphi^{-1})(\varphi(p))$$

- RANG $L = \dim M$ - IMERZIJA
- RANG $L = \dim N$ - SUBMERZIJA

ULAGANJE:

1) f JE IMERZIJA
2) f JE „1-1"
3) $f: M \to f(M)$ JE HOMEOMORFIZAM

NA OSNOVU TEOREME O RANGU

1) f JE IMERZIJA $\Rightarrow$ POSTOJE KARTE

$$(\psi \circ f \circ \varphi^{-1})(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$$

INKLUZIJA

$$m \leq n$$

2) f JE SUBMERZIJA $\Rightarrow$ POSTOJE KARTE

$$(\psi \circ f \circ \varphi^{-1})(x_1, \ldots, x_n, \ldots x_m) = (x_1, \ldots, x_n)$$

$$n \leq m$$

M - GLATKA MNOGOSTRUKOST
$N \subset M$

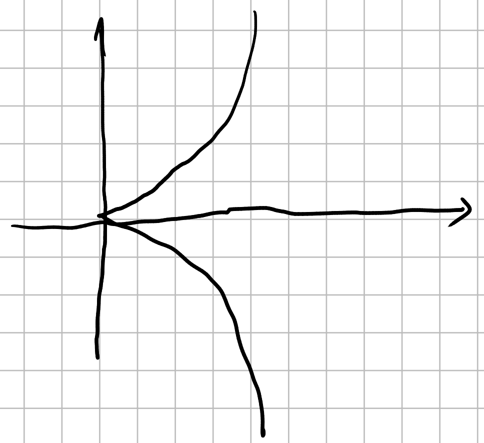
N JE PODMNOGOSTRUKOST AKO:

1) N IMA STRUKTURU GLATKE PODMNOGOSTRUKOSTI
2) INKLUZIJA $i: N \hookrightarrow M$ $i(x) = x$ JE ULAGANJE

PRIMERI:

$f(t) = (t^2, t^3)$

$f : \mathbb{R} \to \mathbb{R}^2$



JESTE MNOGOSTRUKOST DIMENZIJE 1, A NIJE PODMNOGOSTRUKOST U $\mathbb{R}^2$

(1) $f : M \to N$ SVE GLATKO

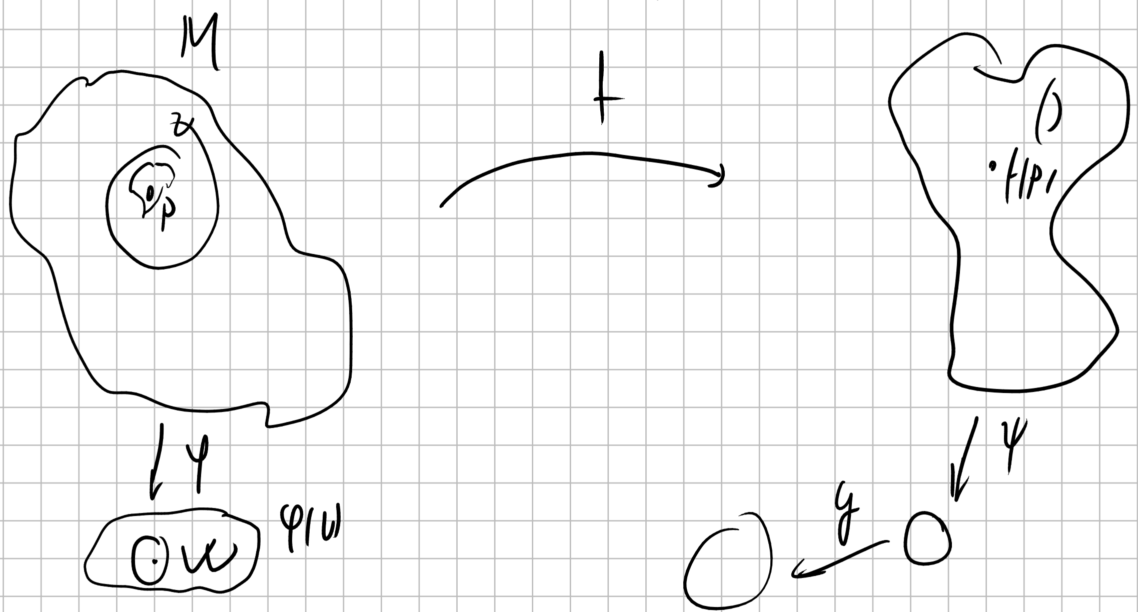
f JE IMERZIJA

$\Rightarrow$ f JE LOKALNO ULAGANJE

HEKA OSOBINA JE LOKALNA, AKO SVAKA TAČKA IMA OTVORENU OKOLINU U KOJOJ VAŽI TA OSOBINA

$\forall p \in M$ $\exists$ OTVORENA OKOLINA U $p \in U$

$f|_U$ JE ULAGANJE



$\varphi(p) = 0 \in \mathbb{R}^m \qquad \psi(f(p)) = 0 \in \mathbb{R}^n$

$\tilde{U} = \varphi(U) \qquad \tilde{V} = \psi(V)$

$\psi \circ f \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ (НЕКЕ ОКОЛИНЕ 0)

f ЈЕ ИМЕРЗИЈА $\Rightarrow$ $D(\psi \circ f \circ \varphi^{-1})$ ИМА МАКСИМАЛАН РАНГ

НА ОСНОВУ ТЕОРЕМЕ О ИМПЛИЦИТНОЈ ФУНКЦИЈИ

ПОСТОЈИ ОКОЛИНА НУЛЕ W И КООРДИНАТЕ g

$$g \circ \psi \circ f \circ \varphi^{-1}|_W = i|_W$$

$\hat{U} = \varphi^{-1}(W) \subset M \qquad \hat{\psi} = g \circ \psi$

$$f|_{\hat{U}} = \hat{\psi}^{-1} \circ i \circ \varphi$$

$\hat{\psi}, \varphi$ – ДИФЕОМОРФИЗМИ $\Rightarrow$ УЛАГАЊА

i – УЛАГАЊЕ

$\Rightarrow f|_{\hat{U}}$ – УЛАГАЊЕ

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(2) $f : M \to N$

$\dim M = \dim N$

N – ПОВЕЗАН

f – ИМЕРЗИЈА

a) f JE OTVORENO PRESLIKAVANJE (SLIKA SVAKOG OTVORENOG SKUPA JE OTVOREN)

$U \subseteq M$ PROIZVOLJAN OTVOREN SKUP

$V = f(U)$

V - OTVOREN ?

$y \in V$ - PROIZVOLJNO

$y = f(x)$ ZA NEKO $x \in U$

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "\varphi"'] & V \arrow[d, "\psi"] \\
\tilde{U} \arrow[r, "g"'] & \tilde{V}
\end{tikzcd}$$

$f = \psi^{-1} \circ g \circ \varphi$

φ, ψ^{-1} - DIFEOMORFIZMI $\Rightarrow$ OTVORENA PRESLIKAVANJA

g JE SA DOBRO ODABRANIM KARTAMA (KAO U PRETHODNOM ZADATKU) IDENTITETA PA JE OTVORENO

$\Rightarrow$ f JE OTVORENO KAO KOMPOZICIJA OTVORENIH PRESLIKAVANJA.

b) M - KOMPAKTNA $\Rightarrow$ f JE SURJEKCIJA

M - OTVOREN U SEBI $\overset{a)}{\Longrightarrow}$ $f(M)$ JE OTVOREN.

M JE KOMPAKTAN $\Rightarrow$ $f(M)$ JE KOMPAKTAN $\Rightarrow$ $f(M)$ JE ZATVOREN

$f(M)$ JE OTVOREN I ZATVOREN

$f(M) \subseteq N$ - POVEZAN

$$f(M) = N$$

v) M - KOMPAKTNA

f - INJEKTIVNO

$\Rightarrow$ f JE VLAGANJE

b) $\Rightarrow$ f JE SURJEKTIVNO $\Rightarrow$ JESTE BIJEKCIJA)

a) $\Rightarrow$ f JE OTVORENO $\Rightarrow$ f^{-1} NEPREKIDNO

$\Rightarrow$ f JE HOMEOMORFIZAM

g) NE POSTOJI IMERZIJA KOMPAKTNE MNO-

GOSTRUKOSTI DIMENZIJE n U $\mathbb{R}^n$

b) $\Rightarrow$ $f(M)$ JE CEO PROSTOR

M KOMPAKTNA $\Rightarrow$ $\mathbb{R}^n$ KOMPAKTAN

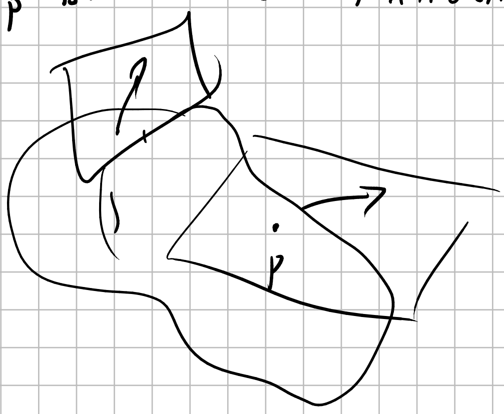
↯

(VITNIJEVA TEOREMA)

TANGENTNI PROSTOR (PONOVO)

1) u $\mathbb{R}^n$

$T_p\mathbb{R}^n$ je tangentni prostor u tački $p \in M$



$T_p\mathbb{R}^n \cong \mathbb{R}^n$ (translacija za p)

$C^\infty(p)$ - skup glatkih funkcija u čijem je domenu tačka p, uz uslov identifikacije funkcija koje se poklapaju u nekoj okolini

$f : U \to \mathbb{R}$ $\qquad g : V \to \mathbb{R}$ $\qquad p \in U \cap V$

$$f \sim g \iff f|_{U\cap V} =_{\mathcal{F}_p} g|_{U \cap V}$$

$\mathcal{F}_p$ - filter okolina tačke p

$C^\infty(p)$ je vektorski prostor nad $\mathbb{R}$.

$X_p \in T_pM$ definiše linearno preslikavanje

$$C^\infty(p) \to \mathbb{R} \qquad f : df(p)X_p$$

Izvod u pravcu vektora

(PRESLIKAVANJE OZNAČAVAMO ISTOM OZNAKOM)

X_p KAO PRESLIKAVANJE IMA SVOJSTVA

1) LINEARNOST

$$X_p(\alpha f + \beta g) = \alpha X_p(f) + \beta X_p(g)$$

2) LAJBNICOVO PRAVILO

$$X_p(f \cdot g) = g(p) X_p(f) + f(p) X_p(g)$$

OVO PRENOSIMO NA MNOGOSTRUKOSTI

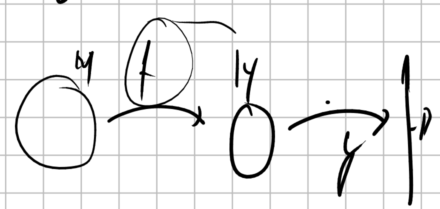
$T_p M$ MOŽEMO DEFINISATI KAO SKUP SVIH LINEARNIH PRESLIKAVANJA

$$X_p : C^\infty(p) \to \mathbb{R}$$

KOJA ZADOVOLJAVAJU LAJBNICOVO PRAVILO.

$f : M \to N$ SVE GLATKO

$p \in M$



$$f^* : C^\infty(f(p)) \to C^\infty(p)$$

$$f^*(g) = g \circ f$$

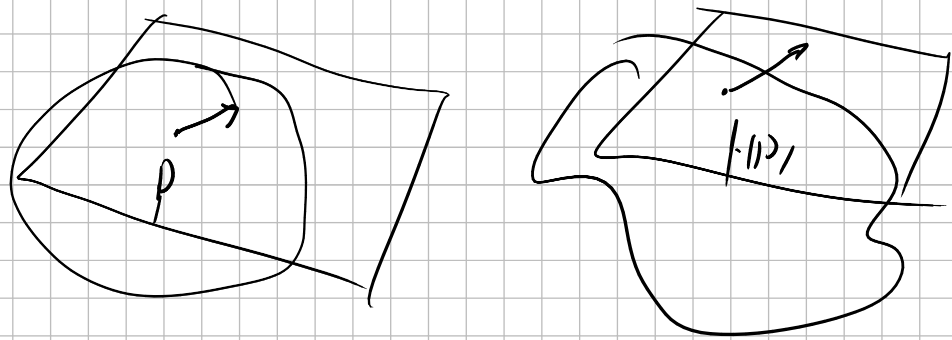
OVO JE HOMOMORFIZAM ALGEBRI

DUALNI HOMOMORFIZAM VEKTORSKIH PROSTORA

$$f_* : T_pM \to T_{f(p)}N$$

$$f_*(X_p)(g) = X_p(f^*(g)) \qquad \forall g \in C^\infty(f(p))$$

NAZIVA SE IZVODOM GLATKOG PRESLIKAVANJA



OZNAKE $f_*(p)$, $Df(p)$, $f'(p)$, ...

(U,φ) - KARTA U OKOLINI p

T_pM JE GENERISAN VEKTORIMA

$$\varphi^{-1}_*(p)\left(\frac{\partial}{\partial x_1}\Big|_{\varphi(p)}\right), \dots, \varphi^{-1}_*(p)\left(\frac{\partial}{\partial x_n}\Big|_{\varphi(p)}\right)$$

(1) $f : M \to N$ SVE GLATKO

$y_0 \in N$

f_* KONSTANTNOG RANGA U OKOLINI $f^{-1}(y_0)$

$\Rightarrow$ $f^{-1}(y_0)$ JE MNOGOSTRUKOST DIMENZIJE

$\dim M$ - RANG f_*

ILI PRAZAN SKUP

$\dim M = m \qquad \text{RANG } f_* = k$

$x_0 \in f^{-1}(y_0)$

NEKA SU DATE KARTE

$(U, \varphi) \quad x_0 \in U \qquad (V, \psi) \quad y_0 \in V$

NA OSNOVU TEOREME O RANGU KARTE SE MOGU IZABRATI TDJ:

$$(\psi \circ f \circ \varphi^{-1})(x_1, \dots, x_n) = (x_1, \dots, x_k, 0, \dots, 0)$$

$\varphi(x_0) = 0 \qquad \psi(y_0) = 0$

U y_0 SE SLIKAJU TAČKE

$x_1 = x_2 = \dots = x_k = 0$

$$\varphi(U \cap f^{-1}(y_0)) = \varphi(U) \cap \{0\} \times \mathbb{R}^{m-k}$$

$\Rightarrow$ $f^{-1}(y_0)$ JE PODMNOGOSTRUKOST DIMENZIJE $n-k$.

$p \in M$

p JE SINGULARNA TAČKA AKO IZVOD $f_*(p)$ NIJE SURJEKTIVAN

AKO JE p SINGULARNA TAČKA $f(p)$ JE SINGULARNA VREDNOST.

$q \in N$ JE REGULARNA VREDNOST AKO NIJE SINGULARNA.

($q \notin f(M)$ ILI $\forall p \in f^{-1}(q)$ p NIJE SINGULARNA TAČKA)

POSLEDICA (PRETHODNOG ZADATKA)

q REGULARNA VREDNOST

$\Rightarrow$ $f^{-1}(q)$ JE MNOGOSTRUKOST

SARDOVA TEOREMA:

MERA SKUPA SVIH KRITIČNIH VREDNOSTI JE NULA.

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(2) $f: M \to N$ GLATKO

K - SKUP REGULARNIH TAČAKA

?

$\Rightarrow$ K JE OTVOREN SKUP

$x \in K$ REGULARNA $\Leftrightarrow$ $f_*(x): T_x M \to T_{f(x)} N$ JE „NA"

$\dim M = m$ $\dim N = n$

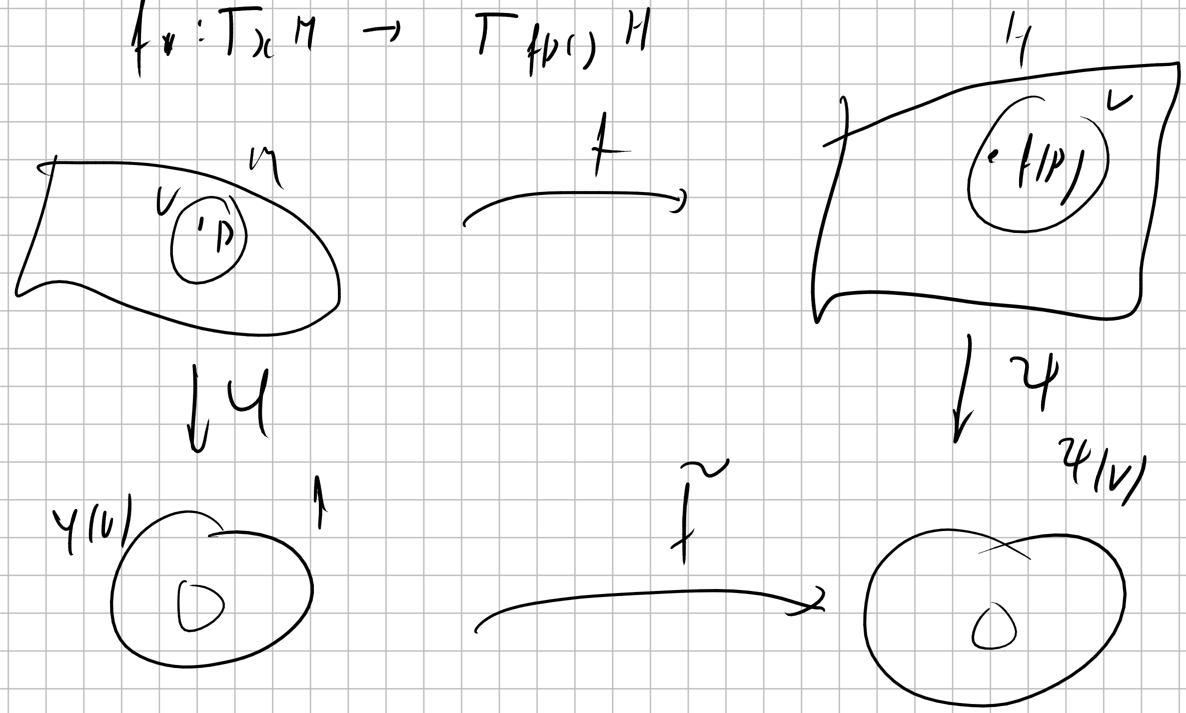
RANG $f_* = n$

$$\dim \operatorname{Ker} f_*(x) + \underbrace{\dim \operatorname{Im} f_*(x)}_{n} = \underbrace{\dim T_x M}_{m}$$

1) $n > m \Rightarrow K = \emptyset$

2) $n = m$

$f_*: T_x M \to T_{f(x)} N$



$D\tilde{f}\,|_{\varphi(x)}$ JE INVERTIBILNO

POSTOJE OKOLINE I NA TIM OKOLINAMA INVERTIBILNO

$\tilde{U} \subseteq \varphi(U)$ $\qquad$ $\tilde{V} \subseteq \psi(V)$

$\varphi^{-1}(\tilde{U})$ TU JE INVERTIBILNO

3) $n < m$ NA ISTI NAČIN SE DOKAZUJE

DIFERENCIJALNE FORME

- PRVO U $\mathbb{R}^3$

$P, Q, R : \mathbb{R}^3 \to \mathbb{R}$ GLATKE FUNKCIJE

IZRAZ

$$P\,dx + Q\,dy + R\,dz$$

NAZIVA SE DIFERENCIJALNA 1-FORMA

1-FORME SU LINEARNI FUNKCIONALI $\in (\mathbb{R}^3)^*$

dx, dy, dz BAZA PROSTORA

(DUALNA BAZI i, j, k U $\mathbb{R}^3$)

α, β 1-FORME

SPOLJAŠNJI PROIZVOD

1) $\alpha \wedge \beta$ KLIN (WEDGE) PROIZVOD

$$\alpha \wedge \beta (\xi, \eta) = \begin{vmatrix} \alpha(\xi) & \beta(\xi) \\ \alpha(\eta) & \beta(\eta) \end{vmatrix}$$

BILINEARNO I ANTISIMETRIČNO

$$\alpha \wedge \beta = - \beta \wedge \alpha$$

2 - FORME

$$P\,dx \wedge dy + Q\,dy \wedge dz + R\,dz \wedge dx$$

3 - FORME

$f \; \underline{dx} \wedge \underline{dy} \wedge \underline{dz}$

$$dx \wedge dx = -dx \wedge dx$$
$$\Downarrow$$
$$dx \wedge dx = 0$$

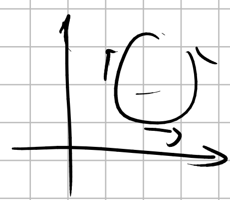
$$dx \wedge dy \wedge dz (\xi, \eta, \nu) = \begin{vmatrix} dx(\xi) & dy(\xi) & dz(\xi) \\ dx(\eta) & dy(\eta) & dz(\eta) \\ dx(\nu) & dy(\nu) & dz(\nu) \end{vmatrix}$$

V - OTVOREN SKUP

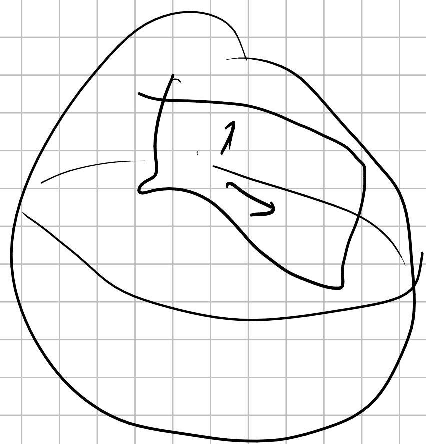
$\Omega^k(V)$ - K - DIFERENCIJALNE FORME

$\downarrow$

VEKTORSKI PROSTOR NAD $\mathbb{R}$



$P = \iint dx\, dy$ $\qquad$ $P = \iint dx \wedge dy$



$U \subseteq \mathbb{R}^3 \qquad \Omega^k(U) = \{0\} \qquad k > 3$

$\Omega^0(V)$ - PROSTOR DIFERENCIJABILNIH FUNKCIJA

$$d : \Omega^k(V) \to \Omega^{k+1}(V)$$

d - SPOLJAŠNJI DIFERENCIJAL

a) d JE LINEARNO PRESLIKAVANJE VEKTORSKIH PROSTORA NAD $\mathbb{R}$

b) ZA $k=0$ d JE DIFERENCIJAL FUNKCIJE

v) LAJBNICOVO PRAVILO

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$$

g) $d(dx) = 0 \quad d(dy) = 0 \quad d(dz) = 0$

$$\eta = y\,dx + zx\,dy \qquad \text{1-Forma}$$

$$d\eta = \frac{\partial \eta}{\partial x}dx + \frac{\partial \eta}{\partial y}dy + \frac{\partial \eta}{\partial z}dz$$

$$d\eta = z\,\underline{dx \wedge dy} + \underline{dy \wedge dx} + x\,dz \wedge dy$$

$$= (z-1)\,dx \wedge dy + x\,dz \wedge dy$$

$$\omega = \log z\, x^2 y \, dx \wedge dy + y z e^{x} \, dx \wedge dz$$

$$d\omega = 2x \log z \, y \, dx \wedge dx \wedge \cancelto{0}{dy} + y z e^{x} \, dx \wedge \cancelto{0}{dx \wedge dz}$$

$$+ \log z \, x^2 \, \cancelto{0}{dy \wedge dx \wedge dy} + z e^{x} \, dy \wedge dx \wedge dz$$

$$+ \frac{x^2 y}{z} \, dz \wedge dx \wedge dy + y e^{x} \, dz \wedge \cancelto{0}{dx} \wedge dz$$

$$= \left(-z e^{x} + \frac{x^2 y}{z} \right) dx \wedge dy \wedge dz$$