ANALIZA 2

VEŽBE

FUNKCIJE VIŠE PROMENLJIVIH

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}$$

DA LI JE f NEPREKIDNO ZAVISI OD METRIKE

$$f: (\mathbb{R}^n, e) \longrightarrow (\mathbb{R}, |.|)$$

EUKLIDSKA METRIKA

f JE NEPREKIDNO AKO JE INVERZNA SLIKA SVAKOG OTVORENOG SKUPA OTVOREN.

KORISTIĆEMO:

f JE NEPREKIDNA U TAČKI $x^\circ = (x_1^\circ, \ldots, x_n^\circ)$ AKO $\forall \varepsilon \ \exists \delta \ \forall x \in \mathbb{R}^n$

$| \quad |$ $$\|x - x^\circ\|_{\mathbb{R}^n} < \delta \Rightarrow |f(x) - f(x^\circ)| < \varepsilon$$

$\| \quad \|$

f JE NEPREKIDNA NA SKUPU $A \subset \mathbb{R}^n$ AKO JE NEPREKIDNA U SVAKOJ TAČKI TOG SKUPA.

f JE NEPREKIDNA U $(x_1^\circ, \ldots, x_n^\circ)$ AKO I SAMO AKO ZA SVAKI NIZ

$$(x_1^k, \dots, x_n^k) \xrightarrow{k \to \infty} (x_1^0, \dots, x_n^0)$$

VAŽI

$$\lim_{k \to \infty} f(x_1^k, \dots, x_n^k) = f(x_1^0, \dots, x_n^0)$$

PONOVLJENI LIMESI (PRIČAMO U 2 DIMENZIJE)

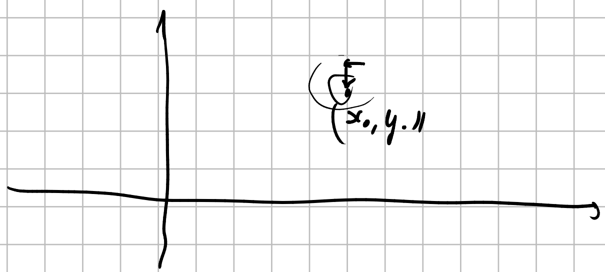
x - FIKSIRANO $\lim_{y \to y_0} f(x, y) = \varphi(x)$ AKO LIMES POSTOJI

y - FIKSIRANO $\lim_{x \to x_0} f(x, y) = \psi(y)$

$$\lim_{x \to x_0} \varphi(x) \overset{?}{\neq} \lim_{y \to y_0} \psi(y)$$

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = \lim_{(x,y) \to (x_0, y_0)} f(x, y)$$

RAZLIČIT ZAPIS



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(1) $$f(x, y) = \frac{x - y}{x + y} \qquad x \neq -y$$

GRANIČNE VREDNOSTI U TAČKI (0,0)

x - FIKSIRANO $$\lim_{y \to 0} \frac{x - y}{x + y} = \begin{cases} 1, & x \neq 0 \\ -1, & x = 0 \end{cases} = \varphi(x)$$

y - FIKSIRANO $$\lim_{x \to 0} \frac{x - y}{x + y} = \begin{cases} 1, & y = 0 \\ -1, & y \neq 0 \end{cases} = \psi(y)$$

$$\lim_{x \to 0} \varphi(x) = 1 \qquad \lim_{x \to 0} \lim_{y \to 0} f(x,y) = 1$$

$$\lim_{y \to 0} \psi(y) = -1 \qquad \lim_{y \to 0} \lim_{x \to 0} f(x,y) = -1$$

PONOVLJENI LIMESI SU RAZLIČITI (POSTOJE OBA)

I) NIZ $\quad (x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right) \xrightarrow[n \to \infty]{} (0,0)$

$$f(x_n, y_n) = \frac{\frac{1}{n} - \frac{1}{n}}{\frac{1}{n} + \frac{1}{n}} = 0 \xrightarrow[n \to \infty]{} \boxed{0}$$

$\neq$

II) NIZ $\quad (a_n, b_n) = \left(\frac{1}{n}, \frac{1}{2n}\right) \xrightarrow[n \to \infty]{} (0,0)$

$$f(a_n, b_n) = \frac{\frac{1}{n} - \frac{1}{2n}}{\frac{1}{n} + \frac{1}{2n}} = \frac{\frac{1}{\cancel{2n}}}{\frac{3}{\cancel{2n}}} \xrightarrow[n \to \infty]{} \boxed{\frac{1}{3}}$$

$\Rightarrow$ NE POSTOJI $\lim\limits_{\substack{x \to 0 \\ y \to 0}} f(x,y)$

2

$$f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$$

$$f(x,y) = \frac{xy}{x^2 + y^2}$$

$$\lim_{x \to 0} f(x,y) = 0$$

$$\lim_{y \to 0} f(x,y) = 0$$

$$\underbrace{\lim_{x \to 0} \lim_{y \to 0} f(x,y) = \lim_{y \to 0} \lim_{x \to 0} f(x,y)}$$

I) NIZ

$$(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right) \xrightarrow[n \to \infty]{} (0,0)$$

$$f(x_n, y_n) = \frac{\frac{1}{n}\cdot\frac{1}{n}}{\frac{1}{n^2}+\frac{1}{n^2}} = \frac{\frac{1}{\cancel{n^2}}}{\frac{2}{\cancel{n^2}}} = \frac{1}{2} \longrightarrow \frac{1}{2}$$

II) NIZ

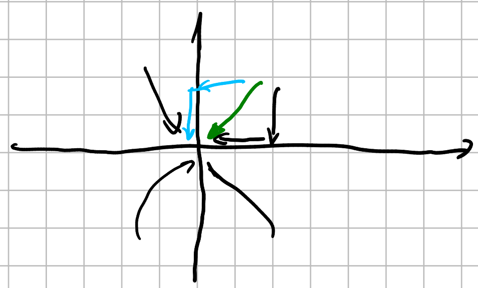
$$(a_n, b_n) = \left(0, \frac{1}{n}\right) \xrightarrow[n\to\infty]{} (0,0)$$

$$f(a_n, b_n) = \frac{0\cdot\frac{1}{n}}{0^2+\frac{1}{n^2}} = 0 \xrightarrow[n\to\infty]{} 0$$

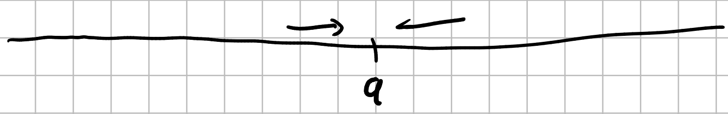
$$\frac{1}{2} \neq 0$$

$\Rightarrow$ NE POSTOJI $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$

AKO PONOVLJENI LIMESI POSTOJE I JEDNAKI SU TO NE OBEZBEĐUJE POSTOJANJE LIMESA.



U $\mathbb{R}$ SMO IMALI LEVI I DESNI LIMES



$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$$

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③

$$f: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$$

$$f(x,y) = \frac{x^2 y}{x^2+y^2}$$

DA LI SE f MOŽE PRODUŽITI DO NEPREKIDNE FUNKCIJE NA $\mathbb{R}^2$?

$f \in C(\mathbb{R}^2 \setminus \{(0,0)\})$ као композиција непрекидних функција

Ако постоји $\lim\limits_{\substack{x \to 0 \\ y \to 0}} f(x,y) = A$

Тада ће непрекидна функција бити:

$$\tilde{f}(x,y) = \begin{cases} f(x,y), & (x,y) \neq (0,0) \\ A, & (x,y) = (0,0) \end{cases} \qquad \tilde{f} : \mathbb{R}^2 \to \mathbb{R}$$

$$f(x,y) = \frac{x^2 y}{x^2 + y^2}$$

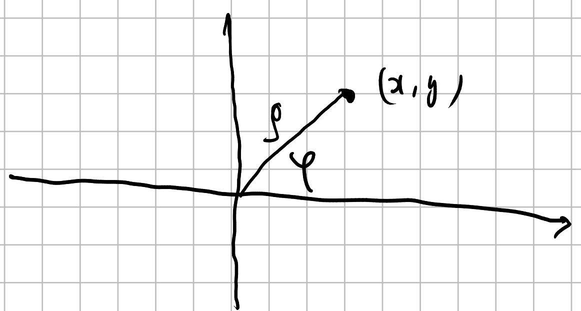
I начин

$$0 \leq |f(x,y)| = \left| \frac{x^2 y}{x^2+y^2} \right| = |y| \overbrace{\frac{x^2}{x^2+y^2}}^{\leq 1} \leq |y| \xrightarrow[\substack{x \to 0 \\ y \to 0}]{} 0$$

0

II начин

Прелазак на поларне координате



$x = \rho \cos\varphi \qquad y = \rho \sin\varphi$

$(x,y) \to (0,0) \iff \rho \to 0$

$$f(x,y) = \frac{x^2 y}{x^2+y^2} \rightsquigarrow f(\rho,\varphi) = \frac{\cancel{\rho^2} \cos^2\varphi \, \rho \sin\varphi}{\cancel{\rho^2}}$$

$$= \rho \cos^2\varphi \sin\varphi$$

$$\lim_{\rho \to 0} f(\rho, \varphi) = \lim_{\rho \to 0} \underbrace{\rho}_{\to 0} \underbrace{\cos^2\varphi \sin\varphi}_{\leq 1 \text{ OGRANIČENO}} = 0$$

NA DVA NAČINA SMO POKAZALI DA LIMES POSTOJI

$$\tilde{f}(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

$$\tilde{f} \in C(\mathbb{R}^2)$$

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(4)
$$\lim_{\substack{x \to +\infty \\ y \to +\infty}} \frac{x+y}{x^2 - xy + y^2}$$

I NAČIN

$x^2 - xy + y^2$

$$0 \leq (x-y)^2 = x^2 - 2xy + y^2 \quad / + xy$$
$$xy \leq x^2 - xy + y^2$$
$$\frac{1}{xy} \geq \frac{1}{x^2 - xy + y^2}$$

$$0 \leq \frac{x+y}{x^2-xy+y^2} \leq \frac{x+y}{xy} = \underbrace{\frac{1}{y}}_{\to 0} + \underbrace{\frac{1}{x}}_{\to 0} \xrightarrow[y \to +\infty]{x \to +\infty} 0$$

II NAČIN

POLARNE KOORDINATE

$x = \rho \cos\varphi \quad y = \rho \sin\varphi$

$$\frac{x+y}{x^2-xy+y^2} = \frac{\cancel{\rho}\cos\varphi + \cancel{\rho}\sin\varphi}{\rho^{\cancel{2}}\cos^2\varphi - \rho\cos\varphi\cancel{\rho}\sin\varphi + \rho^{\cancel{2}}\sin^2\varphi}$$

$$= \frac{\cos\varphi + \sin\varphi}{\rho\,(1 - \cos\varphi\sin\varphi)} = \frac{1}{\rho}\cdot\frac{\cos\varphi + \sin\varphi}{1 - \cos\varphi\sin\varphi} \xrightarrow[\rho\to\infty]{} 0$$

$\frac{1}{\rho} \xrightarrow{\rho\to\infty} 0$

Да ли је ограничено $\frac{\cos\varphi + \sin\varphi}{1 - \cos\varphi\sin\varphi}$, $\varphi \in [0, 2\pi]$

$$1 - \cos\varphi\sin\varphi = 0$$

$$\cos\varphi\sin\varphi = 1 \quad ?$$

$2\cos\varphi\sin\varphi = 2$

$\sin 2\varphi = 2$ ↯

не може да буде

не могу и $\sin\varphi = 1$ и $\cos\varphi = 1$

непрекидно па и ограничено на компакту $[0, 2\pi]$

постоје поновљени лимеси и једнаки су

постоји лимес

(5)

$$f(x,y) = x + y\sin\frac{1}{x}$$

$$f : (\mathbb{R}\setminus\{0\}) \times \mathbb{R} \to \mathbb{R}$$

$(0,0)$

$$0 \le |f(x,y)| = \left|x + y\sin\frac{1}{x}\right| \le$$

$$\le |x| + |y|\underbrace{\left|\sin\frac{1}{x}\right|}_{\le 1} \le |x| + |y| \xrightarrow[\substack{x\to 0\\ y\to 0}]{} 0$$

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0$$

Поновљени лимеси не постоје

$$x_n = \frac{1}{2n\pi} \qquad y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$$

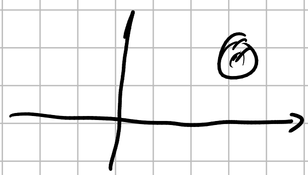
y - фиксирано $\quad f(y, x_n) = x_n + y\sin\frac{1}{x_n}$

$$= \frac{1}{2n\pi} + y \underbrace{\sin 2n\pi}_{=0} \xrightarrow[n\to\infty]{} 0$$

$$f(y_n, y) = y_n + y \sin \frac{1}{y_n}$$

$$= \frac{1}{2n\pi + \frac{\pi}{2}} + y \sin\left(2n\pi + \frac{\pi}{2}\right) = y$$

НЕ ПОСТОЈИ $\lim\limits_{x\to 0} f(x,y)$



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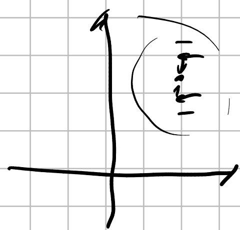
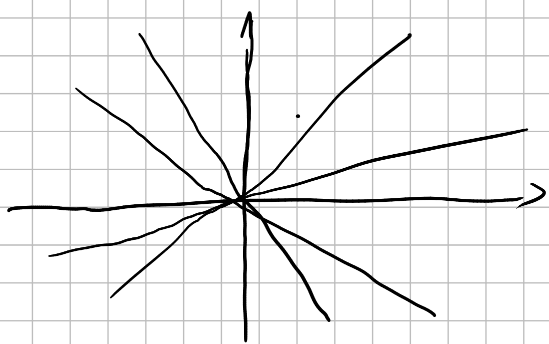
(6)



$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

ДОКАЗАТИ ДА ЈЕ f НЕПРЕКИДНА КАО ФУНКЦИЈА ЈЕДНЕ ПРОМЕНЉИВЕ ДУЖ СВАКОГ ПРАВЦА КОЈИ ПРОЛАЗИ КРОЗ $(0,0)$, А НИЈЕ НЕПРЕКИДНА КАО ФУНКЦИЈА ДВЕ ПРОМЕНЉИВЕ У $(0,0)$



KOJI PRAVCI PROLAZE KROZ $(0,0)$

PRAVE $y = kx$ i PRAVA $x = 0$

$y = kx$

$$f(x, kx) = \begin{cases} \dfrac{\cancel{x^2}\, kx}{x^{4} + k^2 \cancel{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} \dfrac{kx}{x^2 + k^2} \\ 0, & x = 0 \end{cases} = \varphi(x)$$

$$\lim_{x\to 0} \varphi(x) = \lim_{x\to 0} \frac{kx}{x^2+y^2} = 0 = \varphi(0)$$

φ JE NEPREKIDNA FUNKCIJA

PRAVA $x = 0$

$$f(0,y) = \begin{cases} \frac{0^2 y}{0^4+y^2}, & y \neq 0 \\ 0, & y = 0 \end{cases} = 0 = \psi(y)$$

$\psi(y)$ JE NEPREKIDNA

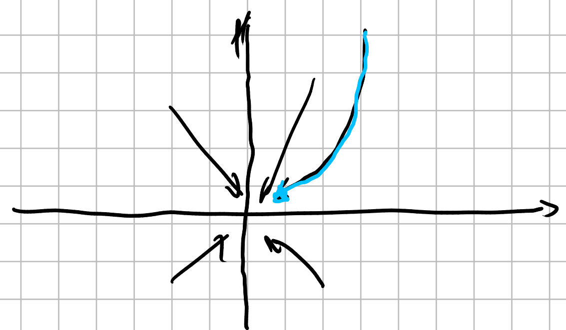
UOČIMO DVA NIZA

I NIZ $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n^2}\right) \xrightarrow[n\to\infty]{} (0,0)$

$$f(x_n, y_n) = \frac{\frac{1}{n^2}\cdot\frac{1}{n^2}}{\frac{1}{n^4}+\frac{1}{n^4}} = \frac{\cancel{\frac{1}{n^4}}}{\cancel{\frac{2}{n^4}}} = \frac{1}{2} \xrightarrow[n\to\infty]{} \frac{1}{2}$$

II NIZ $(a_n, b_n) = \left(\frac{1}{n}, \frac{1}{n}\right) \xrightarrow[n\to\infty]{} (0,0)$

$$f(a_n, b_n) = \frac{\frac{1}{n^2}\cdot\frac{1}{n}}{\frac{1}{n^4}+\frac{1}{n^2}} = \frac{\overbrace{\frac{1}{n}}^{\to 0}}{\underbrace{\frac{1}{n^2}}_{\to 0}+1} \xrightarrow[n\to\infty]{} 0$$



$f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = \arctan(x+y)$

DOKAZATI DA JE f RAVNOMERNO NEPREKIDNA

LAGRANŽOVA TEOREMA

$$a>b \qquad \frac{\arctan a - \arctan b}{a-b} = \arctan'(\xi) = \frac{1}{1+\xi^2} \qquad \xi \in (b,a)$$

$$\arctan a - \arctan b = \frac{1}{1+\xi^2}(a-b) \le a-b$$

$$|f(x_1,y_1) - f(x_2,y_2)| =$$

$$= |\arctan(x_1+y_1) - \arctan(x_2+y_2)| \le$$

$$\le |x_1+y_1-(x_2+y_2)| = |x_1-x_2+y_1-y_2|$$

$$\le |x_1-x_2| + |y_1-y_2|$$

$$\le \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} + \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

$$= 2\, d((x_1,y_1),(x_2,y_2))$$

$\Rightarrow$ f JE LIPŠICOVA SA LIPŠICOVOM KONSTANTOM $L \le 2$

$\Rightarrow$ f JE RAVNOMERNO NEPREKIDNA.

$$a \neq 0 \qquad \lim_{\substack{x\to 0\\ y\to a}} \frac{\sin xy}{x} = \lim_{\substack{x\to 0\\ y\to a}} \overset{\to 1}{\frac{\sin xy}{xy}} \cdot \overset{\to a}{y} \xrightarrow[\substack{x\to 0\\ y\to a}]{} a$$

$$a=0 \qquad \lim_{\substack{x\to 0\\ y\to 0}} \frac{\sin xy}{x} = \lim_{\rho\to 0} \overset{\to 1}{\frac{\sin \rho^2\cos\varphi\sin\varphi}{\rho^2\cos\varphi\sin\varphi}} \cdot \overset{\to 0}{\rho} \cdot \overset{\le 1}{\sin\varphi} \xrightarrow[\rho\to 0]{} = a$$

$$\lim_{\substack{x\to\infty \\ y\to\infty}} (x+y)\, e^{-x^2-y^2}$$

I NAČIN

$$(x+y)\, e^{-x^2-y^2} = \underbrace{x\, e^{-x^2}}_{\to 0}\, \underbrace{e^{-y^2}}_{\to 0} + \underbrace{y\, e^{-y^2}}_{\to 0}\, \underbrace{e^{-x^2}}_{\to 0} \xrightarrow[\substack{x\to\infty \\ y\to\infty}]{} 0$$

$$x \to \infty \Rightarrow e^{-x^2} \longrightarrow 0$$

$$x\, e^{-x^2} \xrightarrow[x\to\infty]{} 0$$

$$\frac{x}{e^{x^2}}$$

$$(x+y)\, e^{-x^2-y^2} = (\rho\cos\varphi + \rho\sin\varphi)\, e^{-\rho^2}$$

$$= \underbrace{\rho\, e^{-\rho^2}}_{\to 0} \underbrace{(\cos\varphi + \sin\varphi)}_{\leq 7} \xrightarrow[\rho\to\infty]{} 0$$

DIFERENCIJABILNOST FUNKCIJA VIŠE PROMENLJIVIH

ANALIZA 1

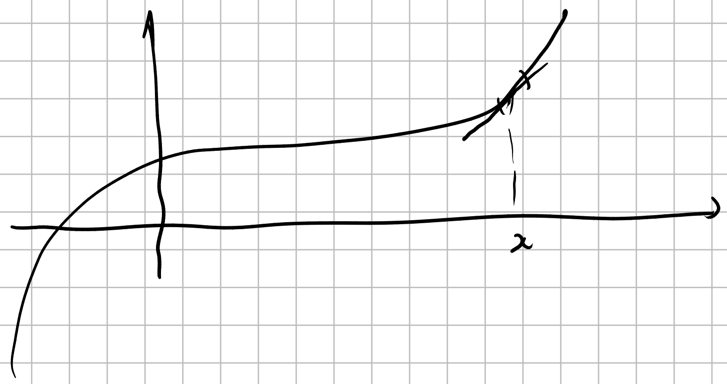
IZVOD FUNKCIJE JE:

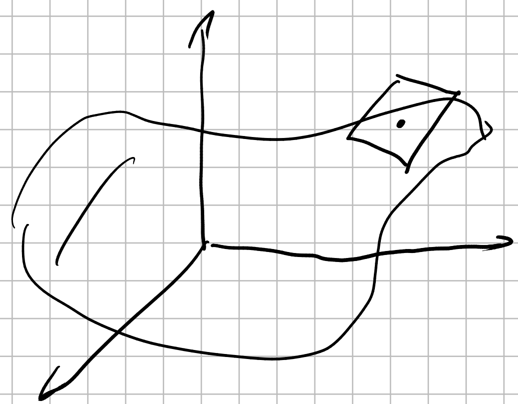
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$ AKO POSTOJI

REKLI SMO DA JE TAKVA FUNKCIJA DIFERENCIJABILNA

$$\Leftrightarrow \quad f(x+h) = f(x) + f'(x)\, h + o(|h|), \quad h \to 0$$

ZA DEFINICIJU DIFERENCIJABILNOSTI





DEFINICIJA:

$$f: \mathbb{R}^n \to \mathbb{R} \qquad a = (a_1, \ldots, a_n) \in \mathbb{R}^n$$

— SAMO NA i-TOM MESTU $+h$

$$\frac{\partial f}{\partial x_i}(a) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_i + h, \ldots, a_n) - f(a)}{h}$$

DEFINICIJA:

$f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$ A - OTVOREN

$a \in A$ h DOVOLJNO MALO $a + h \in A$

f JE DIFERENCIJABILNA U a AKO JE

$$f(a+h) = f(a) + L \cdot h + o(\|h\|), \quad h \to 0$$

$L = (L_1, \ldots, L_n)$ — LINEARNI OPERATOR

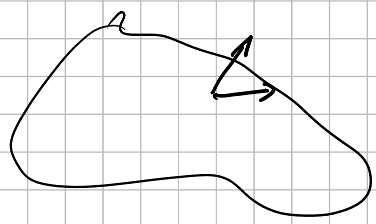
STAV:

AKO JE f DIFERENCIJABILNA U TAČKI a

a) f ЈЕ НЕПРЕКИДНА У a

b) f ИМА ПАРЦИЈАЛНЕ ИЗВОДЕ У a

$$L = df(a) = \left(\frac{\partial f}{\partial x_1}(a), \ldots, \frac{\partial f}{\partial x_n}(a) \right)$$



PARCIJALNI IZVODI. DIFERENCIJAL

$A \subseteq \mathbb{R}^n \qquad A \in \tau_{\mathbb{R}^n}$

$a \in A \qquad a = (a_1, \ldots, a_n)$

$$f : A \to \mathbb{R}$$

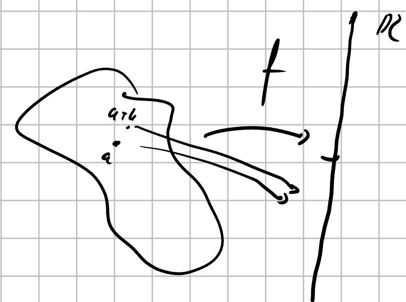
Parcijalni izvod funkcije f po promenljivoj x_i u tački a je

$$\frac{\partial f}{\partial x_i}(a) = \lim_{h_i \to 0} \frac{f(a_1, \ldots, a_i + h_i, \ldots, a_n) - f(a)}{h_i}$$

$h = (h_1, \ldots, h_n)$

$a + h \in A$

Priraštaj funkcije



$$\Delta f(a, h) = f(a+h) - f(a)$$

f je diferencijabilna u tački a ako važi

$$f(a+h) - f(a) = \Delta f(a, h) = L \cdot h + o(h)$$

$o(h) \qquad \lim_{h \to 0} \frac{o(h)}{\|h\|} = 0$

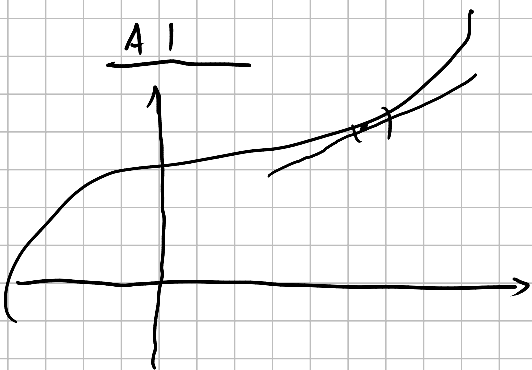
$L = (L_1, \ldots, L_n)$ – konstante

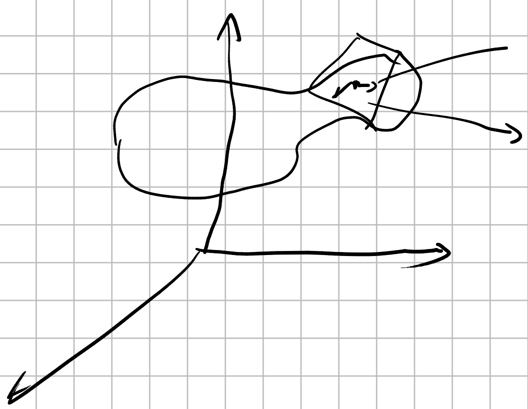
$$Lh = l_1 h_1 + \dots + l_n h_n$$

$$h \mapsto Lh = l_1 h_1 + \dots + l_n h_n$$

DIFERENCIJAL FUNKCIJE f U TAČKI a

$$df(a)(h) = L_1 h_1 + \dots + L_n h_n$$





Teorema:

$$f: A \to \mathbb{R}$$

f diferencijabilna u $a \in A$

Tada važi:

1) f je neprekidna u a
2) Postoje parcijalni izvodi i važi

$$df(a)(h) = \frac{\partial f}{\partial x_1}(a) h_1 + \dots + \frac{\partial f}{\partial x_n}(a) h_n$$

$$L_i = \frac{\partial f}{\partial x_i}(a)$$

Teorema:

$$f: A \to \mathbb{R} \qquad A \in \tau_{\mathbb{R}^n}$$

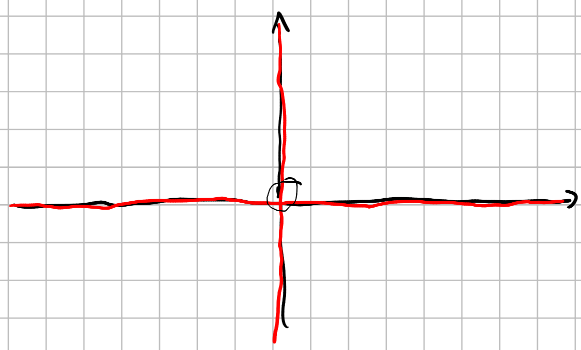
$a \in A$

$\underline{U}$ - neka okolina tačke a

Ako postoje parcijalni izvodi u okolini U i neprekidni su u tački a tada je f DIFERENCIJABILNA U TAČKI a.

PRIMER:

$$f(x,y)=\begin{cases}1, & xy\neq 0\\ 0, & xy=0\end{cases}$$



POSTOJE PARCIJALNI IZVODI U $(0,0)$

$$\frac{\partial f}{\partial x}(0,y)=0 \qquad \frac{\partial f}{\partial y}(x,0)=0$$

DEFINICIJA:

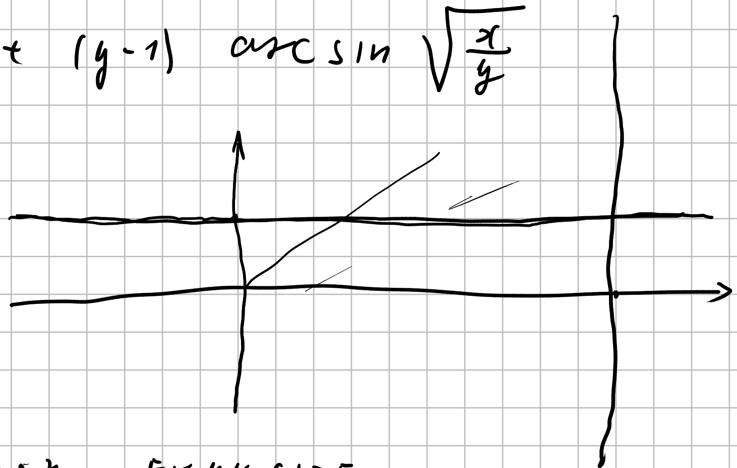
TOTALNI DIFERENCIJAL FUNKCIJE f U TAČKI a JE:

$$df(a)=\frac{\partial f}{\partial x_1}(a)\,dx_1+\dots+\frac{\partial f}{\partial x_n}(a)\,dx_n$$

1

$$f(x,y)=x+(y-1)\arcsin\sqrt{\frac{x}{y}}$$

$$\frac{\partial f}{\partial x}(x,1)$$



ŠTA JE DOMEN FUNKCIJE

$y\neq 0 \qquad \frac{x}{y}\geq 0$

$-1\leq\sqrt{\frac{x}{y}}\leq 1 \qquad \sqrt{\frac{x}{y}}\leq 1 \qquad 0\leq\frac{x}{y}\leq 1$

$x,y>0$

$x\leq y$

y JE FIKSIRANO NPR. $\frac{\partial f}{\partial x}(x,1)$ $y=1$

$$f\big|_{y=1} = x + \underbrace{0 \arcsin\sqrt{\frac{x}{1}}}_{=0} = x$$

$x \le y$ $\qquad$ $0 < x \le 1$

$x, y < 0$ $\qquad$ $\frac{x}{y} \le 1 \quad / y$

$x \ge y$

$$\frac{\partial f}{\partial x}(x,1) = \lim_{h \to 0} \frac{f(x+h,1) - f(x)}{h} = \lim_{h \to 0} \frac{\cancel{x}+h - \cancel{x}}{h} = \boxed{1}$$

---

(1)

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

f IMA PARCIJALNE IZVODE U SVIM TAČKAMA A NIJE NEPREKIDNA U TAČKI $(0,0)$

DA NIJE NEPREKIDNA VEĆ SMO POKAZALI

$$\left(\frac{1}{n}, \frac{1}{n}\right) \xrightarrow[n \to \infty]{} (0,0) \qquad f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{2} \xrightarrow[n \to \infty]{} 0$$

ZA $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{xy}{x^2+y^2}\right) = \frac{y(x^2+y^2) - xy \cdot 2x}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{xy}{x^2+y^2}\right) = \frac{x(x^2+y^2) - 2y^2 x}{(x^2+y^2)^2}$$

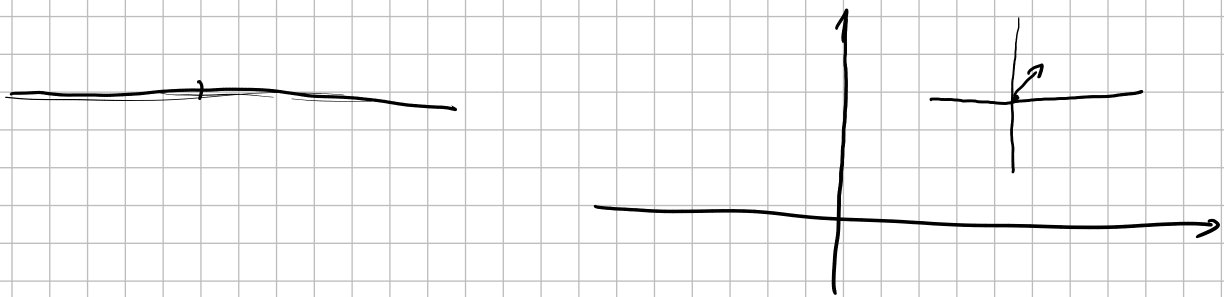
ZA $(x,y)=(0,0)$ MORAMO PO DEFINICIJI

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\overbrace{\frac{0y}{x^2+y^2}}^{=0} - 0}{h} = 0$$

ANALOGNO:

$$\frac{\partial f}{\partial y}(0,0) = 0$$

POSTOJE PARCIJALNI IZVODI U SVIM TAČKAMA $\mathbb{R}^2$, A FUNKCIJA NIJE NEPREKIDNA



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(3)

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$f(x,y) = \sqrt[4]{xy}$$

POSTOJE PARCIJALNI IZVODI I NEPREKIDNI SU A FUNKCIJA NIJE DIFERENCIJABILNA U $(0,0)$

$x \neq 0$ $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt[4]{xy}\right) = \sqrt[4]{y}\,\frac{\partial}{\partial x}\left(x^{\frac{1}{4}}\right) = \sqrt[4]{y}\,\frac{1}{4}\,x^{-\frac{3}{4}} = \frac{1}{4}\sqrt[4]{\frac{y}{x^3}}$$

$x = 0$ $$f(0,y) = \sqrt[4]{0y} = 0$$

U $(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\sqrt[4]{h\cdot 0} - 0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = 0$$

P.P.S. f JE DIF. U $(0,0)$

$$f(h,k) \overset{?}{=} \underset{0}{\underset{\parallel}{f(0,0)}} + \underset{0}{\underset{\parallel}{\frac{\partial f}{\partial x}(0,0)}}\, h + \underset{0}{\underset{\parallel}{\frac{\partial f}{\partial y}(0,0)}} \cdot k + o(\|(h,k)\|)$$

$$\underset{\sqrt[4]{hk}}{\underset{\parallel}{f(h,k)}} \overset{?}{=} o(\|(h,k)\|) \qquad \|(h,k)\| = \sqrt{h^2+k^2}$$

$$\Rightarrow \lim_{\substack{h\to 0\\ k\to 0}} \frac{\sqrt[4]{hk}}{\sqrt{h^2+k^2}} \overset{?}{=} 0 \qquad \left(\frac{1}{n}, \frac{1}{n}\right)$$

$$h = \rho\cos\varphi \qquad k = \rho\sin\varphi$$

$$\lim_{\rho\to 0} \frac{\sqrt[4]{\rho^2(\cos\varphi+\sin\varphi)}}{\rho} \overset{?}{=} 0$$

TREBALO BI ZA SVAKO φ

$$\lim_{\rho\to 0} \frac{1}{\sqrt{\rho}} \cancel{(\cos\varphi+\sin\varphi)} \overset{?}{=} 0$$

$$\varphi = 0 \qquad \lim_{\rho\to 0} \frac{1}{\sqrt{\rho}} = \infty \neq 0$$

$\Rightarrow$ f NIJE DIFERENCIJABILNA U TOČKI $(0,0)$.

---

(3)

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$f(x,y) = \sqrt[3]{x^3+y^3} \qquad x^3 + y^3 = 0$$

$f \in C(\mathbb{R}^2)$ KAO KOMPOZICIJA NEPREKIDNIH

DA LI JE DIFERENCIJABILNA KAO KOMPOZICIJA?

$\sqrt[3]{}$ NIJE DIF U 0

$$f \in \mathcal{D}(\mathbb{R}^2 \setminus \{x^3+y^3=0\})$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt[3]{x^3+y^3}\right) = \frac{1}{\cancel{3}}\frac{\cancel{3}x^2}{\sqrt[3]{(x^3+y^3)^2}} \neq 0 \ ?$$

$$\frac{\partial f}{\partial y} = \frac{y^2}{\sqrt[3]{(x^3+y^3)^2}} \qquad x^3+y^3 \neq 0$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\sqrt[3]{h^3+0^3}-0}{h}$$

$$= \lim_{h\to 0}\frac{h}{h} = 1$$

ZBOG SIMETRIJE PO x i y

$$\frac{\partial f}{\partial y}(0,0) = 1$$

AKO JE f DIFERENCIJABILNA U $(0,0)$

$$f(h,k) = f(0,0) + \frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k + o(\|(h,k)\|)$$

$$\sqrt[3]{h^3+k^3} = 0 + h + k + o(\|h,k\|)$$

$$\lim_{\substack{h\to 0\\ k\to 0}}\frac{\sqrt[3]{h^3+k^3}-h-k}{\sqrt{h^2+k^2}} \overset{?}{\cancel{=}} 0$$

$$\left(\frac{1}{n},\frac{1}{n}\right) \xrightarrow[n\to\infty]{} (0,0)$$

$$\lim_{n\to\infty}\frac{\sqrt[3]{\frac{1}{n^3}+\frac{1}{n^3}}-\frac{1}{n}-\frac{1}{n}}{\sqrt{\frac{1}{n^2}+\frac{1}{n^2}}} = \frac{\frac{\sqrt[3]{2}}{\cancel{n}}-\frac{1}{\cancel{n}}-\frac{1}{\cancel{n}}}{\frac{\sqrt{2}}{\cancel{n}}} = \frac{\sqrt[3]{2}-2}{\sqrt{2}} \neq 0$$

$\Rightarrow$ f NIJE DIF. U $(0,0)$

(4)

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$f \in C(\mathbb{R}^2 \setminus \{(0,0)\})$ као композиција

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\rho\to 0} \underbrace{\rho^2}_{\to 0} \underbrace{\sin\frac{1}{\rho^2}}_{\leq 1} = 0 = f(0,0)$$

$$\Rightarrow f \in C(\mathbb{R}^2)$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{(h^2+0^2)\sin\frac{1}{h^2+0^2} - 0}{h} = \lim_{h\to 0} \frac{h^{\cancel{2}}\overbrace{\sin\frac{1}{h^2}}^{\leq 1}}{\cancel{h}} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = 0$$

$(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left((x^2+y^2)\sin\frac{1}{x^2+y^2}\right) =$$

$$= 2x\sin\frac{1}{x^2+y^2} + \cancel{(x^2+y^2)}\cos\frac{1}{x^2+y^2}\,\frac{-2x}{(x^2+y^2)^{\cancel{2}}}$$

$$= 2x\sin\frac{1}{x^2+y^2} - \frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = 2y\sin\frac{1}{x^2+y^2} - \frac{2y}{x^2+y^2}\cos\frac{1}{x^2+y^2}$$

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \overset{?}{\in} C(\mathbb{R}^2)$$

NIZ (x_n, y_n) $\qquad \frac{1}{x_n^2+y_n^2} = 2n\pi$

$$x_n = y_n = \frac{1}{2\sqrt{n\pi}}$$

$$\frac{\partial f}{\partial x}(x_n, y_n) = \frac{1}{\sqrt{n\pi}} \overbrace{\sin 2n\pi}^{0} - \frac{\frac{1}{\sqrt{n\pi}}}{\frac{1}{2n\pi}} \overbrace{\cos 2n\pi}^{1}$$

$$= -2\sqrt{n\pi} \xrightarrow[n\to\infty]{} 0 = \frac{\partial f}{\partial x}(0,0)$$

$\frac{\partial f}{\partial x}$ i $\frac{\partial f}{\partial y}$ NISU NEPREKIDNI U TAČKI $(0,0)$

$f \in \mathcal{D}(\mathbb{R}^2 \setminus \{(0,0)\})$ KAO KOMPOZICIJA

ISPITUJEMO DIF. U $(0,0)$

$$f(h,k) = o(\|h,k\|) \qquad \left(f(0,0)=0,\ \frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0 \right)$$

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{(h^2+k^2)\sin\frac{1}{h^2+k^2}}{\sqrt{h^2+k^2}} = \lim_{\rho\to 0} \frac{\rho^{\cancel{2}} \sin\frac{1}{\rho^2}}{\cancel{\rho}} = 0$$

$\Rightarrow \quad f \in \mathcal{D}(\mathbb{R}^2)$

5

UOČDA NIJE DIF. U $(3,4)$

$$f: \mathbb{R}^2 \to \mathbb{R}$$

$$f(x,y) = \begin{cases} \sqrt{(x-3)^2+(y-4)^2} + x\sin\frac{y}{x^2+y^2}, & (x,y)\neq(0,0) \\ 5, & (x,y)=(0,0) \end{cases}$$

a) NEPREKIDNOST

b) DIFERENCIJABILNOST

а) $f \in C(\mathbb{R}^2 \setminus \{(0,0)\})$ као композиција

непрекидних функција.

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = f(0,0)$$

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\substack{x\to 0\\ y\to 0}} \underbrace{\sqrt{(x-3)^2+(y-4)^2}}_{\to 5} + \underbrace{x \sin\frac{y}{x^2+y^2}}_{\to 0} = 5$$

$$\Rightarrow \quad f \in C(\mathbb{R}^2)$$

б) диференцијабилност

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{\sqrt{(h-3)^2+4^2} + \underbrace{h\sin\frac{0}{h^2+0^2}}_{=0} - 5}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{9-6h+h^2+16}-5}{h}$$

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha$$

$$= \lim_{h\to 0} \frac{\sqrt{25-6h+h^2}-5}{h} =$$

$$= \lim_{h\to 0} \frac{5\sqrt{1-\frac{6}{25}h+\frac{1}{25}h^2}-5}{h}$$

$$= 5 \lim_{h\to 0} \frac{\left(1-\frac{6}{25}h+\frac{1}{25}h^2\right)^{1/2}-1}{\cancel{h}\left(-\frac{6}{25}h+\frac{1}{25}h^2\right)} \to \frac{1}{2} \qquad \left(-\frac{6}{25}\cancel{h}+\frac{1}{25}\cancel{h}\right)$$

$$= -\frac{6}{5}$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{\sqrt{3^2+(h-4)^2} + 0\sin\frac{h}{0^2+h^2} - 5}{h} =$$

$$= \ldots -$$

DA LI JE JEDINA SUMNJIVA TAČKA ZA DIFERENCIJABILNOST TAČKA (0,0)?

$$f(x,y) = g(x,y) + h(x,y)$$

$$g(x,y) = \sqrt{(x-3)^2 + (y-4)^2}$$

$$h(x,y) = \begin{cases} x \sin \frac{y}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

SUMNJIVE TAČKE

(0,0) ZA FUNKCIJU h

(3,4) ZA FUNKCIJU g

$f \in \mathcal{D}(\mathbb{R}^2 \setminus \{(0,0), (3,4)\})$ KAO ZBIR DVE DIF.

(3,4)

h JE DIF U (3,4)

f JE DIF U $(3,4) \Leftrightarrow g$ DIF U $(3,4)$

$$g(x,y) = \sqrt{(x-3)^2 + (y-4)^2}$$

SMENA PROMENLJIVE

$$X = x-3 \qquad Y = y-4$$

$$g(X,Y) = \sqrt{X^2+Y^2}$$

$X=0 \quad Y=0$

$$\frac{\partial g}{\partial X}(0,0) = \lim_{h\to 0} \frac{\sqrt{h^2+0^2} - 0}{h} = \lim_{h\to 0} \frac{|h|}{h} \text{ NE POSTOJI}$$

$$\frac{\partial g}{\partial x}(0,0) = \lim_{h \to 0} \operatorname{sgn} h \quad - \text{ ne postoji} \qquad \begin{cases} \to 1, & h>0 \\ \to -1, & h<0 \end{cases}$$

$\frac{\partial g}{\partial x}$ ne postoji u $(0,0)$

$\Rightarrow$ $g(x,y)$ nije dif. u $(0,0)$

$g(x,y)$ nije dif. u $(3,4)$

$f(x,y)$ nije dif. u $(3,4)$

[Zbir jedne dif. funkcije i jedne koja nije]

Za domaći završiti tačku $(0,0)$

$$h(x,y) = \begin{cases} x \sin \frac{y}{x^2+y^2} \\ 0 \end{cases}$$

Podsjetiti se diferencijabilnosti i odnosa sa operacijama u Analizi 1.

Definicija:

$f: U \to \mathbb{R}$, $U \subset \mathbb{R}^n$

$a \in U$

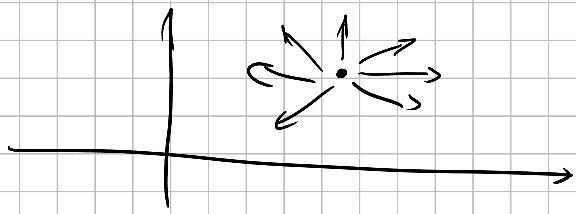
$\vec{l} \in \mathbb{R}^n \setminus \{0\}$ — ne nula vektor

Izvod funkcije f u tački a u pravcu vektora $\vec{l}$ je

$$\frac{\partial f}{\partial \vec{l}}(a) = \lim_{h \to 0} \frac{f(a + h\vec{l}) - f(a)}{h}$$

PARCIJALNI IZVODI SU SPECIJALNI SLUČAJ

ZA $\vec{l} = (1,0)$ i $\vec{l} = (0,1)$



(1)

$$A = \{ (x,y,z) \in \mathbb{R}^3 \mid x \neq 0 \}$$

$$f : A \to \mathbb{R}$$

$$f(x,y,z) = e^{\frac{z}{x}}$$

$$a = (3, 0, -1)$$

$$\vec{l} = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right)$$

$$\frac{\partial f}{\partial \vec{l}}(a) = \lim_{h \to 0} \frac{f(a + h\vec{l}) - f(a)}{h} =$$

$$= \lim_{h \to 0} \frac{e^{\frac{-1 + \frac{2}{3}h}{3 + \frac{2}{3}h}} - e^{-\frac{1}{3}}}{h} =$$

$$\frac{\frac{2}{3}h - 1}{\frac{2}{3}h + 3} = \frac{2h - 3}{2h + 9} = \frac{2h + 9 - 12}{2h + 9}$$

$$= 1 - \frac{12}{2h + 9}$$

$$\frac{\partial f}{\partial \vec{l}}(a) = \lim_{h \to 0} \frac{e^{1 - \frac{12}{2h+9}} - e^{-\frac{1}{3}}}{h} = \dots$$

domaći

DEFINICIJA:

$$f: \mathbb{R}^n \to \mathbb{R}$$

GRADIENT FUNKCIJE f U TAČKI a JE

$$\operatorname{grad} f(a) = \left(\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_n}(a) \right)$$

$$\operatorname{grad} f = \nabla \cdot f$$

∇ – OPERATOR NABLA

$$\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$$

$$\operatorname{grad} f : \mathbb{R}^n \to \mathbb{R}^n$$

VEKTORSKO VREDNOSNA FUNKCIJA

f JE BILA SKALARNO VREDNOSNA (KODOMEN $\mathbb{R}$)

TVRĐENJE:

$$\frac{\partial f}{\partial \vec{l}}(a) = \operatorname{grad} f(a) \cdot \vec{l} \qquad \vec{l} = (l_1, \dots, l_n)$$

$$= \left(\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_n}(a) \right) (l_1, \dots, l_n)$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)\, l_i$$

IZVOD U PRAVCU VEKTORA:

$$f : U \to \mathbb{R}$$

$$U \subseteq \mathbb{R}^n$$

$a \in U$

$\vec{l} \in \mathbb{R}^n \setminus \{0\}$

$$\frac{\partial f}{\partial \vec{l}}(a) = \lim_{t \to 0} \frac{f(a + t\vec{l}) - f(a)}{t}$$

(1)

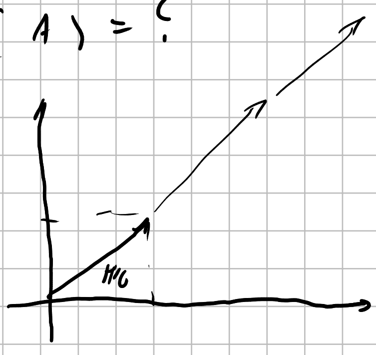
$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$f(x, y) = x^2 - y^2$$

$A = (1, 1)$

$\vec{l}$ vektor koji sa pozitivnim delom x-ose gradi ugao od $\frac{\pi}{6}$,

$$\frac{\partial f}{\partial \vec{l}}(A) = ?$$



Uzimamo da je $\|\vec{l}\| = 1$

$$\vec{l} = \left(\frac{\sqrt{3}}{2}, \frac{1}{2} \right)$$

$$\frac{\partial f}{\partial \vec{l}}(A) = \lim_{t \to 0} \frac{f(A + t\vec{l}) - f(A)}{t} \qquad A = (1,1) \quad f = x^2 - y^2$$

$$= \lim_{t \to 0} \frac{\left(1 + t\frac{\sqrt{3}}{2}\right)^2 - \left(1 + t\frac{1}{2}\right)^2 - \cancel{1^2} + \cancel{1^2}}{t}$$

$$= \lim_{t \to 0} \frac{\overset{\to 0}{\frac{3}{4}t^{\cancel{2}}} + \sqrt{3}\cancel{t} + \cancel{1} - \overset{\to 0}{\frac{1}{4}t^{\cancel{2}}} - \cancel{t} - \cancel{1}}{\cancel{t}} = \boxed{\sqrt{3} - 1}$$

GRADIJENT FUNKCIJE

$$\operatorname{grad} f(a) = \left(\frac{\partial f}{\partial x_1}(a), \ldots, \frac{\partial f}{\partial x_n}(a)\right)$$

$$\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$$

$$\operatorname{grad} f = \nabla f$$

$$\frac{\partial f}{\partial \vec{l}} = \operatorname{grad} f \cdot \vec{l} \quad ?$$

TVRĐENJE!

AKO JE f DIFERENCIJABILNA U TAČKI a TADA POSTOJI IZVOD U PRAVCU PROIZVOLJNOG VEKTORA U TAČKI a I VAŽI

$$\frac{\partial f}{\partial \vec{l}}(a) = \operatorname{grad} f(a) \cdot \vec{l}$$

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(2)

$$f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$$

$$f(x,y) = \frac{x^3 y^2}{x^4 + y^4}$$

a) DA LI SE f MOŽE PRODUŽITI DO NEPREKIDNE FUNKCIJE NA $\mathbb{R}^2$?

b) AKO SE MOŽE PRODUŽITI DO NEPREKIDNE ISPITATI DIFERENCIJABILNOST TAKO DOBIJENE FUNKCIJE.

v) IZRAČUNATI PARCIJALNE IZVODE I PROVERITI DA LI SU NEPREKIDNI

g) IZRAČUNATI IZVOD U PRAVCU PROIZVOLJNOG VEKTORA GDE GOD POSTOJI

a)

$$f(x,y) = \frac{x^3 y^2}{x^4 + y^4}$$

$f \in C\left(\mathbb{R}^2 \setminus \{(0,0)\}\right)$ KAO KOMPOZICIJA NEPREKIDNIH

AKO POSTOJI $A \in \mathbb{R}$ TDJ.

$$\lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) = A$$

f SE MOŽE NEPREKIDNO PRODUŽITI

$$\lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 y^2}{x^4 + y^4} = \Big/ \text{POLARNE KOORDINATE} \Big)$$

$$= \lim_{\rho \to 0} \frac{\cancel{\rho^5} \cos^3\varphi \, \rho^2 \sin^2\varphi}{\cancel{\rho^4} \cos^4\varphi + \cancel{\rho^4} \sin^4\varphi} =$$

$$= \lim_{\rho \to 0} \rho \underbrace{\frac{\cos^3\varphi \sin^2\varphi}{\cos^4\varphi + \sin^4\varphi}}_{\text{DALJE JE OGRANIČENO}}$$

$\forall \varphi \quad \cos^4\varphi + \sin^4\varphi \neq 0$

(NEMOGU I SIN I COS BITI ISTOVREMENO NULA)

$$\frac{\cos^3\varphi \sin^2\varphi}{\cos^4\varphi + \sin^4\varphi} \in C([0,2\pi])$$

PA DOSTIŽE MINIMUM I MAKSIMUM

TJ. $\exists M \in \mathbb{R}$ $\quad \left| \frac{\cos^3\varphi \sin^2\varphi}{\cos^4\varphi + \sin^4\varphi} \right| \le M$

$$\Rightarrow \lim_{\rho \to 0} \rho \frac{\cos^3\varphi \sin^2\varphi}{\cos^4\varphi + \sin^4\varphi} = 0$$

FUNKCIJA ZADATA SA:

$$\tilde{f}(x,y) = \begin{cases} f(x,y) &, (x,y) \neq (0,0) \\ 0 &, (x,y) = (0,0) \end{cases}$$

JE NEPREKIDNA NA $\mathbb{R}^2$

b) DIFERENCIJABILNOST

$f \in \mathcal{D}(\mathbb{R}^2 \setminus \{(0,0)\})$ KAO KOMPOZICIJA.

POTREBNO JE ISPITATI DIFERENCIJABILNOST SAMO U TAČKI $(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\overbrace{\frac{h^3 0^2}{h^4 + 0^4} - 0}^{=0}}{h} = 0$$

SLIČNO

$$\frac{\partial f}{\partial y}(0,0) = 0$$

$$f(0,0) = 0$$

PA f JE DIF. U $(0,0)$ AKO JE

$$f(x,y) \overset{?}{=} o(\|(x,y)\|) \quad , \quad \begin{matrix} x \to 0 \\ y \to 0 \end{matrix}$$

$$\frac{x^3 y^2}{x^4+y^4} \overset{?}{=} o\left(\|(x,y)\|\right)$$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3 y^2}{(x^4+y^4)\sqrt{x^2+y^2}} \overset{?}{=} 0$$

POLARNE KOORDINATE

$$\lim_{\rho\to 0} \frac{\cancel{\rho^5}\cos^3\varphi \sin^2\varphi}{(\rho^4\cos^4\varphi+\rho^4\sin^4\varphi)\,\cancel{\rho}} = \lim_{\rho\to 0}\frac{\cos^3\varphi\sin^2\varphi}{\cos^4\varphi+\sin^4\varphi} \overset{?}{=} 0$$

$\forall \varphi$

$$\varphi = \frac{\pi}{4}$$

$$\frac{\cos^3\frac{\pi}{4}\sin^2\frac{\pi}{4}}{\cos^4\frac{\pi}{4}+\sin^4\frac{\pi}{4}} = \frac{\frac{\sqrt{2}\cdot 2}{8}\;\;\frac{1}{2}}{\frac{4}{16}+\frac{4}{16}} \neq 0$$

$\Rightarrow$ f NIJE DIFERENCIJABILNA U TAČKI $(0,0)$

b) PARCIJALNI IZVODI

$$\frac{\partial f}{\partial x}(0,0) = 0 \qquad \frac{\partial f}{\partial y}(0,0) = 0$$

$$(x,y) \neq (0,0)$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^3y^2}{x^4+y^4}\right) = \frac{3x^2y^2(x^4+y^4) - x^3y^2\,4x^3}{(x^4+y^4)^2}$$

$$= \frac{3x^6y^2+3x^2y^6-4x^6y^2}{(x^4+y^4)^2}$$

$$= \frac{3x^2y^6-x^6y^2}{(x^4+y^4)^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^3y^2}{x^4+y^4}\right) = \frac{2yx^3(x^4+y^4)-x^3y^2\,4y^3}{(x^4+y^4)^2}$$

$$= \frac{2x^7y + 2y^5x^3 - 4x^3y^5}{(x^4+y^4)^2}$$

$$= \frac{2x^7y - 2x^3y^5}{(x^4+y^4)^2}$$

Да ли су парцијални изводи непрекидни ван тачке $(0,0)$? Јесу непрекидни (композиција непрекидних функција)

Парцијални изводи нису непрекидни у тачки $(0,0)$ јер би у том случају f била диференцијабилна функција, а показали смо да није.

g) Извод у правцу

$(x,y) \neq (0,0)$ $\qquad \vec{l} = (l_1, l_2)$

$\vec{l}$ - произвољан јединични вектор

$$\frac{\partial f}{\partial \vec{l}} = \frac{3x^2y^6 - x^6y^2}{(x^4+y^4)^2}\, l_1 + \frac{2x^7y - 2x^3y^5}{(x^4+y^4)^2}\, l_2$$

Извод у правцу у тачки $(0,0)$

$$\frac{\partial f}{\partial \vec{l}}(0,0) = \lim_{t\to 0} \frac{f(t\vec{l}) - \overset{=0}{f(0)}}{t}$$

$$= \lim_{t\to 0} \frac{f(tl_1, tl_2)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{t^3 l_1^3\, t^2 l_2^2}{t^4 l_1^4 + t^4 l_2^4}}{t} = \frac{l_1^3\, l_2^2}{l_1^4 + l_2^4}$$

Ravnomerna neprekidnost

$f : X \to Y$

f je ravnomerno neprekidno

$\forall \varepsilon > 0 \ \exists \delta > 0 \quad \forall x, y \in X \quad d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$

Teorema: (Kantor)

$f : K \to Y$

K - kompaktan

f neprekidna $\Rightarrow$ f ravnomerno neprekidna

---

f r.n. na A

i $B \subseteq A$

$\Rightarrow$ f je r.n. na B

Teorema:

$f : \underset{\mathbb{R}^n}{\overset{\cap}{A}} \to \mathbb{R}$

A - konveksan i otvoren skup

$f \in \mathcal{D}(A)$

$\exists k > 0 \quad \forall a \in A \quad \left| \frac{\partial f}{\partial x_i}(a) \right| < k \qquad 1 \le i \le n$

$\Rightarrow$ f je r.n. na A

f je r.n. i g je r.n. $\Rightarrow f \pm g$ r.n. } Dokaz domaći

f je r.n. i g nije r.n.

$f \pm g$ nije r.n.

$f \cdot g$?

$f(x) = x \quad g(x) = x$ — jesu r.n.

$f \cdot g\,(x) = x^2$ — nije r.n.

Ako su f i g ograničene i r.n. tada je i $f \cdot g$ r.n.

f, g - ograničene

$\exists M_1 \quad \forall x \quad |f(x)| < M_1$

$\exists M_2 \quad \forall x \quad |g(x)| < M_2$

f, g su r.n.

$\varepsilon > 0$ dato

$\exists \delta_1 \quad \forall x, y \quad d(x,y) \leq \delta_1 \quad |f(x) - f(y)| < \frac{\varepsilon}{2M_2}$

$\exists \delta_2 \quad \forall x, y \quad d(x,y) \leq \delta_2 \quad |g(x) - g(y)| < \frac{\varepsilon}{2M_1}$

$\delta = \min\{\delta_1, \delta_2\}$

$d(x,y) < \delta \quad |f \cdot g(x) - f \cdot g(y)| =$

$$|f(x)g(x) - f(y)g(y)| =$$

$$|f(x)g(x) - f(x)g(y) + f(x)g(y) + f(y)g(y)| \le$$

$$\le |f(x)|\,|g(x) - g(y)| + |g(y)|\,|f(x) - f(y)|$$

$$\le M_1 \frac{\varepsilon}{2M_1} + M_2 \frac{\varepsilon}{2M_2} \le \varepsilon$$

---

f р.н. на A
и f р.н. на B

$A \cup B$?

$f(x) = [x]$

$A = [0,1)$ $\quad B = [1,2)$

f је р.н. и на A и на B (константна)

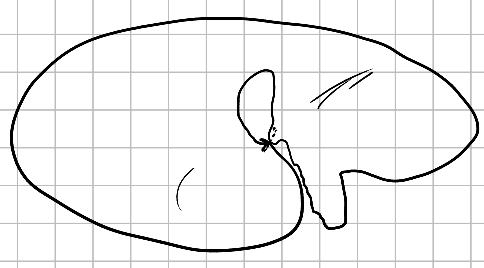
f је прекидна на $A \cup B$

Ако додамо услов да је

$A \cap B \neq \emptyset$ или $d(A,B) > 0$

f је р.н. на унији

У А2 код прикованије



① ISPITATI R.N. NA SKUPU

$$D = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1 \}$$

FUNKCIJA:

a) $f(x,y) = \sin \frac{\pi}{1-x^2-y^2}$

b) $g(x,y) = \sin \frac{\pi}{2-x^2-y^2}$

a)

DOKAZUJEMO DA f NIJE R.N.

HOĆEMO DA NAĐEMO NIZOVE

a_n, b_n

$d(a_n, b_n) \xrightarrow{n\to\infty} 0$

$d(f(a_n), f(b_n)) \cancel{\xrightarrow{n\to\infty}} 0$

$$f(x,y) = \sin \frac{\pi}{1-x^2-y^2}$$

$$\frac{\partial f}{\partial x} = \cos \frac{\pi}{1-x^2-y^2} \quad \pi \quad \frac{2x}{(1-x^2-y^2)^2}$$ → MOŽE DA TEŽI 0 NA D

OVO NIJE OGRANIČENO NA D

a_n BIRAMO TDD. $f(a_n) = \sin 2n\pi$

$a_n = (x_n, y_n) \qquad b_n = (\tilde{x}_n, \tilde{y}_n)$

$y_n = \tilde{y}_n = 0$

$$x_n = \sqrt{1-\frac{1}{2n}}$$

$$f(a_n) = f((x_n, y_n)) = \sin \frac{\pi}{\cancel{1}-\cancel{1}+\frac{1}{2n}} = \sin 2n\pi = 0$$

$$\tilde{x}_n = \sqrt{1-\frac{1}{2n+\frac{1}{2}}}$$

$$f(b_n) = \sin \frac{\pi}{\cancel{1}-\cancel{1}+\frac{1}{2n+\frac{1}{2}}} = \sin\left(2n+\frac{1}{2}\right)\pi = \sin\frac{\pi}{2} = 1$$

$$f(b_n) - f(a_n) = 1 \; \cancel{\xrightarrow[n\to\infty]{}} \; 0$$

$$d(a_n, b_n) = d\left(\left(\sqrt{1-\tfrac{1}{2n}}, 0\right), \left(\sqrt{1-\tfrac{1}{2n+\frac{1}{2}}}, 0\right)\right)$$

$$= \sqrt{\left(1-\frac{1}{2n}-\left(1-\frac{1}{2n+\frac{1}{2}}\right)\right)^2 + (0-0)^2} =$$

$$= \left|\cancel{1}-\frac{1}{2n}-\cancel{1}+\frac{1}{2n+\frac{1}{2}}\right| = \left|\frac{2}{4n+1}-\frac{1}{2n}\right|$$

$$= \left|\frac{\cancel{4n}-\cancel{4n}-1}{(4n+1)\,2n}\right| = \frac{1}{(4n+1)\,2n} \xrightarrow[n\to\infty]{} 0$$

$$d(a_n, b_n) \xrightarrow[n\to\infty]{} 0$$

b)

$$g(x,y) = \sin\frac{\pi}{2-x^2-y^2}$$

$$g \in C(\overline{D}) \qquad \overline{D} : x^2+y^2 \leq 1$$

$\overline{D}$ je kompaktan skup

Kantor $\Rightarrow g$ je r. n. na $\overline{D}$

$\Rightarrow$ g JE R.N. NA $D \subseteq \overline{D}$

---

(2)

$$f : A \to \mathbb{R}$$

$$A = \{ (x,y) \in \mathbb{R}^2 \mid 0 < x^2+y^2 \leq 1 \}$$

$$f(x,y) = \frac{x^3+y^3}{x^2+y^2}$$

NIJE TEŠKO PROVERITI !

$$\lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) = 0$$

SLEDI DA JE NEPREKIDNA FUNKCIJA :

$$\tilde{f}(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

$$\tilde{f} \in C(\overline{A})$$

$\overline{A}$ JE KOMPAKTAN

$\Rightarrow$ $\tilde{f}$ JE R.N. NA $\overline{A}$ (KANTOR)

$\tilde{f}$ JE R.N. NA $A \subseteq \overline{A}$

$\tilde{f}|_A = f$ $\Rightarrow$ f JE R.N. NA A

③

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

ISPITATI R.N. NA $\mathbb{R}^2$

$(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x} = \frac{3x^2(x^2+y^2) - (x^3+y^3)\,2x}{(x^2+y^2)^2} =$$

$$= \frac{3x^4 + 3x^2y^2 - 2x^4 - 2xy^3}{(x^2+y^2)^2}$$

$$= \frac{x^4 + 3x^2y^2 - 2xy^3}{(x^2+y^2)^2}$$

PRELAZIMO NA POLARNE KOORDINATE

$$|f(\rho,\varphi)| = \left| \frac{\cancel{\rho^4}\cos^4\varphi + 3\rho^4\cos^2\varphi\sin^2\varphi - 2\rho^4\cos\varphi\sin^3\varphi}{\cancel{\rho^4}} \right|$$

$$= |\cos^4\varphi + 3\cos^2\sin^2\varphi - 2\cos\varphi\sin^3\varphi|$$

$$\leq \cos^4\varphi + 3\cos^2\varphi\sin^2\varphi + 2|\cos\varphi\sin\varphi| \leq 6$$

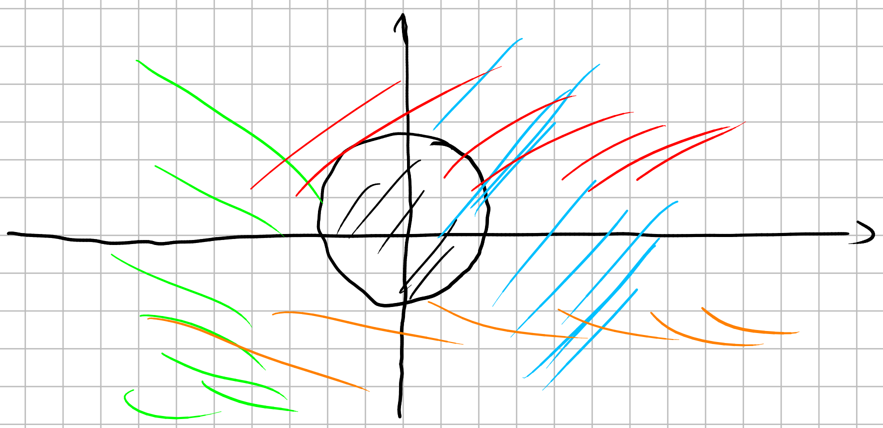
$$\left|\frac{\partial f}{\partial x}\right| \leq 6$$

SIMETRIČNO JE PO x I y

$$\left|\frac{\partial f}{\partial y}\right| \leq 6$$

PREDSTAVIĆEMO $\mathbb{R}^2$ KAO UNIJU OTVORENIH I KONVEKSNIH SKUPOVA

$$A = \{(x,y) \in \mathbb{R}^2 \mid |(x,y)| < 1\}$$



$$B_1 := \{(x,y) \in \mathbb{R}^2 \mid x > 0\}$$

$B_2 \qquad x < 0$

$C_1 \qquad y > 0$

$C_2 \qquad y < 0$

$$A \cup B_1 \cup B_2 \cup C_1 \cup C_2 = \mathbb{R}^2$$

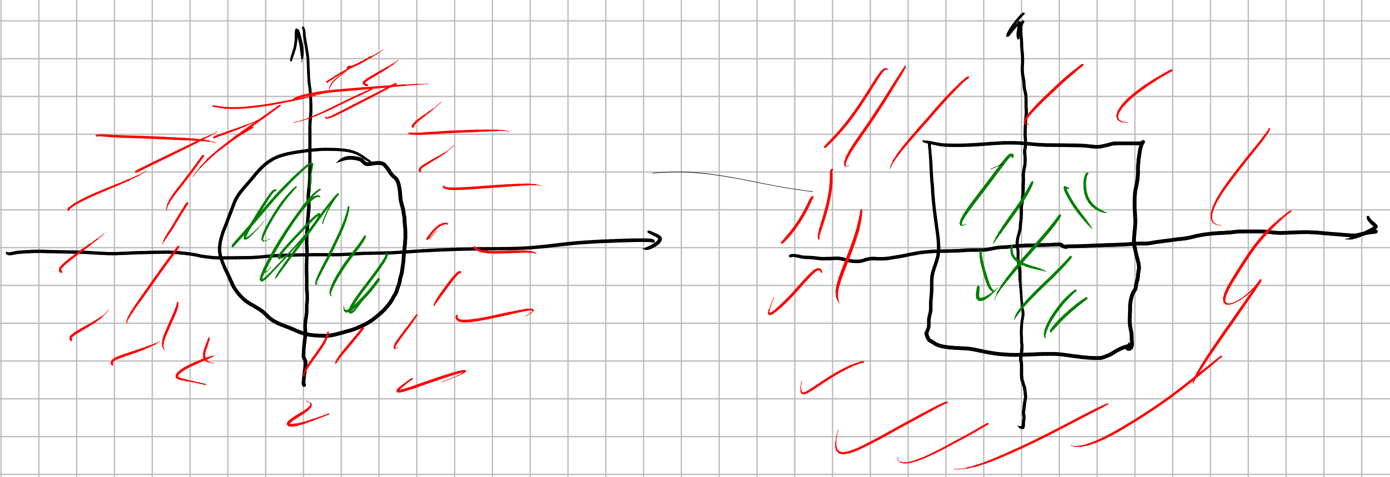
f JE R.N. NA A (PRETHODNI ZADATAK)

f JE R.N. NA B_1, B_2, C_1, C_2 (TEOREMA – OGRANIČENI PARCIJALNI IZVODI)

$\Rightarrow$ f JE R.N. $A \cup B_1 \cup B_2 \cup C_1 \cup C_2 = \mathbb{R}^2$

✓✓

SMETE DA KORISTITE PODELU:



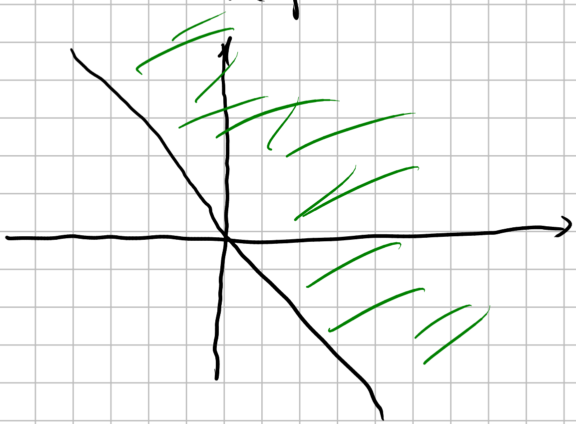
4

$$f(x,y) = x \lg(x+y)$$

ISPITATI R.I.N. NA DOMENU

DOMEN FUNKCIJE JE

$$x+y>0$$



$$\frac{\partial f}{\partial x} = \lg(x+y) + x\,\frac{1}{x+y}$$

NIJE OGRANIČENO

$$a_n = \left(n, \frac{1}{n} - n\right)$$

$$b_n = \left(n, \frac{2}{n} - n\right)$$

$$d(a_n, b_n) = \sqrt{(\cancel{n} - \cancel{n})^2 + \left(\frac{1}{n} - \cancel{n} - \left(\frac{2}{n} - \cancel{n}\right)\right)^2}$$

$$= \left| \frac{1}{n} - \frac{2}{n} \right| = \frac{1}{n} \xrightarrow[n \to \infty]{} 0$$

$$d(f(a_n), f(b_n)) = | f(a_n) - f(b_n) | =$$

$$= \left| n \log \frac{1}{n} - n \log \frac{2}{n} \right| =$$

$$= n \left| \log \frac{1}{n} - \log \frac{2}{n} \right| = n \left| \log \frac{\frac{1}{n}}{\frac{2}{n}} \right| = n \log 2 \xrightarrow[n \to \infty]{} \infty$$

$d(a_n, b_n) \xrightarrow[n \to \infty]{} 0$ i $d(f(a_n), f(b_n)) \not\xrightarrow[n \to \infty]{} 0$

$\Rightarrow f$ НИЈЕ РАВНОМЕРНО НЕПРЕКИДНА

NORMIRANI VEKTORSKI PROSTORI

① V, W - NORMIRANI VEKTORSKI PROSTORI

$L : V \to W$ - LINEARNO

EKVIVALENTNO JE:

1) L JE NEPREKIDNO U 0
2) L JE NEPREKIDNO
3) L JE RAVNOMJERNO NEPREKIDNO
4) L JE LIPŠICOVO
5) L JE LINEARNO OGRANIČENO PRESLIKAVANJE

$$\sup_{x \in S(0,1)} \| L(x) \| < +\infty$$

DOKAZ:

$$5) \Rightarrow 4) \Rightarrow 3) \Rightarrow 2) \Rightarrow 1) \Rightarrow 5)$$

$5) \Rightarrow 4)$

L JE LINEARNO OGRANIČENO PRESLIKAVANJE

$\forall x \in S(0,1) \quad \|L(x)\| \leq C \|x\|$

$x \notin S(0,1)$ $\quad \frac{x}{\|x\|} \in S(0,1)$

$\|x\| \neq 0 \quad \frac{\|L(x)\|}{\|x\|} = \left\| L\left(\frac{x}{\|x\|}\right)\right\| \leq C \left\|\frac{x}{\|x\|}\right\| = C \frac{\|x\|}{\|x\|}$

$$\|L(x)\| \leq c\,\|x\|$$

$\|x\| = 0$ — TRIVIJALNO

$x_1, x_2 \in V$

$$\|L(x_1) - L(x_2)\| = \|L(x_1 - x_2)\| \leq C\,\|x_1 - x_2\|$$

$\Rightarrow L$ JE LIPŠICOVO SA KONSTANTOM $\leq C$

4) $\Rightarrow$ 3) SVAKO LIPŠICOVO PRESLIKAVANJE JE I RAVNOMERNO NEPREKIDNO

$\varepsilon > 0$

$\delta = \dfrac{\varepsilon}{C}$

$\|x_1 - x_2\| \leq \delta = \dfrac{\varepsilon}{C}$ $\qquad$ δ NE ZAVISI OD x_1, x_2

$$\|L(x_1) - L(x_2)\| < \varepsilon$$

3) $\Rightarrow$ 2) RAVNOMERNO NEPREKIDNO JE NEPREKIDNO

2) $\Rightarrow$ 1) AKO JE NEPREKIDNO SVUGDE NEPREKIDNO JE I U 0.

1) $\Rightarrow$ 5)

L JE NEPREKIDNO U 0

$\forall \varepsilon > 0 \quad \exists \delta > 0$

$$\|x - 0\| < \delta \Rightarrow \|L(x) - L(0)\| < \varepsilon$$

$$\|x\| < \delta \Rightarrow \|L(x)\| < \varepsilon$$

$x \in V$ PROIZVOLJNO

$0 < \lambda < \delta$ $\qquad$ TADA JE

$$\left\|\frac{\lambda x}{\|x\|}\right\| < \delta$$

$$\Rightarrow \left\|L\left(\frac{\lambda x}{\|x\|}\right)\right\| < \varepsilon$$

$$\|$$

$$\lambda\, L\left(\frac{x}{\|x\|}\right) < \varepsilon$$

$$\|L(x)\| \le \frac{\varepsilon}{r}\|x\| \qquad x \neq 0$$

$$C = \frac{\varepsilon}{r}$$

TO JE OGRANIČEN $\Rightarrow$ 5)

---

(2) $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

SVAKA NORMA:

$$\|\cdot\| : \mathbb{K}^r \to \mathbb{R}_+$$

JE LIPŠICOVO PRESLIKAVANJE

ZA NORMU VAŽI NEJEDNAKOST TROUGLA

$$\|x+y\| \le \|x\| + \|y\|$$

$$\|x\| = \|x+y-y\| \le \|x-y\| + \|y\|$$

$$\|x\| - \|y\| \le \|x-y\| \qquad 1)$$

ANALOGNO:

$$\|y\| = \|y+x-x\| \le \|x-y\| + \|x\|$$

$$\|y\| - \|x\| \le \|x-y\| \qquad 2)$$

1), 2) $\Rightarrow$ $\big|\, \|x\| - \|y\| \,\big| \le \|x-y\|$

$\Rightarrow$ $\|\cdot\|$ JE LIPŠICOVA $\quad$ $Lip = 1$

---

(3) $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

$|\cdot| : \mathbb{K}^r \to \mathbb{R}_+$ - EUKLIDSKA NORMA

$\|\cdot\|$ - PROIZVOLJNA DRUGA NORMA

$$\|\cdot\| \asymp |\cdot|$$

$\asymp$ - RELACIJA EKVIVALENCIJE

DOVOLJNO JE POKAZATI DA JE SVAKA NORMA EKVIVALENTNA JEDNOJ PROIZVOLJNOJ

IZABRAĆEMO $\|.\|_\infty$

$$x = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$$

$e_1, \dots, e_n$ — BAZNI VEKTORI

$$\|x\| = \|x_1 e_1 + \dots + x_n e_n\| \leq$$

$$\leq \|x_1 e_1\| + \dots + \|x_n e_n\| =$$

$$= |x_1| \, \|e_1\| + \dots + |x_n| \, \|e_n\| \leq (*)$$

$$|x_1|, |x_2|, \dots, |x_n| < \|x\|_\infty$$

$$(*) \leq \|x\|_\infty \|e_1\| + \dots + \|x\|_\infty \|e_n\|$$

$$\leq \|x\|_\infty \underbrace{(\|e_1\| + \dots + \|e_n\|)}_{B}$$

$$\Rightarrow \|x\| \leq B \|x\|_\infty$$

TREBA NAM JOŠ

$$\|x\| \geq A \|x\|_\infty$$

$\|.\|$ JE NEPREKIDNA U TOPOLOGIJI GENERISANOJ SA $\|.\|_\infty$

$$| \|x\| - \|y\| | \leq \|x - y\| \leq \underline{B} \, \|x-y\|_\infty \Rightarrow \text{LIPŠICOVA KONSTANTA } B$$

$S(0,1)$ JE KOMPAKTAN SKUP

($S(0,1)$ JE OGRANIČEN I ZATVOREN)

$\|.\|$ JE NEPREKIDNA i $S(0,1)$ JE KOMPAKT

$\Rightarrow$ $\|\ \|$ POSTIŽE MINIMUM NA $S(0,1)$

$$\exists x_m \quad 0 \leq m = \|x_m\| \leq \|x\| \quad \forall x \in S(0,1)$$

$$m \leq \left\| \frac{x}{\|x\|_\infty} \right\|$$

$$m \|x\|_\infty \leq \|x\|$$

$$m \|x\|_\infty \leq \|x\| \leq B \|x\|_\infty$$

$$\|\ \| \simeq \|\ \|_\infty$$

VAŽI I DA JE $\|.\| \simeq |.|$

IZ TRANZITIVNOSTI

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(4) SABIRANJE n VEKTORA U NORMIRANOM VEKTORSKOM PROSTORU JE LIPŠICOVO

(NIJE REČENO KOJA JE NORMA, ALI NA OSNOVU PRETHODNOG ZADATKA SVEJEDNO JE, AKO JE LIPŠICOVO U JEDNOJ NORMI BIĆE I U DRUGOJ)

$x_1, x_2, \dots, x_n \in V$ $\quad$ V - NORMIRANI V. P.

$$L : (x_1, \dots, x_n) \longmapsto x_1 + x_2 + \dots + x_n$$

$$L : V^n \longrightarrow V$$

$x, y \in V^n$

NA ZBIRU IMAMO NORMU U STARTU

TREBA DA POKAŽEMO

$$\|L(x) - L(y)\|_V \leq c\ \|x - y\|_\Theta \longrightarrow$$ NEKA NORMA NA PROIZVODU

$$\|L(x) - L(y)\| = \|x_1 + x_2 + \dots + x_n - (y_1 + y_2 + \dots + y_n)\| =$$

$x = (x_1, \dots, x_n)$

$y = (y_1, \dots, y_n)$

$$= \|x_1 - y_1 + x_2 - y_2 + \dots + x_n - y_n\| \leq$$

$$\leq \|x_1 - y_1\| + \|x_2 - y_2\| + \dots + \|x_n - y_n\|$$

$$= \|x - y\|_1$$

L JE LIPŠICOVO U 1-NORMI $L_p = 1$

$$\leq \|x_1 - y_1\| + \|x_2 - y_2\| + \dots + \|x_n - y_n\|$$

$$= \sqrt{\|x_1 - y_1\|^2} + \dots + \sqrt{\|x_n - y_n\|^2}$$

$$\leq \sqrt{\|x_1 - y_1\|^2 + \dots + \|x_n - y_n\|^2} + \dots + \sqrt{\|x_1 - y_1\|^2 + \dots + \|x_n - y_n\|^2}$$

$$\leq n \, \|x - y\|_2$$

---

(5)

V - NORMIRANI VEKTORSKI PROSTOR NAD $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

$L : \mathbb{K}^n \to V$ - LINEARNO

POKAZATI:

$$\|L\|_\infty \leq \sqrt{\sum_{i=1}^{n} \|L(e_i)\|^2}$$

$e_1, \dots, e_n$ - STANDARDNA BAZA

NA PROIZVODU PODRAZUMEVAMO EUKLIDSKU NORMU

$x \in \mathbb{K}^n \qquad x = (x_1, \dots, x_n)$

$$x = x_1 e_1 + \dots + x_n e_n$$

$$\|L(x)\| = \|L(x_1 e_1 + \dots + x_n e_n)\| = \|L(\sum_{i=1}^{n} x_i e_i)\|$$

$$= \|\sum_{i=1}^{n} L(x_i e_i)\|$$

$$\leq \sum_{i=1}^{n} \|L(x_i e_i)\|$$

KOŠI - ŠVARCOVA NEJEDNAKOST

$$= \sum_{i=1}^{n} |x_i| \, \|L(e_i)\| \leq$$

$$\leq \sqrt{\sum_{i=1}^{n} |x_i|^2} \sqrt{\sum_{i=1}^{n} \|L(e_i)\|^2}$$

$$= \|x\|_2 \sqrt{\sum_{i=1}^{n} \|L(e_i)\|^2}$$

OVO VAŽI ZA SVAKO x, PA JE

$$\frac{\|L(x)\|}{\|x\|} \leq \sqrt{\sum_{i=1}^{n} \|L(e_i)\|^2}$$

$$\|L\|_\infty \leq \sqrt{\sum_{i=1}^{n} \|L(e_i)\|^2}$$

---

⑥ V, W - NORMIRANI V. P.

$\{x_n\}_{n \in \mathbb{N}}$ - NIZ U V

$L : V \to W$ - LINEARNO OGRANIČENO PRESLIKAVANJE

$\sum_{n=1}^{\infty} x_n$ KONVERGIRA $\Rightarrow$ $\sum_{n=1}^{\infty} L(x_n)$ KONVERGIRA

$$L\left(\sum_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} L(x_n)$$

POKAZAĆEMO DA JE

$\sum_{n=1}^{\infty} L(x_n)$ KOŠIJEV

KOŠI - IVANI

$$\left\| \sum_{k=n}^{m} L(x_k) \right\| = \left\| L\left(\sum_{k=n}^{m} x_k\right) \right\| \leq$$

$$\leq \|L\| \left\| \sum_{k=n}^{m} x_k \right\|$$

$\sum_{k=1}^{\infty} x_k$ - KOŠIJEV $\qquad \left\| \sum_{k=n}^{m} \right\| \leq \frac{\varepsilon}{\|L\|}$

$$\left\| \sum_{k=n}^{m} L(x_k) \right\| \leq \varepsilon$$

$\Rightarrow$ KONVERGIRA

$$L\left(\sum_{k=1}^{n} x_k\right) = \sum_{k=1}^{n} L(x_k) \quad / \lim_{n \to \infty}$$

$$\lim_{n \to \infty} L\left(\sum_{k=1}^{n} x_k\right) = \lim_{n \to \infty} \sum_{k=1}^{n} L(x_k)$$

$\overset{?}{\|}$ ZAŠTO SME

$$L \lim_{n \to \infty} \sum_{k=1}^{n} x_k = \sum_{k=1}^{\infty} L(x_k)$$

$$L\left(\sum_{k=1}^{\infty} x_k\right) = \sum_{k=1}^{\infty} L(x_k)$$

7) Ograničeno višelinearno preslikavanje (uopštenje linearnog preslikavanja)

$$\underbrace{(X_1, \|\cdot\|_1),\ (X_2, \|\cdot\|_2),\ \ldots,\ (X_n, \|\cdot\|_n),\ (Y, \|\cdot\|_Y)}_{\text{normirani vektorski prostori}}$$

$$L : X_1 \times \ldots \times X_n \to Y$$

višelinearno preslikavanje znači linearno po svakoj promenljivoj.

Uvodimo normu višelinearnog preslikavanja

$$\|L\| = \sup_{\substack{\|x_1\|_1 = 1 \\ \|x_2\|_2 = 1 \\ \vdots \\ \|x_n\|_n = 1}} \|L(x_1, \ldots, x_n)\|_Y$$

$$= \sup_{\substack{h_1 \neq 0 \\ \vdots \\ h_n \neq 0}} \frac{\|L(h_1, \ldots, h_n)\|_Y}{\|h_1\|_1 \cdots \|h_n\|_n}$$

$$= \inf \{ C \geq 0 \mid \|L(x_1, \ldots, x_n)\| \leq C \|x_1\|_1 \cdots \|x_n\|_n \}$$

Potpuno isto se pokazuje da je ekvivalentno

- L je neprekidno
- L je ograničeno

Primer:

$$B : l_1(V) \times L(V, W) \to l_1(W)$$

$$B(\{x_n\}_{n \in \mathbb{N}}, L) = \{L(x_n)\}_{n \in \mathbb{N}}$$

$$l_1(V) = \{ \{x_n\}_{n \in \mathbb{N}} \subseteq V \mid \sum_{n=1}^{\infty} \|x_n\| < +\infty \}$$

B je bilinearno i neprekidno preslikavanje

I) Linearno po prvoj promenljivoj

$$B(\{x_n\}+\{y_n\}, L) = \{L(x_n+y_n)\}_{n\in\mathbb{N}}$$

$$= \{L(x_n)\}_{n\in\mathbb{N}} + \{L(y_n)\}_{n\in\mathbb{N}}$$

$$B(c\{x_n\}, L) = c\{L(x_n)\}$$

II) Linearno je po drugoj promenljivoj

$$B(\{x_n\}, L+G) = \ldots = \{L(x_n)\} + \{G(x_n)\}$$

$$B(\{x_n\}, cL) = c\{L(x_n)\}$$

$$\|B(\{x_n\}, L)\| = \sum_{n=1}^{\infty} |L(x_n)| \leq$$

$$\leq \sum_{n=1}^{\infty} \|L\| \, |x_n| = \|L\| \, \|x_n\|$$

$\Rightarrow \|B\| < +\infty \Rightarrow$ B je neprekidno

Unitarni (predhilbertovi) prostori

X – vektorski prostor nad $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

Skalarni proizvod u X je preslikavanje

$$\langle \cdot, \cdot \rangle : X \times X \longrightarrow X$$

sa osobinama

1) Aditivnost po prvoj promenljivoj

$$\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$$

2) Homogenost po prvoj promenljivoj

$$\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$$

3) HERMITSKA SIMETRIJA

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

4) POZITIVNA DEFINITNOST

$$\langle x, x \rangle \geq 0$$

5) NEDEGENERISANOST

$$\langle x, x \rangle = 0 \iff x = 0$$

N1 4) IMA SMISLA U KOMPLEKSNOM SLUČAJU ZBOG 3

$$\langle x, x \rangle = \overline{\langle x, x \rangle}$$

$$\Rightarrow \langle x, x \rangle \in \mathbb{R}$$

N2 VAŽI I ADITIVNOST PO DRUGOJ PROMENLJIVOJ

$$\langle x, y_1 + y_2 \rangle = \overline{\langle y_1 + y_2, x \rangle} =$$

$$= \overline{\langle y_1, x \rangle + \langle y_2, x \rangle} =$$

$$= \overline{\langle y_1, x \rangle} + \overline{\langle y_2, x \rangle} =$$

$$= \langle x, y_1 \rangle + \langle x, y_2 \rangle$$

N3

$$\langle x, \lambda y \rangle = \overline{\lambda} \langle x, y \rangle$$

$$\langle x, \lambda y \rangle = \overline{\langle \lambda y, x \rangle} = \overline{\lambda \langle y, x \rangle}$$

$$= \overline{\lambda} \, \overline{\langle y, x \rangle} = \overline{\lambda} \langle x, y \rangle$$

PAR $(X, \langle \cdot, \cdot \rangle)$ SE NAZIVA

UNITARNI ILI PREDHILBERTOV PROSTOR.

PRIMERI UNITARNIH PROSTORA!

1) $\mathbb{R}^n$

$$\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n$$

2) $\mathbb{C}^n$

$$\langle z, w\rangle = z_1 \overline{w_1} + \cdots + z_n \overline{w_n}$$

3) $\ell^2(\mathbb{C})$

PROSTOR KOMPLEKSNIH NIZOVA TAKVIH DA JE

$$\sum_{n=1}^{\infty} |z_n|^2 < \infty$$

$$\langle \{z_n\}, \{w_n\}\rangle = \sum_{n=1}^{\infty} z_n \overline{w_n}$$

4) $C[a,b]$

$$\langle f, g\rangle = \int_a^b f(t)\overline{g(t)}\,dt$$

OSOBINE SKALARNOG PROIZVODA!

1) $$\left\langle \sum_i \lambda_i x_i, \sum_j \nu_j y_j \right\rangle = \sum_i \sum_j \lambda_i \overline{\nu_j} \langle x_i, y_j\rangle$$

$$\left\langle \sum_i \lambda_i x_i, \sum_j \nu_j y_j \right\rangle = \sum_i \left\langle \lambda_i x_i, \sum_j \nu_j y_j \right\rangle$$

$$= \sum_i \sum_j \langle \lambda_i x_i, \nu_j y_j\rangle$$

$$= \sum_i \sum_j \lambda_i \overline{\nu_j} \langle x_i, y_j\rangle$$

2) $$\langle x+y, x+y\rangle = \langle x, x\rangle + 2\,\mathrm{Re}\langle x, y\rangle + \langle y, y\rangle$$

$$\langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle$$

$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$

$$= \langle x, x \rangle + \langle x, y \rangle + \overline{\langle x, y \rangle} + \langle y, y \rangle$$

$$= \langle x, x \rangle + 2 \operatorname{Re} \langle x, y \rangle + \langle y, y \rangle$$

3) $$\langle x+y, x-y \rangle = \langle x, x \rangle - 2i \operatorname{Im} \langle x, y \rangle - \langle y, y \rangle$$

4) $$4 \operatorname{Re} \langle x, y \rangle = \langle x+y, x+y \rangle - \langle x-y, x-y \rangle$$

5) $$\operatorname{Im} \langle x, y \rangle = \operatorname{Re} \langle x, iy \rangle$$

(DOKAZ DOMAĆI 3), 4), 5) - RAČUN)

X - VEKTORSKI PROSTOR NAD POLJEM $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

SKALARNI PROIZVOD

$$\langle \cdot , \cdot \rangle : X \times X \to \mathbb{K}$$

OSOBINE:

1) LINEARNOST PO I PROMJENLJIVOJ

$$\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$$

$$\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$$

2) HERMITSKA SIMETRIJA

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

3) POZITIVNA DEFINITNOST

$$\langle x, x \rangle \geq 0$$

4) NEDEGENERISANOST

$$\langle x, x \rangle = 0 \iff x = 0$$

ADITIVNOST PO II PROMJENLJIVOJ

$$\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2$$

$$\langle x, \beta y \rangle = \overline{\beta} \langle x, y \rangle$$

⋮

$(X, \langle \cdot, \cdot \rangle)$ — UNITARNI (PREDHILBERTOV) PROSTOR

PRIMERI:

1) $\mathbb{R}^n$

$$\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n$$

2) $\mathbb{C}^n$

$$\langle z, w \rangle = z_1 \overline{w_1} + \dots + z_n \overline{w_n}$$

3) $\ell^2(\mathbb{C})$

$\sum_{n=1}^{\infty} |z_n|^2$ — KONVERGIRA

$$\langle \{z_n\}, \{w_n\} \rangle = \sum_{n=1}^{\infty} z_n \overline{w_n}$$

4) $C[a, b]$

$$\langle f, g \rangle = \int_a^b f(t) \overline{g(t)} \, dt$$

⋮

① KOŠI–ŠVARCOVA NEJEDNAKOST

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

NAPOMENA:

JEDNAKOST VAŽI AKO I SAMO AKO SU VEKTORI x i y LINEARNO ZAVISNI

$x, y \in X$ – VEKTORI $\quad \xi \in \mathbb{R}$

$$0 \leq \langle x + \xi y, x + \xi y \rangle = \langle x, x + \xi y \rangle + \langle \xi y, x + \xi y \rangle =$$

$$= \langle x, x \rangle + \langle x, \xi y \rangle + \langle \xi y, x \rangle + \langle \xi y, \xi y \rangle =$$

$$= \langle x, x \rangle + \xi \langle x, y \rangle + \xi \overline{\langle x, y \rangle} + \xi^2 \langle y, y \rangle =$$

$$= \langle x,x\rangle + 2\xi \operatorname{Re}\langle x,y\rangle + \xi^2 \langle y,y\rangle \geq 0$$

$$\cancel{4} \operatorname{Re}\langle x,y\rangle^2 \leq \cancel{4}\langle x,x\rangle \langle y,y\rangle$$

AKO JE $\mathbb{K} = \mathbb{R}$ KRAJ DOKAZA

AKO JE $\mathbb{K} = \mathbb{C}$

$$\langle x,y\rangle = e^{i\theta} |\langle x,y\rangle|$$

$$|\langle x,y\rangle| = \langle x, e^{i\theta} y\rangle = \operatorname{Re}\langle x, e^{i\theta} y\rangle$$

$$\leq \sqrt{\langle x,x\rangle}\sqrt{\langle e^{i\theta}y, e^{i\theta}y\rangle}$$

$$= \sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle}$$

SPECIJALNO:

$$\left|\sum_{n=0}^{\infty} z_n \overline{w_n}\right| \leq \left(\sum_{n=0}^{\infty} |z_n|^2\right)^{1/2} \left(\sum_{n=0}^{\infty} |w_n|^2\right)^{1/2}$$

$$\left|\int_a^b f(t)\,\overline{g(t)}\,dt\right| \leq \left(\int_a^b f(t)^2\,dt\right)^{1/2} \left(\int_a^b |g(t)|^2\,dt\right)^{1/2}$$

---

(2) $(X, \langle\cdot,\cdot\rangle)$ - UNITARNI PROSTOR

$$\|x\| := \sqrt{\langle x,x\rangle}$$

DOKAZATI NEJEDNAKOST TROUGLA

$$\langle x+y, x+y\rangle = \langle x,x\rangle + 2\operatorname{Re}\langle x,y\rangle + \langle y,y\rangle$$

$$\leq \langle x,x\rangle + 2\sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle} + \langle y,y\rangle$$

$$\leq \left(\sqrt{\langle x,x\rangle} + \sqrt{\langle y,y\rangle}\right)^2$$

$$|\langle x,y\rangle| \leq \|x\| + \|y\|$$

③ JEDNAKOST PARALELOGRAMA ↗ $\|\cdot\|$ POTIČE IZ $\langle \cdot,\cdot \rangle$

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

$$+\begin{cases} \|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \cancel{\langle x,y\rangle} + \cancel{\langle y,x\rangle} + \|y\|^2 \\ \|x-y\|^2 = \langle x-y, x-y\rangle = \|x\|^2 - \cancel{\langle x,y\rangle} - \cancel{\langle y,x\rangle} + \|y\|^2 \end{cases}$$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

VAŽI I OBRAT TVRĐENJA:

$\|\cdot\|$ NAD $\mathbb{K}$ ZADOVOLJAVA JEDNAKOST PARALELOGRAMA $\Rightarrow$ POSTOJI SKALARNI PROIZVOD $\langle \cdot,\cdot \rangle$ TAKAV DA JE $\|x\|^2 = \langle x,x\rangle$

NAD $\mathbb{R}$

$$\langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$

NAD $\mathbb{C}$

$$\langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) + \frac{i}{4}\left(\|x+iy\|^2 - \|x-iy\|^2\right)$$

(PROVERITI)

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④ ORTOGONALNOST

X - UNITARNI PROSTOR

$x, y \in X$

x i y SU ORTOGONALNI AKO JE $\langle x,y\rangle = 0$

x i y SU ORTONORMIRANI AKO SU ORTOGONALNI i $\|x\| = \|y\| = 1$

PRIMER:

$$C[-\pi, \pi]$$

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt$$

1) Ortogonalni su vektori $\sin nx$ i $\sin mx$

$m, n \in \mathbb{N}$ $\quad m \neq n$

$$\langle \sin nx, \sin mx \rangle = \int_{-\pi}^{\pi} \sin nx\, \overline{\sin mx}\, dx$$

$$= \int_{-\pi}^{\pi} \frac{1}{2}\left(\cos(nx - mx) - \cos(nx + mx)\right) dx$$

$$= \frac{1}{2}\left(\int_{-\pi}^{\pi} \cos(n-m)x\, dx - \int_{-\pi}^{\pi} \cos(n+m)x\, dx\right)$$

$$= \frac{1}{2}\left(\frac{1}{n-m} \underbrace{\sin(n-m)x}_{\to 0}\Big|_{-\pi}^{\pi} - \frac{1}{n+m} \underbrace{\sin(n+m)x}_{\to 0}\Big|_{-\pi}^{\pi}\right) = 0$$

$$\langle \sin nx, \sin nx \rangle = \int_{-\pi}^{\pi} \sin^2 nx\, dx = \int_{-\pi}^{\pi} \frac{1 - \cos 2nx}{2}\, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} dx - \frac{1}{2}\int_{-\pi}^{\pi} \underbrace{\cos 2nx\, dx}_{\to 0} = \pi$$

$$\|\sin nx\| = \sqrt{\pi}$$

2) $\cos nx$ i $\cos mx$ su ortogonalni za

$n, m \in \mathbb{N}$ $\quad n \neq m$

$$\langle \cos nx, \cos mx \rangle = \int_{-\pi}^{\pi} \cos nx\, \cos mx\, dx$$

$$= \int_{-\pi}^{\pi} \frac{1}{2}\left(\cos(n-m)x + \cos(n+m)\right) dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \cos(n-m)x \, dx + \frac{1}{2} \int_{-\pi}^{\pi} \cos(n+m)x \, dx$$

$$= \frac{1}{2} \frac{1}{n-m} \cancel{\sin(n-m)x}^{0} \Big|_{-\pi}^{\pi} + \frac{1}{2} \frac{1}{n+m} \cancel{\sin(n+m)x}^{0} \Big|_{-\pi}^{\pi}$$

$$= 0$$

$$\langle \cos nx, \cos nx \rangle = \pi$$

3) $\cos nx$ $\sin mx$ su ortogonalni

$$\langle \sin mx, \cos nx \rangle = \int_{-\pi}^{\pi} \sin mx \cos nx \, dx$$

$$= \int_{-\pi}^{\pi} \frac{1}{2} \left(\sin(m+n)x + \sin(m-n)x \right) dx =$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \sin(m+n)x \, dx + \frac{1}{2} \int_{-\pi}^{\pi} \sin \underbrace{(m-n)}_{=0,\ m=n} x \, dx$$

$$= \frac{1}{2} \frac{1}{m+n} \left(- \cancel{\cos(m+n)x}^{0} \right) \Big|_{-\pi}^{\pi} + \frac{1}{2} \frac{1}{m-n} \left(\cancel{-\cos(m-n)x}^{0} \right) \Big|_{-\pi}^{\pi}$$

$$= 0$$

$\langle x, y \rangle = 0$ pišemo i $x \perp y$

S - skup

$x \perp S \iff x \perp y$, za $\forall y \in S$

$$S^{\perp} = \{ x \mid x \perp S \}$$

1) $S^{\perp}$ je vektorski prostor iz linearnosti $\langle \cdot , \cdot \rangle$

$x, y \in S^{\perp}$

$z \in S \qquad x \perp z \quad \text{i} \quad y \perp z$

$$\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle = 0$$

$$\langle \alpha x, z \rangle = \alpha \langle x, z \rangle = 0$$

$$x, y \in S^{\perp} \Rightarrow x+y \in S^{\perp} \qquad \alpha x \in S^{\perp}$$

2) $S^{\perp}$ JE ZATVOREN

OVO SLEDI IZ NEPREKIDNOSTI $\langle \cdot , \cdot \rangle$

NEPREKIDNOST SLEDI IZ KOŠI-ŠVARCA

$$|\langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle| = |\langle x_1 - x_2, y_1 \rangle - \langle x_2, y_1 - y_2 \rangle|$$

$$\leq \|x_1 - x_2\| \|y_1\| + \|x_2\| \|y_1 - y_2\|$$

$\Rightarrow$ NEPREKIDNOST

$\{y_n\}_{n \in \mathbb{N}}$ NIZ U $S^{\perp}$

$$\lim_{n \to \infty} y_n = y$$

$$\langle x, y \rangle = \langle x, \lim_{n \to \infty} y_n \rangle = \lim_{n \to \infty} \overbrace{\langle x, y_n \rangle}^{=0} = 0$$

$$\langle x, y \rangle = 0 \Rightarrow y \in S^{\perp}$$

$\Rightarrow S^{\perp}$ JE ZATVOREN

$S^{\perp}$ - ZATVOREN VEKTORSKI PROSTOR

$$((S^{\perp})^{\perp})^{\perp} = S^{\perp}$$

NE VAŽI UVEK

$$(S^{\perp})^{\perp} \cancel{=} S$$

u $\mathbb{R}^2$

$$S = \{(1,0)\}$$

$$S^{\perp} = \{(0,y) \mid y \in \mathbb{R}\}$$

$$(S^{\perp})^{\perp} = \{(x,0) \mid x \in \mathbb{R}\} \neq S$$

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⑤ PITAGORINA TEOREMA

X - UNITARNI VEKTORSKI PROSTOR

$x_1, \dots, x_n \in X$

MEĐUSOBNO ORTOGONALNI VEKTORI

$\langle x_i, x_j \rangle = 0$ za $i \neq j$

TADA VAŽI:

$$\left\| \sum_{k=1}^{n} x_k \right\|^2 = \sum_{k=1}^{n} \|x_k\|^2$$

$\langle x_k, x_j \rangle = 0$ za $k \neq j$

$$\left\langle \sum_{k=1}^{n} x_k, \sum_{j=1}^{n} x_j \right\rangle = \sum_{k=1}^{n} \sum_{j=1}^{n} \langle x_k, x_j \rangle$$

$$= \sum_{k=1}^{n} \langle x_k, x_k \rangle = \sum_{k=1}^{n} \|x_k\|^2$$

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⑥ HILBERTOVI PROSTORI

UNITARNI PROSTOR $(X, \langle \cdot, \cdot \rangle)$ SE NAZIVA HILBERTOVIM AKO JE KOMPLETAN METRIČKI PROSTOR U ODNOSU NA METRIKU

$$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y \rangle}$$

PRIMER:

$$L^2[a,b]$$

KOMPLETIRANJE PROSTORA $\left(C[a,b], \|\cdot\|_2\right)$

$$\|f\|_2 = \left(\int_a^b (f(t))^2\,dt\right)^{1/2}$$

$\|\cdot\|_2$ JE IZVEDENA IZ SKALARNOG PROIZVODA

$$\langle f,g\rangle = \int_a^b f(t)\,g(t)\,dt$$

U PROSTORIMA KONAČNE DIMENZIJE SVAKO LINEARNO PRESLIKAVANJE SE MOŽE ZADATI MATRICOM. HOĆEMO DA PROŠIRIMO NA BESKONAČNO DIMENZIONE.

TEOREMA: (RISOVA O REPREZENTACIJI LINEARNIH FUNKCIONALA)

X - HILBERTOV PROSTOR

X^* - NJEGOV DUALNI PROSTOR

(PROSTOR SVIH NEPREKIDNIH PRESLIKAVANJA $X \to \mathbb{K}$. OVO JE VEKTORSKI PROSTOR NAD $\mathbb{K}$)

ZA $\forall a^* \in X^*$ POSTOJI JEDINSTVEN VEKTOR $a \in X$ T.D.

$$a^*(x) = \langle x, a\rangle \qquad \forall x \in X$$

$$\phi : X^* \to X$$

$$a^* \longmapsto a$$

ϕ IMA SVOJSTVA

1) Антилинеарност

$$\phi(\lambda a^* + \mu b^*) = \bar{\lambda}\, \phi(a^*) + \bar{\mu}\, \phi(b^*)$$

2) Сурјективност

$$\phi(X^*) = X$$

3) Изометричност

$$\|\phi(a^*)\| = \|a^*\|$$

норма преко sup

Хилбертов простор је изоморфан свом дуалу.

У Банаховим просторима не важи (немамо скаларни производ), не постоји природно пресликавање $X \longrightarrow X^*$

Увек постоји:

$$\psi : X \longrightarrow (X^*)^* \qquad \psi(x)(x^*) = x^*(x)$$

али ψ не мора бити изоморфизам

$$X \subset (X^*)^*$$

Ако „=" простор се назива рефлексиван

(Хилбертови простори су рефлексивни)

Диференцирање у нормираним просторима

$$f(a+h) - f(a) = Df(a)\, h + o(h)$$

↑ линеарно

① X - normirani vektorski prostor nad $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

$V \subseteq X$

$a \in \operatorname{int} V$

$f, g : V \to \mathbb{K}$

$f \ \mathcal{D}(a) \qquad g \ \mathcal{D}(a)$

1) $fg \ \mathcal{D}(a)$

$$D(f \cdot g)(a) = f(a)\, Dg(a) + Df(a)\, g(a)$$

2) $g(a) \neq 0 \qquad \frac{f}{g} \ \mathcal{D}(a)$

$$D\left(\frac{f}{g}\right)(a) = \frac{Df(a)\, g(a) - f(a)\, Dg(a)}{g(a)^2}$$

1) $$(f \cdot g)(a+h) - (f \cdot g)(a) \overset{?}{=} L \cdot h + o(h)$$

$$(f \cdot g)(a+h) - (f \cdot g)(a) = f(a+h)\, g(a+h) - f(a)\, g(a) = (*)$$

$f, g \ \mathcal{D}(a)$

$$(*) = \big(f(a) + Df(a)\, h + o(h)\big)\big(g(a) + Dg(a)\, h + o(h)\big) - f(a)\, g(a)$$

$$= \cancel{f(a) g(a)} + f(a)\, Dg(a)\, h + \underbrace{f(a)\, o(h)}_{= o(h)} + Df(a)\, h\, g(a) + \underbrace{Df(a)\, Dg(a)\, h^2}_{= o(h)}$$

$$+ \underbrace{Df(a)\, h \cdot o(h)}_{= o(h)} + o(h)\,(\quad) - \cancel{f(a)\, g(a)}$$

$$= f(a)\, Dg(a)\, h + g(a)\, Df(a)\, h + o(h)$$

$$= \underbrace{\big(f(a)\, Dg(a) + g(a)\, Df(a)\big)}_{D(fg)(a)}\, h + o(h)$$

b) $g(a) \neq 0$

$$D\left(\frac{1}{g}\right)(a) = ? \qquad \frac{1}{g(a+h)} - \frac{1}{g(a)} = L \cdot h + o(h)$$

$$\frac{1}{g(a+h)} - \frac{1}{g(a)} = \frac{1}{g(a) + Dg(a)h + o(h)} - \frac{1}{g(a)} =$$

$$= \frac{\cancel{g(a)} - \cancel{g(a)} - Dg(a)h + o(h)}{g(a)^2 + g(a)Dg(a)h + o(h)} = \frac{-Dg(a)\, h}{g(a)^2 + g(a)Dg(a)h + o(h)} + o(h)$$

$$\parallel$$

$$\frac{-Dg(a)\, h}{g(a)^2} + o(h)$$

$$\frac{Dg(a)\, h}{g(a)^2} - \frac{Dg(a)\, h}{g(a)^2 + g(a)Dg(a)h + o(h)} =$$

$$\frac{\cancel{g(a)^2 Dg(a) h} + g(a)^2 Dg(a)^2 \underbrace{h^2}_{= o(h)} + o(h) - \cancel{g(a) Dg(a) h}}{g(a)^2 \left(g(a)^2 + g(a)Dg(a)h + o(h)\right)} = o(h)$$

$$D\left(\frac{1}{g}\right)(a) = \frac{-Dg(a)}{g(a)^2}$$

$$D\left(\frac{f}{g}\right)(a) = D\left(f \cdot \frac{1}{g}\right)(a) =$$

$$= f(a)\, D\left(\frac{1}{g}\right)(a) + Df(a)\frac{1}{g(a)}$$

$$= f(a)\frac{-Dg(a)}{g(a)^2} + \frac{Df(a)}{g(a)}$$

$$= \frac{g(a)Df(a) - f(a)Dg(a)}{g(a)^2}$$

(2) X, Y – NORMIRANI VEKTORSKI PROSTORI

$f : X \to Y$

f – LOKALNO KONSTANTNO PRESLIKAVANJE

$\forall a \in X$ $\exists\, \mathcal{U}$ – okolina tačke a $\quad f|_{\mathcal{U}} = C_{\mathcal{U}} \in Y$

$\Rightarrow \quad Df(x) = 0 \quad \forall x \in X$

$a \in X$ PROIZVOLJNA TAČKA

$\mathcal{U}(a) \qquad f|_{\mathcal{U}(a)} = C_{\mathcal{U}}$

IZABEREMO h DOVOLJNO MALO DA JE $\quad a+h \in \mathcal{U}(a)$

$0 = C_{\mathcal{U}} - C_{\mathcal{U}} = f(a+h) - f(a) = Df(a)\, h + o(h)$

$Df(a) = 0$

IZ JEDINSTVENOSTI DIFERENCIJALA SLEDI ZAKLJUČAK

VAŽI I OBRAT: $\forall x \in X \quad Df(x) = 0 \quad \Rightarrow$

$\Rightarrow f$ JE LOKALNO KONSTANTNO PRESLIKAVANJE

DA BI BILO GLOBALNO KONSTANTNO POTREBNA JE POVEZANOST.

---

(3) X, Y – NORMIRANI VEKTORSKI PROSTORI

$f : X \to Y$

f – NEPREKIDNO LINEARNO PRESLIKAVANJE

$\Rightarrow \quad Df(x) = f \quad \forall x \in X$

SLEDI IZ LINEARNOSTI f I JEDINSTVENOSTI DIFERENCIJALA

$$f(x+h) = f(x) + f(h) = f(x) + \underbrace{f(h)}_{Df(a)\,h} + o(h)$$

<u>primer:</u>

$$X = Y = \mathbb{R}$$

Linearno preslikavanje je oblika

$$f(x) = ax$$

$$f'(x) = a$$

$a = [a]$ - matrica linearnog preslikavanja

$$h \mapsto ah$$

---

(4) X, Y, Z - normirani vektorski prostori

$V \subseteq X$

$a \in \operatorname{int} V$

$f : V \to Y$

$L \in \mathcal{L}(Y, Z)$

f d (a)

$D(L \circ f)(a) = L \circ Df(a)$

$$(L \circ f)(a+h) - (L \circ f)(a) = L(f(a+h)) - L(f(a))$$

$$= L\big(f(a+h) - f(a)\big) = L\big(Df(a)h + o(h)\big)$$

$$= L(Df(a)\,h) + L(o(h))$$

$$= L(Df(a))\,h + L(o(h))$$

$$= (L \circ Df(a))\,h + L(o(h))$$

$$\|L(o(h))\| \leq \|L\|\,\|o(h)\| = o(h)$$

$$= \left(L \circ Df(a) \right) h + o(|h|)$$

$$\parallel$$

$$D(L \circ f)(a)$$

ПОДСЕЋАЊЕ:

ДИФЕРЕНЦИРАНО У НОРМИРАНОМ ПРОСТОРУ

$$f(a+h) - f(a) = Df(a)\cdot h + o(h)$$

ЛИНЕАРНО

- $f \ \mathcal{D}\ a$ i $g\ \mathcal{D}\ (a)$

  $(f\cdot g)\ \mathcal{D}\ (a)$ $\qquad D(f\cdot g)(a) = f(a)\,Dg(a) + Df(a)\cdot g(a)$

  $g \neq 0 \qquad \left(\frac{f}{g}\right) \mathcal{D}\ (a) \qquad D\left(\frac{f}{g}\right)(a) = \dfrac{Df(a)\,g(a) - Dg(a)\,f(a)}{g(a)^2}$

- f ЛОКАЛНО КОНСТАНТНО $\Rightarrow$ $Df(x) = 0 \quad \forall x$

- АКО ЈЕ f НЕПРЕКИДНО ЛИНЕАРНО ПРЕСЛИКАВАЊЕ

  $Df(x) = f$

- $f: V \to Y$

  $L \in \mathcal{L}(Y, Z)$

  $D(L\circ f)_{(a)} = L \circ Df(a)$

① $X, Y_1, \ldots, Y_k$ - NORMIRANI VEKTORSKI PROSTORI

$V \subseteq X$

$a \in \operatorname{Int} V$

$f : V \to Y_1 \times \ldots \times Y_k$

TADA VRIJEDI!

$f = (f_1, \ldots, f_k)$ Ⓓ$(a) \iff (\forall i \in \{1, \ldots, k\})$ f_i Ⓓ(a)

Dokaz:

$\Rightarrow$: f Ⓓ(a)

$$f(a+h) - f(a) = Df(a) \cdot h + o(h)$$

$$\| $$

$$(f_1(a+h) - f_1(a), \ldots, f_k(a+h) - f_k(a)) = Df(a)h + o(h)$$

$$\| $$

$(L_1, \ldots, L_k)$ L_i - linearan

$$f_1(a+h) - f_1(a) = L_1 \cdot h + o(h)$$

$$\vdots$$

$$f_k(a+h) - f_k(a) = L_k \cdot h + o(h)$$

$\Leftarrow$ $\forall i$ f_i Ⓓ(a)

$$f_i(a+h) - f_i(a) = Df_i(a)h + o(h)$$

$$f(a+h) - f(a) = (f_1(a+h), f_2(a+h), \ldots, f_k(a+h)) - (f_1(a), \ldots, f_k(a))$$

$$= (f_1(a+h) - f_1(a), \ldots, f_k(a+h) - f_k(a))$$

$$= (Df_1(a), \ldots, Df_k(a))h + o(h)$$

(2) $L \in \mathcal{L}(X_1, \dots, X_k; Y)$

L - K-LINEARNO PRESLIKAVANJE

$L: X_1 \times \dots \times X_k \to Y$

TADA ZA SVAKO:

$a = (a_1, \dots, a_k)$ | $h = (h_1, \dots, h_k)$

$$DL(a)\,h = L(h_1, a_2, \dots, a_k) + L(a_1, h_2, a_3, \dots, a_k) + \dots + L(a_1, a_2, \dots, h_k)$$

PRIMER: $k = 2$

$$L(a_1 + h_1, a_2 + h_2) - L(a_1, a_2) =$$

$$= L(a_1, a_2 + h_2) + L(h_1, a_2 + h_2) - L(a_1, a_2) =$$

$$= \cancel{L(a_1, a_2)} + L(a_1, h_2) + L(h_1, a_2) + L(h_1, h_2) - \cancel{L(a_1, a_2)} =$$

$$= L(h_1, a_2) + L(a_1, h_2) + \underbrace{L(h_1, h_2)}_{= o(h)} \qquad h = (h_1, h_2)$$

$$\|L(h_1, h_2)\| \leq \|L\|\, h^2$$

$$DL(a)\,h = L(h_1, a_2) + L(a_1, h_2)$$

$$L(a_1 + h_1, \dots, a_k + h_k) - L(a_1, \dots, a_k) =$$

$$= L(a_1, a_2 + h_2, \dots, a_k + h_k) + L(h_1, a_2 + h_2 + \dots + a_k + h_k) - L(a_1, \dots, a_k) =$$

$$= L(a_1, a_2, \dots, a_k + h_k) + L(a_1, h_2, a_3 + h_3 + \dots, a_k + h_k) - L(a_1, \dots, a_k)$$

$= \dots$

③ $X, Y_1, \dots, Y_k$ – normirani vektorski prostori

$\emptyset \neq V \subseteq X$

$a \in \operatorname{Int} V$

$f : V \to Y_1 \times \dots \times Y_k$

$f \; \mathcal{D}(a)$

$L \in \mathcal{L}(Y_1, \dots, Y_k ; Z)$

$\Rightarrow \quad (L \circ f) \; \mathcal{D}(a)$

$$D(L \circ f)(a) = L(Df_1(a), f_2(a), \dots, f_k(a))$$
$$+ \dots + L(f_1(a), f_2(a), \dots, Df_k(a))$$

$$(L \circ f)(a+h) - (L \circ f)(a) =$$

$$L(f(a+h)) - L(f(a)) =$$

$$L(f_1(a+h), f_2(a+h), \dots, f_k(a+h)) - L(f_1(a), \dots, f_k(a)) =$$

$$= L(Df_1(a)h + o(h), \dots, Df_k(a)h + o(h)) - L(f_1(a), \dots, f_k(a)) =$$

iz prethodnog zadatka

$$= L(Df_1(a), f_2(a), \dots, f_k(a)) + \dots + L(f_1(a), \dots, Df_k(a))$$

Primjeri:

1) $f_1, \dots, f_k : \langle a-\delta, a+\delta \rangle \to \mathbb{C}$

$f_i \; \mathcal{D}(a) \quad \forall i \in \{1, \dots, k\}$

$L : \mathbb{C}^k \to \mathbb{C}$

$L(z_1, \dots, z_k) = z_1 \cdots z_k$

$$(f_1 f_2 \cdots f_k)'(a) = f_1'(a) f_2(a) \cdots f_k(a) + f_1(a) f_2'(a) \cdots f_k(a) + \dots + f_1(a) f_2(a) \cdots f_k'(a)$$

2) $(Y, \langle \cdot, \cdot \rangle)$ - UNITARNI PROSTOR

$f_1, f_2 : V \to Y$

$V \subseteq X \qquad a \in \operatorname{Int} V$

$f_1, f_2 \; \mathcal{D}(a)$

$$\frac{d}{dx}\Big|_{x=a} \langle f_1(x), f_2(x) \rangle = \langle f_1'(a), f_2(a) \rangle + \langle f_1(a), f_2'(a) \rangle$$

3) VEKTORSKI PROIZVOD

$f_1, f_2 : V \to \mathbb{R}^3$

$f_1, f_2 \; \mathcal{D}(a) \qquad a \in \operatorname{Int} V \;\; \Rightarrow$

$\times$ - VEKTORSKI PROIZVOD

$$(f_1 \times f_2)'(a) = f_1'(a) \times f_2(a) + f_1(a) \times f_2'(a)$$

4) MEŠOVITI PROIZVOD

$f_1, f_2, f_3 : V \to \mathbb{R}^3$

$V \subseteq X \qquad a \in \operatorname{Int} V$

$f_1, f_2, f_3 \; \mathcal{D}(a)$

$[\cdot, \cdot, \cdot]$ - MEŠOVITI PROIZVOD

$$[f_1, f_2, f_3]'(a) = [f_1'(a), f_2(a), f_3(a)] + [f_1(a), f_2'(a), f_3(a)] + [f_1(a), f_2(a), f_3'(a)]$$

5) DETERMINANTA

$f_1, \dots, f_k : V \to \mathbb{C}^k \qquad f_i \in \mathcal{D}(a) \; \forall i$

$[f_1(x), \dots, f_k(x)]_{k \times k}$

$$\frac{d}{dx}\Big|_{x=a} [f_1(x), \ldots, f_k(x)] =$$

$$= [Df_1(x), f_2(x), \ldots, f_k(x)] +$$

$$+ [f_1(x), Df_2(x), \ldots, f_k(x)] +$$

$$\vdots$$

$$+ [f_1(x), f_2(x), \ldots, Df_k(x)]$$

Primer: $k=2$

$$A(x) = \begin{vmatrix} a(x) & b(x) \\ c(x) & d(x) \end{vmatrix} = a(x)\,d(x) - b(x)\,c(x)$$

$$A'(x) = \underline{a'(x)\,d(x)} + \overline{a(x)\,d'(x)} - \overline{b'(x)\,c(x)} - \underline{b(x)\,c'(x)}$$

$$= \begin{vmatrix} a'(x) & b(x) \\ c'(x) & d(x) \end{vmatrix} + \begin{vmatrix} a(x) & b'(x) \\ c(x) & d'(x) \end{vmatrix}$$

---

(4) X, Y – Banahovi prostori

$$G := \{ L \in \mathcal{L}(X,Y) \mid \exists L^{-1} \in \mathcal{L}(Y,X) \}$$

G – invertibilna linearna preslikavanja

a) G je otvoren skup

b) Naći izvod preslikavanja

$$f : G \longrightarrow \mathcal{L}(Y, X)$$

$$f(L) = L^{-1}$$

a)

$\cdot\ L \in G$

$\varepsilon > 0 \qquad \|A\| < \varepsilon \implies L - A \in G$

SETIMO SE DA VRIJEDI:

$$(1-g)^{-1} = \sum_{n=0}^{\infty} g^n \qquad \|g\| < 1$$

$$L^{-1} \Big\backslash \left(1 - AL^{-1}\right)^{-1} = \sum_{n=0}^{\infty} \left(AL^{-1}\right)^n$$

$$L^{-1}\left(1 - AL^{-1}\right)^{-1} = L^{-1} \sum_{n=0}^{\infty} \left(AL^{-1}\right)^n$$

$$\underbrace{(L - A)^{-1}}_{\text{INVERZ OD } L-A} = L^{-1} \sum_{n=0}^{\infty} \left(AL^{-1}\right)^n$$

AKO OVO KONVERGIRA

$$\|AL^{-1}\| < 1$$

$$\|A\| < \frac{1}{\|L^{-1}\|}$$

$$B\left(L, \frac{1}{\|L^{-1}\|}\right) \in G$$

KAKO JE L PROIZVOLJNO G JE OTVOREN

b)

$$f(L) = L^{-1}$$

$$f(L+h) - f(L) \stackrel{?}{=} Df \cdot h + \omega(h)$$

$$f(L+h) - f(L) = \frac{1}{L+h} - \frac{1}{L}$$

$$= (L+h)^{-1} - L^{-1} =$$

$$= \left(L\left(1 + L^{-1}h\right)\right)^{-1} - L^{-1} =$$

$$= L^{-1} \underbrace{\left(1 + L^{-1}h\right)}^{-1} - L^{-1} =$$

$$= \sum_{n=0}^{\infty} (-L^{-1}h)^n L^{-1} - L^{-1}$$

izdvojimo sabirke $n=0$ i $n=1$

$$= \cancel{L^{-1}} - L^{-1}hL^{-1} + \sum_{n=2}^{\infty} (-L^{-1}h)^n - \cancel{L^{-1}}$$

$$= -L^{-1}hL^{-1} + \underbrace{\sum_{n=2}^{\infty} (-L^{-1}h)^n}_{= o(|h|)}$$

$$\left\| \sum_{n=2}^{\infty} (-L^{-1}h)^n \right\| = \sum_{n=2}^{\infty} \|L^{-1}h\|^n \le \frac{\|L^{-1}\| \, \|h\|^2}{1 - \|L^{-1}\|} = o(h)$$

$$f(L) = L^{-1}$$

$$Df\,h = -L^{-1}hL^{-1}$$

Specijalno:

X - normirani vektorski prostor

Glavna linearna grupa

$$GL(X) = \{ L \in \mathcal{L}(X,X) \mid \exists L^{-1} \in \mathcal{L}(X,X) \}$$

X - Banachov

$GL(X)$ je otvoren u $\mathcal{L}(X,X)$

$$f : GL(X) \to GL(X)$$

$$f : L \to L^{-1}$$

$$Df(L)h = -L^{-1}hL^{-1}$$

Ovo je uopćenje iz A1:

$$f : \mathbb{R}_+ \to \mathbb{R}_+ \qquad f(x) = x^{-1} = \frac{1}{x}$$

$$f'(x) = \frac{-1}{x^2}$$

$$f'(x)\,h = \frac{-1}{x^2}\,h = \underbrace{-\frac{1}{x}\; h\; \frac{1}{x}}$$

---

5. $L \in \mathcal{L}(X, X)$ – LINEARNI OPERATOR

L JE NILPOTENTAN AKO $\exists k \in \mathbb{N}$ $L^k = 0$

DOKAZATI:

L - NILPOTENTAN $\Rightarrow$ $\mathbb{1} - L \in GL(X)$

$$L^k = 0 \Rightarrow \forall n > k \quad L^n = 0$$

1) $S_n = \mathbb{1} + \cancel{L} + \dots + \cancel{L^n}$ $\quad n > k$

$$L S_n = \cancel{L} + \cancel{L^2} + \dots + \cancel{L} + L^{n+1}$$

$$(\mathbb{1} - L) S_n = \mathbb{1} - \cancel{L^{n+1}}^{\,=0} \qquad n+1 > k \qquad L^{n+1} = 0$$

$$(\mathbb{1} - L)^{-1} = S_n = \sum_{i=0}^{k-1} L^i$$ – KONAČNA SUMA KONVERGIRA

POSTOJI $(\mathbb{1} - L)^{-1} \Rightarrow \mathbb{1} - L \in GL(X)$

PRIMER:

OPERATOR DIFERENCIRANJA:

$$D : \mathbb{K}_p[x] \longrightarrow \mathbb{K}_p[x]$$

---

6. NAĆI IZVOD PRESLIKAVANJA

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$f(x, y) = (-y, x)$$

$$f(x+h_1, y+h_2) - f(x,y) =$$

$$= (-(y+h_2), x+h_1) - (-y, x) =$$

$$= (-h_2, h_1) = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{Df} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = Df \, h + o(h)$$

POZNATO NAM JE: $f: \mathbb{R}^n \to \mathbb{R}^m$

f JE DIF U $x \in \mathbb{R}^n$ AKO

$$f(x+h) - f(x) = L\,h + o(h)$$

LINEARNO PRESLIKAVANJE
MOŽEMO ZADATI MATRICOM

$$f: \mathbb{R}^n \to \mathbb{R}^m$$

$$f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), f_2(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$$

JAKOBIJEVA MATRICA

$$L(x) = Df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & & & \vdots \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1}(x) & \dots & \dots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}$$

AKO JE $m = n$ POSTOJI DETERMINANTA
JAKOBIJEVE MATRICE I NAZIVA SE JAKOBIJAN.

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$g : \mathbb{R}^m \longrightarrow \mathbb{R}^k$

f Ⓓ (x_0)

g Ⓓ $(f(x_0))$

$\Rightarrow (g \circ f)$ Ⓓ (x_0), VAŽI,

$$\underset{k \times n}{D(g \circ f)(x_0)} = \underset{k \times m}{D(g(f(x_0))} \cdot \underset{m \times n}{Df(x_0)}$$

---

① PRELAZAK NA POLARNE KOORDINATE U $\mathbb{R}^2$

$$A = \{ (\rho, \varphi) \in \mathbb{R}^2 \mid \rho > 0 \quad 0 \leq \varphi < 2\pi \}$$

$$f(\rho, \varphi) = (\underbrace{\rho \cos\varphi}_{f1}, \underbrace{\rho \sin\varphi}_{f2})$$

$$df = \begin{bmatrix} \frac{\partial f1}{\partial \rho} & \frac{\partial f1}{\partial \varphi} \\ \frac{\partial f2}{\partial \rho} & \frac{\partial f2}{\partial \varphi} \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\varphi & -\rho \sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{bmatrix}$$

J. JAKOBIJAN

$$J = \det \begin{bmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{bmatrix} = \rho\cos^2\varphi + \rho\sin^2\varphi = \rho$$

$$J = \rho$$

(2) CILINDRIČNE KOORDINATE

$$f((\rho, \varphi, z)) = (\rho\cos\varphi, \rho\sin\varphi, z))$$

$$df = \begin{bmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad J = \rho$$

---

(3) SFERNE KOORDINATE

$$f: \mathbb{R}^3 \to \mathbb{R}^3$$

$$\rho > 0 \qquad 0 \le \varphi \le \pi \qquad 0 \le \theta < 2\pi$$

$$f(\rho, \varphi, \theta) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$$

$$df = \begin{bmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{bmatrix}$$

$$\det(df) = \boxed{\rho^2\sin\varphi} = J$$

---

(4) $f: \mathbb{R}^3 \to \mathbb{R}$

$u: \mathbb{R}^2 \to \mathbb{R}$

$$u(x,y) = f(x^2+y^2, x^2-y^2, 2xy)$$

NAĆI TOTALNI DIFERENCIJAL FUNKCIJE u

$u: \mathbb{R}^2 \to \mathbb{R}$

$$du = \left[\frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y} \right]$$

$$u = (f \circ g)(x, y)$$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "g"] \arrow[rr, bend right, "u"] & \mathbb{R}^3 \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$$g(x, y) = (x^2 + y^2,\ x^2 - y^2,\ 2xy)$$

$$f : \mathbb{R}^3 \to \mathbb{R}$$

$$\frac{\partial f}{\partial x_1} \qquad \frac{\partial f}{\partial x_2} \qquad \frac{\partial f}{\partial x_3}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(f(\overbrace{x^2 + y^2}^{x_1},\ \overbrace{x^2 - y^2}^{x_2},\ \overbrace{2xy}^{x_3})\right)$$

$$= \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial x} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial x} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial x}$$

$$= 2x\frac{\partial f}{\partial x_1} + 2x\frac{\partial f}{\partial x_2} + 2y\frac{\partial f}{\partial x_3}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(f(x^2 + y^2,\ x^2 - y^2,\ 2xy)\right)$$

$$= 2y\frac{\partial f}{\partial x_1} - 2y\frac{\partial f}{\partial x_2} + 2x\frac{\partial f}{\partial x_3}$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$du = \left(2x\frac{\partial f}{\partial x_1} + 2x\frac{\partial f}{\partial x_2} + 2y\frac{\partial f}{\partial x_3}\right)dx + \left(2y\frac{\partial f}{\partial x_1} - 2y\frac{\partial f}{\partial x_2} + 2x\frac{\partial f}{\partial x_3}\right)dy$$

II Način

$$u = f \circ g$$

$$Du(x) = Df(g(x)) \cdot Dg(x)$$

$$Df(g(x)) = \left[\frac{\partial f}{\partial x_1} g(x) \quad \frac{\partial f}{\partial x_2} g(x) \quad \frac{\partial f}{\partial x_3} g(x) \right]$$

$$g(x, y) = (x^2 + y^2 \ , x^2 - y^2 \ , 2xy)$$

$$Dg(x) = \begin{bmatrix} 2x & 2y \\ 2x & -2y \\ 2y & 2x \end{bmatrix}$$

$$du(x) = Du(x) = \left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \frac{\partial f}{\partial x_3} \right] \begin{bmatrix} 2x & 2y \\ 2x & -2y \\ 2y & 2x \end{bmatrix}$$

$$= \left[2x \frac{\partial f}{\partial x_1} + 2x \frac{\partial f}{\partial x_2} + 2y \frac{\partial f}{\partial x_3} \ , \ 2y \frac{\partial f}{\partial x_1} - 2y \frac{\partial f}{\partial x_2} + 2x \frac{\partial f}{\partial x_3} \right]$$

IZVODI VIŠEG REDA

DEFINICIJA:

$$f: \underset{\mathbb{R}^n}{\overset{\cap}{A}} \longrightarrow \mathbb{R}$$

a) AKO POSTOJI $\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(a)$ NAZIVAMO GA DRUGI PARCIJALNI IZVOD U TAČKI a

OZNAKE:

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(a) \qquad i \neq j$$

$$\frac{\partial^2 f}{\partial x_i^2}(a) \qquad i = j$$

b) n-TI IZVOD SE DEFINIŠE INDUKTIVNO, KAO IZVOD n-1 IZVODA.

(1)

$$f(x,y) = \arcsin\sqrt{\frac{x}{y}}$$

NAĆI MEŠOVITE DRUGE PARCIJALNE IZVODE

$$\frac{\partial^2 f}{\partial y\, \partial x} \qquad \frac{\partial^2 f}{\partial x\, \partial y}$$

– ŠTA JE DOMEN f

1) $y \neq 0$

2) $\frac{x}{y} \geq 0 \qquad (x \geq 0 \text{ i } y > 0) \text{ ili } (x < 0 \text{ i } y < 0)$

3) $\sqrt{\frac{x}{y}} \leq 1$

$\left|\frac{x}{y}\right| \leq 1 \qquad |x| \leq |y|$

Domen:

$\{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq y\} \cup \{ y \leq x \leq 0\} \setminus \{(0,0)\}$

Radimo za $0 \leq x < y$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\arcsin\sqrt{\frac{x}{y}}\right) = \frac{1}{\sqrt{1-\frac{x}{y}}}\,\frac{1}{2\sqrt{\frac{x}{y}}}\cdot\frac{1}{y} =$$

$$= \frac{1}{2\,\frac{\sqrt{y-x}}{\sqrt{y}}\,\frac{\sqrt{x}}{\sqrt{y}}\,y} = \frac{1}{2\sqrt{x}\sqrt{y-x}}$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{2\sqrt{x}\sqrt{y-x}}\right) =$$

$$= \frac{1}{2\sqrt{x}}\,\frac{\partial}{\partial y}\left(\sqrt{y-x}\right)^{-\frac{1}{2}} = \frac{-1}{4\sqrt{x}}\,(y-x)^{-\frac{3}{2}} = \frac{-1}{4\sqrt{x}\sqrt{(y-x)^3}}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\arcsin\sqrt{\frac{x}{y}}\right) = \frac{1}{\sqrt{1-\frac{x}{y}}}\,\frac{1}{2\sqrt{\frac{x}{y}}}\,\frac{-x}{y^2} =$$

$$= \frac{-x}{2\,\frac{\sqrt{y-x}}{\cancel{\sqrt{y}}}\,\frac{\sqrt{x}}{\cancel{\sqrt{y}}}\,y^{\cancel{2}}} = \frac{-\sqrt{x}}{2\sqrt{y-x}\;y}$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-\sqrt{x}}{2\sqrt{y-x}\;y}\right) =$$

$$= -\frac{1}{2y}\,\frac{\frac{1}{2\sqrt{x}}\sqrt{y-x} - \sqrt{x}\,\frac{1}{2\sqrt{y-x}}(-1)}{y-x}\;\frac{\sqrt{y-x}}{\sqrt{y-x}} =$$

$$= -\frac{1}{4y}\,\frac{\frac{y-x}{\sqrt{x}} + \sqrt{x}}{\sqrt{(y-x)^3}}\cdot\frac{\sqrt{x}}{\sqrt{x}} = -\frac{1}{4\cancel{y}}\,\frac{\cancel{y}\cancel{-x}+\cancel{x}}{\sqrt{(y-x)^3}\;\sqrt{x}} = \frac{-1}{4\sqrt{x}\sqrt{(y-x)^3}}$$

DOBIJA SE : $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

DA LI TO UVEK VAŽI ?

2

$$f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2} & , \ (x,y) \neq (0,0) \\ 0 & , \ (x,y) = (0,0) \end{cases}$$

POKAZATI DA POSTOJE MEŠOVITI DRUGI PARCIJALNI IZVODI U TAČKI $(0,0)$, ALI DA SE RAZLIKUJU.

$(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left((xy)\cdot\left(\frac{x^2-y^2}{x^2+y^2}\right)\right) = y\,\frac{x^2-y^2}{x^2+y^2} + xy\,\frac{2x(x^2+y^2)-(x^2-y^2)2x}{(x^2+y^2)^2}$$

$$= y\,\frac{x^2-y^2}{x^2+y^2} + xy\,\frac{\cancel{2x^3}+2xy^2-\cancel{2x^3}+2xy^2}{(x^2+y^2)^2}$$

$$= y\,\frac{x^2-y^2}{x^2+y^2} + \frac{4x^2y^3}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = x\,\frac{x^2-y^2}{x^2+y^2} - \frac{4y^2x^3}{(x^2+y^2)^2} \qquad \text{(IZ SIMETRIJE)}$$

$(x,y) = (0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h\cdot 0\,\frac{h^2-0^2}{h^2+0^2}}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = 0$$

$$\frac{\partial f}{\partial x} = \begin{cases} y\,\dfrac{x^2-y^2}{x^2+y^2} + \dfrac{4x^2y^3}{(x^2+y^2)^2} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial y\,\partial x}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) =$$

$$= \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h\to 0}\frac{h\,\frac{0^2-h^2}{0^2+h^2} + \cancelto{0}{\frac{4\,0^3h^3}{(0^2+h^2)^2}} - 0}{h} = -1$$

$$\frac{\partial^2 f}{\partial y\,\partial x}(0,0) = -1$$

$$\frac{\partial f}{\partial y} = \begin{cases} x\,\dfrac{x^2-y^2}{x^2+y^2} - \dfrac{4x^3y^2}{(x^2+y^2)^2} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \lim_{h\to 0}\frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h}$$

$$= \lim_{h\to 0}\frac{h\,\frac{h^2-0^2}{h^2+0^2} - \cancelto{0}{\frac{4h^3 0^2}{(h^2+0^2)^2}} - 0}{h} = 1$$

$$\frac{\partial^2 f}{\partial x\,\partial y}(0,0) = 1 \neq -1 = \frac{\partial^2 f}{\partial y\,\partial x}$$

<u>TEOREMA:</u>

$$f: A \to \mathbb{R} \qquad A \in \tau_{\mathbb{R}^n}$$

AKO f IMA MEŠOVITE DRUGE PARCIJALNE IZVODE U TAČKI a I NEPREKIDNI SU U a TADA SU I JEDNAKI

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a)$$

POSLEDICA!

$$f \in C^n(A)$$

$\dfrac{\partial^n f}{\partial x_{i_1} \ldots \partial x_{i_n}}(a)$ NE MENJA PRI PROIZVOLJNOJ PERMUTACIJI INDEKSA.

DOMAĆI:

1. $f(x,y) = y^x$

2. $f(x,y) = \begin{cases} xy, & |y| \leq |x| \\ -xy, & |y| > |x| \end{cases}$

U $(0,0)$ SE RAZLIKUJU MEŠOVITI PARCIJALNI IZVODI DRUGOG REDA

PARCIJALNI IZVODI SLOŽENE FUNKCIJE

$\theta : \mathbb{R}^2 \to \mathbb{R}^2$

$f : \mathbb{R}^2 \to \mathbb{R}$

$\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$

$$\tilde{f} = f \circ \theta$$

TRAŽIMO PARCIJALNE IZVODE FUNKCIJE $\tilde{f}$!

$$\begin{tikzcd} (x,y) \arrow[r, mapsto] \arrow[rr, bend right, "\tilde{f}"'] & (\xi(x,y), \eta(x,y)) \arrow[r, mapsto, "f"] & \mathbb{R} \end{tikzcd}$$

$$\tilde{f}' = \begin{bmatrix} \frac{\partial \tilde{f}}{\partial x} & \frac{\partial \tilde{f}}{\partial y} \end{bmatrix} \qquad \theta' = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix}$$

$$f' = \begin{bmatrix} \frac{\partial f}{\partial \xi} & \frac{\partial f}{\partial \eta} \end{bmatrix}$$

$$\tilde{f} = f \circ \theta$$

$$\tilde{f}' = f' \cdot \theta'$$

$$\begin{bmatrix} \frac{\partial \tilde{f}}{\partial x} & \frac{\partial \tilde{f}}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial \xi} & \frac{\partial f}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}, & \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} \end{bmatrix}$$

$$\boxed{\begin{aligned} \frac{\partial \tilde{f}}{\partial x} &= \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial \tilde{f}}{\partial y} &= \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} \end{aligned}}$$

DRUGI PARCIJALNI IZVODI $\tilde{f}$!

$$\frac{\partial^2 \tilde{f}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \tilde{f}}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}\right) =$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}\right) =$$

$$= \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \xi}\right)}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \eta}\right)}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}$$

[illegible]

POSMATRAMO $\frac{\partial f}{\partial \xi}$ i $\frac{\partial f}{\partial \eta}$ KAO NOVE FUNKCIJE

$$\begin{tikzcd}
(x,y) \arrow[r, mapsto, "\theta"] \arrow[rr, bend right, "u"'] & (\xi(x,y), \eta(x,y)) \arrow[r, mapsto, "\frac{\partial f}{\partial \xi}"] & \frac{\partial f}{\partial \xi}(\xi(x,y), \eta(x,y))
\end{tikzcd}$$

$$u = \frac{\partial f}{\partial \xi} \circ \theta \qquad u' = \left(\frac{\partial f}{\partial \xi}\right)' \cdot \theta'$$

$$u' = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{bmatrix}$$

$$\left(\frac{\partial f}{\partial \xi}\right)' = \left[\frac{\partial}{\partial \xi}\left(\frac{\partial f}{\partial \xi}\right), \frac{\partial}{\partial \eta}\left(\frac{\partial f}{\partial \xi}\right)\right] = \left[\frac{\partial^2 f}{\partial \xi^2} \quad \frac{\partial^2 f}{\partial \eta \partial \xi}\right]$$

$$\theta' = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix}$$

$$\frac{\partial u}{\partial x} = \left[\frac{\partial^2 f}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 f}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial x}, \underbrace{\frac{\partial^2 f}{\partial \xi^2}\frac{\partial \xi}{\partial y} + \frac{\partial^2 f}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial y}}_{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \xi}\right)} \right]$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \xi}\right) = \frac{\partial^2 f}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 f}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial^2 \tilde{f}}{\partial x^2} = \frac{\partial^2 f}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + \underline{\frac{\partial^2 f}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x}} + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} +$$
$$+ \underline{\frac{\partial^2 f}{\partial \eta \partial \xi}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}} + \frac{\partial^2 f}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2$$

$$\boxed{\frac{\partial \tilde{f}}{\partial x^2} = \frac{\partial^2 f}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 f}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 f}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}}$$

$$\boxed{\frac{\partial \tilde{f}}{\partial y^2} = \frac{\partial^2 f}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 f}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 f}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}}$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 f}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 f}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial x} + \frac{\partial^2 f}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}}$$

$$+\frac{\partial f}{\partial \xi}\frac{\partial^2 \xi}{\partial x\partial y}+\frac{\partial f}{\partial \eta}\frac{\partial^2 \eta}{\partial x\partial y}$$

(1) $$x^2\frac{\partial^2 f}{\partial x^2}-2xy\frac{\partial^2 f}{\partial x\partial y}-3y^2\frac{\partial^2 f}{\partial y^2}$$

TRANSFORMISATI DATI IZRAZ UVODEĆI NOVE PROMENLJIVE u i v.

$$u=\frac{x}{y}\qquad v=x^3y$$

$y\neq 0$ DA BI u BILO DOBRO DEFINISANO.

$J\neq 0$

$$J=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{1}{y} & \frac{-x}{y^2}\\ 3x^2y & x^3\end{vmatrix}=\frac{x^3}{y}+\frac{3x^3}{y}=\frac{4x^3}{y}$$

$J\neq 0 \iff x\neq 0$

SMENA JE DOBRA U OBLASTIMA KOJE NE SEKU KOORDINATNE OSE.

$$\underbrace{(x,y)\xmapsto{\theta}(u(x,y),v(x,y))\xmapsto{g}g(u,v)}_{f}$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 g}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2+2\frac{\partial^2 g}{\partial u\partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial^2 g}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2+\frac{\partial g}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial g}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$u=\frac{x}{y}\qquad v=x^3y$$

$$\frac{\partial u}{\partial x}=\frac{1}{y}\qquad \frac{\partial v}{\partial x}=3x^2y\qquad \frac{\partial^2 u}{\partial x^2}=0\qquad \frac{\partial^2 v}{\partial x^2}=6xy$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{1}{y^2}\frac{\partial^2 g}{\partial u^2}+6x^2\frac{\partial^2 g}{\partial u\partial v}+9x^4y^2\frac{\partial^2 g}{\partial v^2}+6xy\frac{\partial g}{\partial v}$$

ZAMIJENITI NAZIVE

$$= 16\, uv \frac{\partial^2 g}{\partial u \partial v} + 4u \frac{\partial g}{\partial u}$$

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② NAĆI SVE FUNKCIJE

$$f : \mathbb{R}^2 \to \mathbb{R}$$

TD.

$$d^3 f = 0$$

(SVI TREĆI PARCIJALNI IZVODI 0)

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial^3 f}{\partial y^3} = 0$$

$$\frac{\partial^3 f}{\partial x^3} = 0 \quad \Rightarrow \quad \left.\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= A(y) \\ \frac{\partial^3 f}{\partial y \partial x^2} &= 0 \end{aligned}\right\} \Rightarrow A \in \mathbb{R}$$

$$\frac{\partial^2 f}{\partial x^2} = A$$

$$\frac{\partial f}{\partial x} = Ax + B(y)$$

$$f = \frac{A}{2} x^2 + x B(y) + C(y)$$

$$\frac{\partial f}{\partial y} = x \cdot B'(y) + C'(y)$$

$$\frac{\partial^2 f}{\partial y \partial x} = B'(y) \qquad \frac{\partial^3 f}{\partial y^2 \partial x} = B''(y) = 0$$

$$B(y) = Ey + D$$

$$f(x,y) = \frac{A}{2}x + x(Ey + D) + C(y)$$

$$\frac{\partial^3 f}{\partial y^3} = C'''(y) = 0$$

$$C(y) = \frac{1}{2}Fy^2 + Gy + H$$

$$\boxed{f(x,y) = \alpha x^2 + \beta xy + \gamma y^2 + \delta x + \varepsilon y + \eta}$$

KVADRATNE FORME

$$\Phi(h_1,\dots,h_m) = \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}h_ih_j$$

SVAKOJ KVADRATNOJ FORMI PRIDRUŽUJEMO MATRICU

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix}$$

Φ JE POZITIVNO POLUDEFINITNA $\Leftrightarrow \Phi(h_1,\dots,h_m) \geq 0 \quad \forall (h_1,\dots,h_m) \in \mathbb{R}^n$

Φ JE NEGATIVNO —||— $\Leftrightarrow \Phi(h_1,\dots,h_m) \leq 0$ —||—

Φ JE POZITIVNO DEFINITNA $\Leftrightarrow \Phi(h_1,\dots,h_m) > 0$ —||—

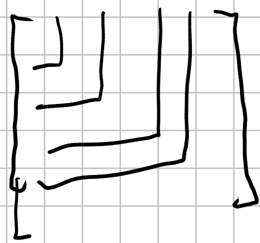
Φ JE NEGATIVNO DEFINITNA $\Leftrightarrow \Phi(h_1,\dots,h_m) < 0$ —||—

SILVESTEROV KRITERIJUM:

1) AKO SU SVI GLAVNI MINORI MATRICE KVADRATNE FORME STROGO POZITIVNI

$a_{11} > 0$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$$



$\vdots$

$$\begin{vmatrix} a_{11} & & \\ & \ddots & \\ a & & a_{nn} \end{vmatrix} > 0$$

Онда је ϕ позитивно дефинитна

2°

Ако је $a_{11} < 0$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$$

мењају наизменично знак

онда је ϕ негативно дефинитна -

Локални екстремуми

$f : A \to \mathbb{R}$

$A \subseteq \mathbb{R}^n$

$a \in A$ и f дефинисана у околини $U(a)$

f у тачки a има локални максимум $\Leftrightarrow f(x) \le f(a)\ \forall x \in U$

минимум $\Leftrightarrow f(x) \ge f(a)\ \forall x \in U$

Теорема:

$f : A \to \mathbb{R}$

$A \in \tau_{\mathbb{R}^n}$

f има парцијалне изводе у тачки a

AKO IMA LOKALNI EKSTREMUM U TAČKI a

ONDA SU SVI PARCIJALNI IZVODI 0

$$\frac{\partial f}{\partial x_1}(a) = \ldots = \frac{\partial f}{\partial x_n}(a) = 0$$

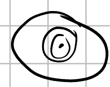
$\downarrow$

STACIONARNE (KRITIČNE) TAČKE

Lokalni ekstremumi

Definicija:

$$f : \underset{\mathbb{R}^n}{\overset{\cap}{A}} \to \mathbb{R}$$

f definisana u $\mathcal{U}(a) \subseteq A$ 

f ima lokalni minimum u a ako $f(x) \geq f(a)$ $x \in \mathcal{U}$

f ima lokalni maksimum u a ako $f(x) \leq f(a)$ $x \in \mathcal{U}$

f ima strogi lokalni minimum u a ako $f(x) > f(a)$ $x \in \mathcal{U}\setminus\{a\}$

f ima strogi lokalni maksimum u a ako $f(x) < f(a)$ $x \in \mathcal{U}\setminus\{a\}$

Teorema:

A - otvoren

f ima parcijalne izvode u $a \in A$

f ima lokalni ekstremum u $a \in A$

$$\frac{\partial f}{\partial x_1}(a) = \frac{\partial f}{\partial x_2}(a) = \dots = \frac{\partial f}{\partial x_n}(a) = 0$$

$f(x) = x^3$ $\quad f'(x) = 3x^2$ $\quad f'(0) = 0$

0 nije ekstremum

TEOREMA:

$$f \mapsto d^2 f(a)$$

MATRICA IZVOD DRUGOG REDA

$$d^2 f(a) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(a) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(a) \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(a) & & \cdots & \frac{\partial^2 f}{\partial x_n^2}(a) \end{bmatrix}$$

Φ - KVADRATNA FORMA PRIDRUŽENA $d^2 f(a)$

1) AKO JE Φ POZITIVNO DEFINITNA $\Rightarrow$ TAČKA LOK. MINIMUMA
2) AKO JE Φ NEGATIVNO DEFINITNA $\Rightarrow$ TAČKA LOK. MAKSIMUMA
3) AKO Φ MENJA ZNAK $\Rightarrow$ NIJE EKSTREMUM

① $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = 2x^4 + y^4 - x^2 - 2y^2$$

$$\frac{\partial f}{\partial x} = 8x^3 - 2x = 2x(4x^2 - 1)$$

$$\frac{\partial f}{\partial y} = 4y^3 - 4y = 4y(y^2 - 1)$$

$\frac{\partial f}{\partial x} = 0 \iff x = 0$ ILI $x = \frac{1}{2}$ ILI $\cancel{x = -\frac{1}{2}}$

$\frac{\partial f}{\partial y} = 0 \iff y = 0$ ILI $y = 1$ ILI $\cancel{y = -1}$

$$f(-x,y) = f(x,-y) = f(-x,-y) = f(x,y)$$

$\Rightarrow$ DOVOLJNO JE POSMATRATI I KVADRANT

A $(0,0)$ B $(0,1)$ C $(\frac{1}{2}, 0)$ D $(\frac{1}{2}, 1)$

OVO SU STACIONARNE TAČKE U I KVADRANTU

$$\frac{\partial^2 f}{\partial x^2} = 24x^2 - 2$$

$$\frac{\partial^2 f}{\partial y^2} = 12y^2 - 4 \qquad d^2(f) = \begin{bmatrix} 24x^2-2 & 0 \\ 0 & 12y^2-4 \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$$

PRIMENJUJEMO SILVESTEROV KRITERIJUM.
(PROŠLI ČAS)

$$d^2 f(A) = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\left.\begin{aligned} A_1 &= -2 < 0 \\ A_1 &= \det \begin{vmatrix} -2 & 0 \\ 0 & -4 \end{vmatrix} = 8 > 0 \end{aligned}\right\}$$ $d^2(f)(A)$ JE POZITIVNO DEFINITNA

$\Rightarrow$ A JE LOKALNI MAKSIMUM

$$d^2(f)(B) = \begin{bmatrix} -2 & 0 \\ 0 & 8 \end{bmatrix}$$

$$\left.\begin{aligned} A_1 &= -2 < 0 \\ \det &\begin{vmatrix} -2 & 0 \\ 0 & 8 \end{vmatrix} = -16 < 0 \end{aligned}\right\}$$ NE MOŽEMO PRIMENITI SILVESTEROV KRITERIJUM

$$d^2(f)(B)(h_1, h_2) = -2h_1^2 + 8h_2^2$$

$$d^2(f)(B)(0,1) = 8 > 0$$

$$d^2(f)(B)(1,0) = -2 < 0$$

$\Rightarrow$ KVADRATNA FORMA MENJA ZNAK $\Rightarrow$ B NIJE LOKALNI EKSTREMUM.

$$d^2(f)(C) = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\left.\begin{aligned} A_1 &= 4 > 0 \\ \det &= -16 < 0 \end{aligned}\right\}$$

$$d^2 f(C)\ (h_1, h_2) = 4h_1^2 - 4h_2^2$$

$$\left.\begin{array}{l} d^2 f(C)\ (1,0) = 4 > 0 \\ d^2 f(C)\ (0,1) = -4 < 0 \end{array}\right\} \begin{array}{l} \text{MENJA ZNAK} \\ C \text{ NIJE EKSTREMUM} \end{array}$$

$$d^2(f)\ (D) = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}$$

$$\left.\begin{array}{l} \Delta_1 = 4 \quad > 0 \\ \det | \ | = 32 \quad > 0 \end{array}\right\} \Rightarrow \text{POZITIVNO DEFINITNA}$$

$\Rightarrow$ D JE LOKALNI MINIMUM

ŠTA JE SLIKA POLJA ?

DOMEN JE $\mathbb{R}^2$ A TO JE POVEZAN SKUP

$f[\mathbb{R}^2]$ MORA BITI POVEZAN SKUP U $\mathbb{R}$

$f[\mathbb{R}^2]$ JE INTERVAL

$$f(x,y) = 2x^4 + y^4 - x^2 - 2y^2$$

AKO PREĐEMO NA POLARNE KOORDINATE

$$f(\rho, \varphi) = 2\rho^4 \cos^4\varphi + \rho^4 \sin^4\varphi - \rho^2\cos^2\varphi - 2\rho^2\sin^2\varphi$$

$$\lim_{\rho \to \infty} f(\rho, \varphi) = +\infty$$

$$M \in \mathbb{R} \quad \exists \rho_0 \quad \forall \rho \geq \rho_0 \quad f(\rho, \varphi) \geq M$$

UOČIMO DISK $B[(0,0), \rho_0]$

$B[(0,0), \rho_0]$ I NA NJEMU FUNKCIJA DOSTIŽE MINIMUM I MAKSIMUM.

MINIMUM MOŽE BITI NA RUBU ILI U NEKOJ TAČKI U UNUTRAŠNJOSTI GDE JE LOKALNI MINIMUM

Д је тачка локалног минимума

$D = \left(\frac{1}{2}, 1\right)$

$$f(D) = 2\,\frac{1}{16} + 1 - \frac{1}{4} - 2 = -\frac{9}{8}$$

$M > 0 \Rightarrow$ Д је глобални минимум

$$f[\mathbb{R}^2] = \left[-\frac{9}{8}, +\infty\right)$$

2)

$$f : \mathbb{R}^3 \to \mathbb{R}$$

$$f(x,y,z) = x^3 + y^3 + z^3 + 12xy + 2z$$

Одредити локалне екстреме и слику домена

$$\frac{\partial f}{\partial x} = 3x^2 + 12y \qquad \frac{\partial f}{\partial y} = 3y^2 + 12x \qquad \frac{\partial f}{\partial z} = 3z^2 - 2$$

$\frac{\partial f}{\partial x} = 0 \qquad 3x^2 + 12y = 0 \qquad x^2 + 4y = 0$

$\frac{\partial f}{\partial y} = 0 \qquad 3y^2 + 12x = 0 \qquad y^2 + 4x = 0$

$\frac{\partial f}{\partial z} = 0 \qquad 3z^2 - 2 = 0 \qquad z = \sqrt{\frac{2}{3}}$ или $z = -\sqrt{\frac{2}{3}}$

$$\left.\begin{aligned} x^2 + 4y &= 0 \\ y^2 + 4x &= 0 \end{aligned}\right\} -$$

$x^2 - y^2 + 4(y - x) = 0$

$(x-y)(x+y) - 4(x-y) = 0$

$(x-y)(x+y-4) = 0$

$x = y$

$x = y$

$x^2 + 4x = 0$

$x = 0$ или $x = -4$

$x + y = 4 \qquad y = 4 - x$

$x^2 + 4(4-x) = 0$

$x^2 - 4x + 16 = 0$

$$x_{1,2} = \frac{4 \pm \sqrt{16 - 4 \cdot 16}}{2}$$

NEMA REALNIH REŠENJA

$x = 0$ i $y = 0$ ILI $x = -4$ i $y = -4$

Dobili smo 4 stacionarne tačke

$A = (0, 0, \sqrt{\frac{2}{3}})$ $C = (-4, -4, \sqrt{\frac{2}{3}})$

$B = (0, 0, -\sqrt{\frac{2}{3}})$ $D = (-4, -4, -\sqrt{\frac{2}{3}})$

$$\frac{\partial^2 f}{\partial x^2} = 6x \qquad \frac{\partial^2 f}{\partial x \partial y} = 12$$

$$\frac{\partial^2 f}{\partial y^2} = 6y \qquad \frac{\partial^2 f}{\partial x \partial z} = 0$$

$$\frac{\partial^2 f}{\partial z^2} = 6z \qquad \frac{\partial^2 f}{\partial y \partial z} = 0$$

$$d^2 f = \begin{bmatrix} 6x & 12 & 0 \\ 12 & 6y & 0 \\ 0 & 0 & 6z \end{bmatrix}$$

$A = (0, 0, \sqrt{\frac{2}{3}})$

$$d^2 f(A) = \begin{bmatrix} 0 & 12 & 0 \\ 12 & 0 & 0 \\ 0 & 0 & 6\sqrt{\frac{2}{3}} \end{bmatrix}$$

$A_1 = 0$ NIJE NI POZITIVNO NI NEGATIVNO DEFINITNO

$$d^2 f(A)(h_1, h_2, h_3) = 24 h_1 h_2 + 6\sqrt{\frac{2}{3}} h_3^2$$

$$\left.\begin{aligned} d^2 f(A)(1, 1, 0) &= 24 > 0 \\ d^2 f(A)(1, -1, 0) &= -24 < 0 \end{aligned}\right\} \Rightarrow$$ MENJA ZNAK NIJE EKSTREMUM

SLIČNO U B MENJA ZNAK

$C(-4, -4, \sqrt{\tfrac{2}{3}})$

$$d^2 f(C) = \begin{bmatrix} -24 & 12 & 0 \\ 12 & -24 & 0 \\ 0 & 0 & 6\sqrt{\tfrac{2}{3}} \end{bmatrix}$$

$A_1 = -24 \quad < 0$

$A_2 = \begin{vmatrix} -24 & 12 \\ 12 & -24 \end{vmatrix} = 24^2 - 12^2 > 0$ SILVESTER НЕ ПОМАЖЕ

$A_3 = 6\sqrt{\tfrac{2}{3}} \qquad A_2 > 0$

$f(x,y,z) = x^3 + y^3 + z^3 + 12xy + 2z$

$y = 0 \qquad z = 0$

$f(x,0,0) = x^3$

$\lim\limits_{x \to +\infty} f(x,0,0) = +\infty$

$\lim\limits_{x \to -\infty} f(x,0,0) = -\infty$

СЛИКА ДОМЕНА ЈЕ ПОВЕЗАН СКУП

$$\boxed{f[\mathbb{R}^3] = \mathbb{R}}$$

3. $f : \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = xy\, e^{-x^2 - y^2}$

НАЂИ СЛИКУ ДОМЕНА

$\mathbb{R}^2$ - ПОВЕЗАН $\qquad f[\mathbb{R}^2] = \underset{\text{ИНТЕРВАЛ}}{I} \subseteq \mathbb{R}$

АКО ПРЕЂЕМО НА ПОЛАРНЕ КООРДИНАТЕ

$$f(\rho,\varphi) = \rho^2 \cos\varphi \sin\varphi \, e^{-\rho^2}$$

$$0 \le |\rho^2 \cos\varphi \sin\varphi \, e^{-\rho^2}| \le \rho^2 e^{-\rho^2} \xrightarrow[\rho \to +\infty]{} 0$$

$$f(x,y) = xy \, e^{-x^2-y^2}$$

$$\frac{\partial f}{\partial x} = y e^{-x^2-y^2} + xy \, e^{-x^2-y^2}(-2x) = e^{-x^2-y^2} y (1-2x^2)$$

$$\frac{\partial f}{\partial y} = e^{-x^2-y^2} x (1-2y^2)$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

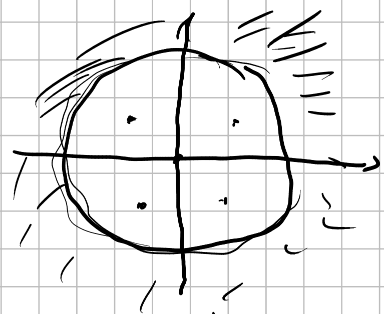
$$y(1-2x^2) = 0 \qquad x(1-2y^2) = 0$$

$$A = (0,0) \qquad B = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \qquad C = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$D = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \qquad E = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

OVO SU STACIONARNE TAČKE

$$f(0,0) = 0 = f(A)$$



$$|f| \le \frac{1}{4} e^{-1}$$

$$f(B) = f(C) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} e^{-\frac{1}{2} - \frac{1}{2}} = \underbrace{\frac{1}{2} e^{-1}}$$

$$f(D) = f(E) = \underbrace{-\frac{1}{2} e^{-1}}$$

$$\lim_{\rho \to \infty} f(\rho,\varphi) = 0$$

$$\exists \rho_0 \quad \forall \rho > \rho_0 \quad |f(\rho,\varphi)| \le \frac{1}{4} e^{-1}$$

SIGURNO NISU GLOBALNI EKSTREMUMI

VAN $B[(0,0), \rho_0]$

$\downarrow$

KOMPAKTAN SKUP PA f DOSTIŽE MINIMUM I MAKSIMUM

STACIONARNE TAČKE SU JEDINI KANDIDATI

ZA GLOBALNO EKSTREMUME

$$f[\mathbb{R}^2] = \left[-\frac{1}{2}e^{-1}, \frac{1}{2}e^{-1}\right]$$

4

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$f(x,y) = (x^2+y^2)\, e^{-x^2-y^2}$$

$$\Big\downarrow$$

$$\rho, \varphi$$

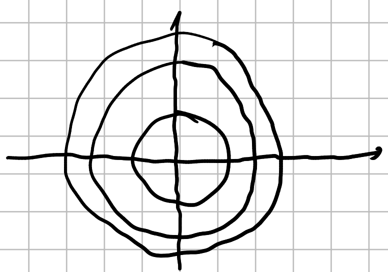
$$f(\rho,\varphi) = \rho^2 e^{-\rho^2}$$

$\uparrow$

NE ZAVISI OD φ

NA SVIM KRUŽNICAMA SA CENTROM U $(0,0)$

f JE KONSTANTNA FUNKCIJA:

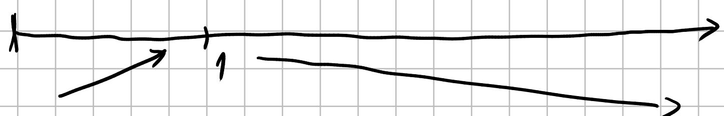


$$f(\rho) = \rho^2 e^{-\rho^2} \qquad 0 \le \rho < +\infty$$

$$f'(\rho) = 2\rho e^{-\rho^2} + \rho^2 e^{-\rho^2}(-2\rho)$$

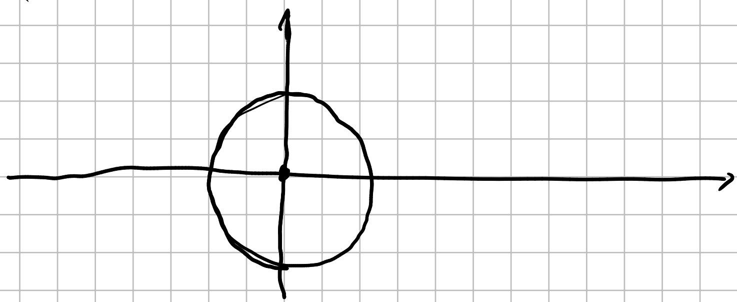
$$= e^{-\rho^2}(2\rho - 2\rho^3) = 2\rho e^{-\rho^2}(1-\rho^2)$$

$$f'(\rho) = 0 \iff \rho = 1 \quad \text{ili} \quad \rho = 1$$



$$\lim_{\rho \to \infty} \rho^2 e^{-\rho^2} = 0 \qquad f(0) = 0$$

$f(1) = e^{-1}$



Lokalni i globalni minimum je u tački $(0,0)$

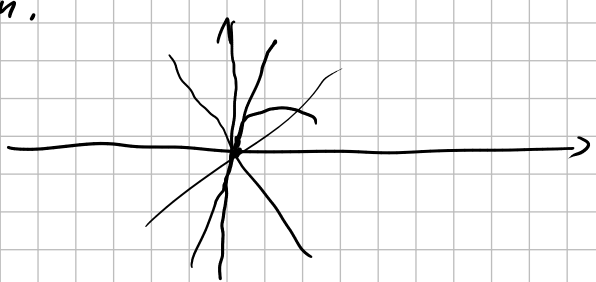
Lokalni i globalni maksimumi su u tačkama kružnice $\rho = 1$ $(x^2 + y^2 = 1)$

Nisu strogi maksimumi (zašto?)

5

$$f(x,y) = (x - y^2)(2x - y^2)$$

Duž svake prave kroz koordinatni početak f ima lokalni minimum, a kao funkcija dve promenljive f nema u $(0,0)$ lokalni minimum.



$$y = kx$$

$$g(x) = (x - k^2x^2)(2x - k^2x^2)$$

$$g(x) = 2x^2 - k^2x^3 - 2k^2x^3 + k^4x^4$$

$$g(x) = k^4x^4 - 3k^2x^3 + 2x^2$$

$$g'(x) = 4k^4x^3 - 9k^2x^2 + 4x$$

$g'(0) = 0$ stacionarna tačka

$$g''(x) = 12k^4x^2 - 18k^2x + 4$$

$$g''(0) = 4 > 0$$

$\Rightarrow$ g има локални минимум у 0.

$2°$ $\quad x = 0$

$$h(y) = y^4$$

Има очигледно минимум у 0

$1°, 2°$ дуж сваке праве кроз координатни почетак у $(0,0)$ је локални минимум.

$$f(x,y) = (x - y^2)(2x - y^2)$$

$$f(x,y) = y^4 - 3xy^2 + 2x^2$$

$$\frac{\partial f}{\partial x} = -3y^2 + 4x \qquad \frac{\partial f}{\partial y} = 4y^3 - 6xy$$

$\Rightarrow$ $(0,0)$ јесте стационарна тачка.

$$\frac{\partial^2 f}{\partial x^2} = 4 \qquad \frac{\partial^2 f}{\partial y^2} = 12y^2 - 6x \qquad \frac{\partial f}{\partial x \partial y} = -6y$$

$$d^2f((0,0)) = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

Не може Силвестер

$$f(h,l) - \overset{=0}{f(0,0)} = \underline{l^4 - 3hl^2 + 2h^2}$$

$$h = \varepsilon \qquad f(\varepsilon, 0) = 2\varepsilon^2 > 0$$

$l = \varepsilon \qquad h = \frac{2}{3}\varepsilon^2$

$$f\left(\frac{2}{3}\varepsilon^2, \varepsilon\right) - f(0,0) = \varepsilon^4 - 3\cdot\frac{2}{3}\varepsilon^4 + \frac{8}{9}\varepsilon^4 = -\frac{1}{9}\varepsilon^4 < 0$$

$\Rightarrow$ f u svakoj okolini tačke $(0,0)$ menja znak, pa $(0,0)$ nije tačka lokalnog ekstremuma.

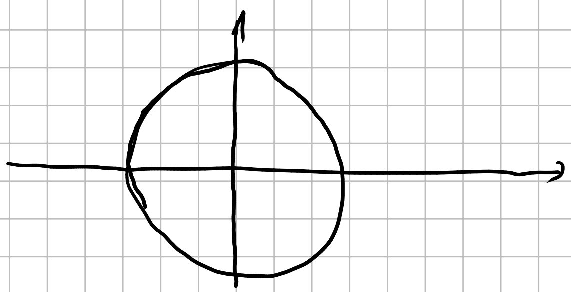
---

(6) $D = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 25 \}$

$$f(x,y) = x^2+y^2-12x+6y$$

$$f : D \to \mathbb{R}$$

$$f[D] = ?$$



D je kompaktan $\Rightarrow$ f dostiže minimum i maksimum na D

$$f[D] = [\alpha, \beta]$$

$f = x^2+y^2-12x+6y$

1) Ispitujemo int D

$\frac{\partial f}{\partial x} = 2x - 12 \qquad \frac{\partial f}{\partial x} = 0 \iff x = 6$

$\frac{\partial f}{\partial y} = 2y + 6 \qquad \frac{\partial f}{\partial y} = 0 \iff y = -3$

Dobili smo kritičnu tačku $(6,-3)$, ali $6^2 + (-3)^2 > 25$ pa ova tačka nije u domenu

$\Rightarrow$ f DOSTIŽE MINIMUM I MAKSIMUM

NA RUBU D.

2) ∂D

$\partial D \quad : \quad x^2 + y^2 = 25$

$\partial D \quad : \quad \rho = 5$

$$f(x,y) = x^2 + y^2 - 12x + 6y$$

$$f(\rho,\varphi)\Big|_{\rho=5} = 25 - 12 \cdot 5 \cos\varphi + 6 \cdot 5 \sin\varphi$$

$$= 25 - 60 \cos\varphi + 30 \sin\varphi$$

$$= 25 + 30 (\sin\varphi - 2\cos\varphi) \qquad \varphi \in [0, 2\pi]$$

$$= 25 + 30\sqrt{5}\left(\frac{1}{\sqrt{5}} \sin\varphi - \frac{2}{\sqrt{5}} \cos\varphi\right)$$

$$\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2 = 1$$

$$\cos\theta = \frac{1}{\sqrt{5}} \qquad \sin\theta = \frac{2}{\sqrt{5}}$$

$$= 25 + 30\sqrt{5}\,(\cos\theta \sin\varphi - \sin\theta \cos\varphi)$$

$$= 25 + 30\sqrt{5}\, \underbrace{\sin(\varphi - \theta)}_{-1 \leq \quad \leq 1}$$

$$\underline{25 - 30\sqrt{5}} \leq f(\varphi) \leq \underline{25 + 30\sqrt{5}}$$

$$f[D] = [25 - 30\sqrt{5},\ 25 + 30\sqrt{5}]$$

⑧

$$D = \{ (x,y) \in \mathbb{R}^2 \mid x \geq 0 , y \geq 0 \}$$

$$f : D \to \mathbb{R}$$

$$f(x,y) = (x + 3y)\, e^{-x-4y}$$

Наћи слику домена

$$f(\rho,\varphi) = (\rho\cos\varphi + 3\rho\sin\varphi)\, e^{-\rho\cos\varphi - 4\rho\sin\varphi)}$$

$\cos\varphi, \sin\varphi \geq 0$ јер су у I квадранту

$$\lim_{\rho\to\infty} f(\rho,\varphi) = (\cos\varphi + 3\sin\varphi)\, \rho\, e^{-\rho(\underbrace{\cos\varphi + 4\sin\varphi}_{>0})}$$

$$= 0$$

$$f(x,y) = \underbrace{(x+3y)}_{\geq 0}\, \underbrace{e^{-x-4y}}_{>0} \geq 0$$

$$f(0,0) = 0$$

Тачка $(0,0)$ је тачка глобалног минимума.

$$\frac{\partial f}{\partial x} = e^{-x-4y} + (x+3y)\, e^{-x-4y}(-1)$$

$$= e^{-x-4y}(1 - x - 3y)$$

$$\frac{\partial f}{\partial y} = 3e^{-x-4y} + (x+3y)\, e^{-x-4y}(-4)$$

$$= e^{-x-4y}(3 - 4x - 11y)$$

$$\frac{\partial f}{\partial x} = 0 \quad \Rightarrow \quad x + 3y = 1 \quad / \cdot 4 \qquad 4x + 12y = 4$$

$$\frac{\partial f}{\partial y} = 0 \quad \Rightarrow \quad 4x + 12y = 3 \qquad ↯ \qquad 4x + 12y = 3$$

Не постоје критичне тачке

у унутрашњости домена

$x = 0 \qquad\qquad y = 0 \qquad\qquad \rho = \rho_0$

$x = 0 \qquad f(0, y) = 3y\, e^{-4y}$

$$g(y) = 3y\, e^{-4y}$$

$$g'(y) = 3\, e^{-4y} + 3y\, e^{-4y}\, (-4)$$

$$g'(y) = 3\, e^{-4y}\, (1 - 12y)$$

$$g'(y) = 0 \iff y = \frac{1}{12}$$

$y = 0 \qquad f(x, 0) = x\, e^{-x}$

$$h(x) = x\, e^{-x}$$

$$h'(x) = e^{-x} + x\, e^{-x}\, (-1)$$

$$= e^{-x}\, (1 - x)$$

$$h'(x) = 0 \iff x = 1$$

$A = (0, \frac{1}{12}) \qquad B(1, 0)$

$$f(0, \tfrac{1}{12}) = (0 + 3 \cdot \tfrac{1}{12})\, e^{-0 - 4\frac{1}{12}} = \frac{1}{4} e^{-3}$$

$$f(1, 0) = e^{-1} \qquad e^{-1} > e^{-3}$$

Može se zaključiti da je tačka $(1,0)$ tačka globalnog maksimuma.

$$f[D] = [0, e^{-1}]$$

Domaći:

1/ $f(x,y) = xy\sqrt{4-x^2-y^2}$

Sliku domena

2) $z(x,y) = x^2 y^3 (6-x-y)$

IMPLICITNO ZADATE FUNKCIJE

TEOREMA:

$A \subseteq \mathbb{R}^2 \qquad A \in \tau$

$(a,b) \in A$

$F: A \to \mathbb{R}$

$F \in C(A)$

Ako postoji $\frac{\partial F}{\partial y}$ na skupu A i neprekidan je i:

$\frac{\partial F}{\partial y}(a,b) \neq 0$

Tada postoji okolina $W = U \times V \ni (a,b)$

$a \in U \qquad (a-\alpha,\ a+\alpha)$

$b \in V \qquad (b-\beta,\ b+\beta)$

i postoji funkcija

$f: U \to V$ tdj.

$f(a) = b$ i $F(x, f(x)) = 0 \quad \forall x \in U$

f se naziva implicitno zadata funkcija

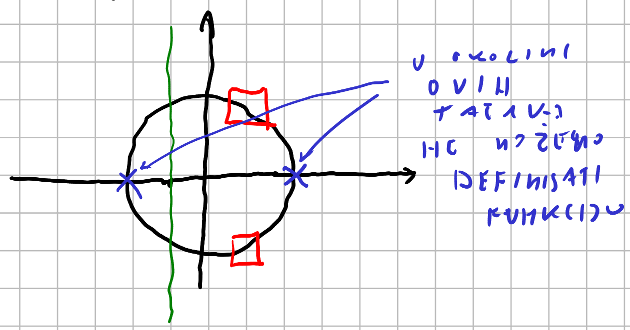
⊕ Ako još važi da postoji $\frac{\partial F}{\partial x}$ na A koji je neprekidan tada je $f \in C^1(A)$

$$f'(x) = -\frac{\frac{\partial F}{\partial x}(x,f(x))}{\frac{\partial F}{\partial y}(x,f(x))}$$

PRIMER:

$$F(x,y) = x^2 + y^2 - 1 \qquad f(x) = \pm\sqrt{1-x^2}$$

$F(x,y)=0$ TO JE ZAPRAVO KRUG



$$\frac{\partial F}{\partial y} = 2y = 0$$

$$y = 0$$

(1) DOKAZATI DA JE SA

$$y + \frac{1}{2}\sin y - x = 0$$

DEFINISANA IMPLICITNO ZADATA FUNKCIJA NA $\mathbb{R}$

I NAĆI $y'(x)$, $y''(x)$, $y'(\pi)$, $y''(\pi)$.

$$F(x,y) = y + \frac{1}{2}\sin y - x$$

- DA SMO MOGLI DA REŠIMO PO y GOTOVO JE PRVI DEO ZADATKA.

- AKO PRIMENIMO TEOREMU:

$$F \in C(\mathbb{R})$$

$$\frac{\partial F}{\partial y} = 1 + \frac{1}{2}\cos y > \frac{1}{2}$$

$$\frac{\partial F}{\partial y} \in C(\mathbb{R}) \qquad \frac{\partial F}{\partial y} \neq 0$$

OVO NAM NIJE DOVOLJNO DA TVRDIMO DA JE GLOBALNO DEFINISANA FUNKCIJA, TEOREMA DAJE SAMO LOKALNU DEFINISANOST.

- MOŽEMO BEZ POMOĆI TEOREME

$$x = x(y) = y + \frac{1}{2}\sin y$$

$$x'(y) = 1 + \frac{1}{2}\cos y > 0$$

$x(y)$ JE STROGO RASTUĆA FUNKCIJA

$\Rightarrow$ x je injektivno

$$\lim_{y \to +\infty} x(y) = \lim_{y \to +\infty} \left(y + \frac{1}{2} \sin y\right) = +\infty$$

$$\lim_{y \to -\infty} x(y) = \lim_{y \to -\infty} \left(y + \frac{1}{2} \sin y\right) = -\infty$$

$\Rightarrow$ x je i surjektivno

(Neprekidna slika intervala je interval)

$\Rightarrow$ x je bijekcija pa postoji

$$x^{-1}(y) = y(x)$$

Definisana je globalno implicitna funkcija.

Diferencijabilnost ?

Ovo je lokalno svojstvo pa za izvod možemo primeniti teoremu.

$$F(x,y) = y + \frac{1}{2} \sin y - x$$

$$F \in C(\mathbb{R}^2)$$

$$\frac{\partial F}{\partial y} = 1 + \frac{1}{2} \cos y \quad \in \quad C(\mathbb{R}^2)$$

$$1 + \frac{1}{2} \cos y > 0$$

$$\frac{\partial F}{\partial x} = -1 \quad \in \quad C(\mathbb{R}^2)$$

$$\Rightarrow \quad \exists \quad f'(x) \in C^1(\mathbb{R})$$

$$f'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{-1}{1+\frac{1}{2}\cos y} = \frac{1}{1+\frac{1}{2}\cos y}$$

$$f''(x) = (f(x)')' = \left(\frac{1}{1+\frac{1}{2}\cos y}\right)'$$

$$= \frac{-\left(-\frac{1}{2}\sin y\right)}{\left(1+\frac{1}{2}\cos y\right)^2} \; y'(x) =$$

$$= \frac{\sin y}{(1+\frac{1}{2}\cos y)^2} \frac{1}{1+\frac{1}{2}\cos y} = \frac{\sin y}{(1+\frac{1}{2}\cos y)^3}$$

$y'(\pi) = ?$ $\quad y''(\pi) = ?$

AKO JE $x = \pi$ KOLIKO JE y?

$$y + \frac{1}{2}\sin y - x = 0$$

$y + \frac{1}{2}\sin y = \pi$ MOŽEMO ZAKLJUČITI $y = \pi$

$$y'(\pi) = \frac{1}{1+\frac{1}{2}\cos \pi} = \frac{1}{1-\frac{1}{2}} = 2$$

$$y''(\pi) = \frac{\sin y}{2(1+\frac{1}{2}\cos y)^3} = 0$$

---

(2)

$$x^2 + xy + y^2 = 3$$

$y' = ?$ $\quad y'' = ?$ $\quad y''' = ?$

$$F(x,y) = x^2 + xy + y^2 - 3$$

$$F \in C(\mathbb{R}^2)$$

$$\frac{\partial F}{\partial y} = x + 2y \in C(\mathbb{R}^2)$$

$x + 2y = 0 \Rightarrow x = -2y$

ZA OVE TAČKE NE MOŽEMO DEFINISATI IMPLICITNO ZADATU FUNKCIJU

DALJE SMATRAMO DA JE $\underline{x \neq -2y}$

$$\frac{\partial F}{\partial x} = 2x + y \in C(\mathbb{R}^2)$$

$$y'(x) = - \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = - \frac{2x+y}{2y+x}$$

$$y''(x) = \left(y'(x) \right)' = \left(- \frac{2x+y}{2y+x} \right)' =$$

$$= - \frac{(2+y')(2y+x) - (2x+y)(2y'+1)}{(2y+x)^2}$$

$$= - \frac{4y + \cancel{2x} + \cancel{2yy'} + xy' - 4xy' - \cancel{2x} - \cancel{2yy'} - y}{(2y+x)^2}$$

$$= - \frac{3y - 3xy'}{(2y+x)^2} = -3 \frac{y - x\frac{2x+y}{2y+x}}{(2y+x)^2}$$

$$= -3 \frac{y(2y+x) - x(2x+y)}{(2y+x)^3} = -3 \frac{2y^2 + \cancel{yx} - 2x^2 - \cancel{xy}}{(2y+x)^3}$$

$$= -6 \frac{y^2 - x^2}{(2y+x)^3}$$

$$f'''(x) = \left(f''(x) \right)' = \left(-6 \frac{y^2 - x^2}{(2y+x)^3} \right)'$$

$$= -6 \frac{(2y' - 2x)(2y+x)^3 - (y^2 - x^2)\, 3(2y+x)^2 (2y'+x)}{(2y+x)^6}$$

$$= \ldots$$

ZAVRŠITI RAČUN ZA DOMAĆI.

Teorema:

$A \in \mathbb{R}^{n+1} \qquad A \in \mathcal{T}$

$(a, b) = (a_1, \ldots, a_n, b) \in A$

$F : A \to \mathbb{R} \qquad f \in C(A)$

$$F = F(x_1, x_2, \dots, x_n, y)$$

$$F(a, b) = 0$$

$\dfrac{\partial F}{\partial y}$ postoji na A i neprekidna je

$$\frac{\partial F}{\partial y}(a, b) \neq 0$$

Tada postoji okolina $\underset{\mathbb{R}^n \,\supseteq}{U} \ni (a_1, \dots, a_n)$ i $V \ni b$

i postoji funkcija

$$f : \underset{\mathbb{R}^n}{\overset{}{U}} \to \underset{\mathbb{R}}{V}, \quad U \subset \mathbb{R}^n, \; V \subset \mathbb{R}$$

$$f(a) = f(a_1, \dots, a_n) = b$$

$$F(x_1, \dots, x_n, f(x_1, \dots, x_n)) = 0 \quad \forall (x_1, \dots, x_n) \in U$$

(+)

Ako još postoje

$$\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \dots, \frac{\partial F}{\partial x_n} \quad \text{na } A$$

i neprekidni su

$$\frac{\partial f}{\partial x_i} = - \frac{\frac{\partial F}{\partial x_i}}{\frac{\partial F}{\partial y}}$$

(1)

Dokazati da je sa:

$$z^3 - xyz + y^2 = 16$$

definisana implicitno zadata funkcija

$z = z(x, y)$ u okolini tačke $(1, 4, 2)$

$$F(x, y, z) = z^3 - xyz + y^2 - 16$$

$$F(1, 4, 2) = 2^3 - 1 \cdot 4 \cdot 2 + 4^2 - 16 = 0$$

$$F(x,y,z) \in C(\mathbb{R}^3)$$

$$\frac{\partial F}{\partial z} = 3z^2 - xy \quad \in C(\mathbb{R}^3)$$

$$\frac{\partial F}{\partial z} \neq 0 \qquad \frac{\partial F}{\partial z} = 0 \iff 3z^2 = xy$$

$$\frac{\partial F}{\partial z}(1,4,2) = 3\cdot 2^2 - 1\cdot 4 = 8 \neq 0$$

$\Rightarrow$ U OKOLINI TAČKE $(1,4,2)$ JESTE DEFINISANA IMPLICITNA FUNKCIJA $z = z(x,y)$.

$$\frac{\partial z}{\partial x}(1,4) \quad i \quad \frac{\partial^2 z}{\partial x^2}(1,4)$$

$$\frac{\partial F}{\partial x} = -yz$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{yz}{3z^2 - xy}$$

$$\frac{\partial z}{\partial x}(1,4) = -\frac{4\cdot 2}{3\cdot 2^2 - 1\cdot 4} = -\frac{8}{8} = \boxed{-1}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{yz}{3z^2 - xy}\right)$$

$$= -\frac{y\frac{\partial z}{\partial x}(3z^2 - xy) - yz\left(6z\cdot\frac{\partial z}{\partial x} - y\right)}{(3z^2 - xy)^2} =$$

$$= -\frac{y\frac{-yz}{3z^2 - xy}(3z^2 - xy) - yz(6z - y)}{(3z^2 - xy)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{4\frac{-4\cdot 2}{\cancel{3\cdot 2^2 - 4\cdot 1}}\cancel{(3\cdot 2^2 - 4\cdot 1)} - 4\cdot 2(6\cdot 2 - 4)}{(3\cdot 2^2 - 4\cdot 1)^2} = -\frac{-16 - 64}{64}$$

$$= \frac{-80}{64} = -\frac{5}{4}$$

② NAĆI DIFERENCIJAL FUNKCIJE z AKO JE ONA IMPLICITNO ZADATA SA

$$\frac{x}{z} = \lg \frac{z}{y} + 1$$

$$z = z(x,y)$$

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

$$F(x,y,z) = \frac{x}{z} - \lg \frac{z}{y} - 1$$ → DOBRO JE DEFINISANO ZA

$$F(x,y,z) = 0$$ $z \neq 0 \quad y \neq 0 \quad \frac{z}{y} > 0$

F JE NEPREKIDNO NA DOMENU.

$$\frac{\partial F}{\partial z} = \frac{-x}{z^2} - \frac{1}{\frac{z}{y}}\,\frac{1}{y} = \frac{-x}{z^2} - \frac{1}{z} = \frac{-x-z}{z^2}$$

$$\frac{\partial F}{\partial z} \neq 0 \iff x + z \neq 0 \qquad x \neq -z$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{\frac{1}{z}}{\frac{-x-z}{z^2}} = \frac{+z}{x+z}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{\frac{1}{y}}{\frac{-x-z}{z^2}} = \frac{z^2}{y(x+z)}$$

$$dz = \frac{z}{x+z}\,dx + \frac{z^2}{y(x+z)}\,dy$$

II NAČIN

DIREKTNO DIFERENCIRAMO.

$$\frac{x}{z} = \lg \frac{z}{y} + 1 \quad / \, d$$

$$\frac{dx \cdot z - x\, dz}{z^2} = \frac{y}{z} \cdot \frac{dz\, y - z\, dy}{y^2}$$

$$dx \cdot \frac{1}{z} - \frac{x}{z^2}\, dz = \frac{1}{z}\, dz - \frac{1}{y}\, dy$$

$\Vert$

Izrazimo dz (mora se dobiti isto)

Teorema

$A \subseteq \mathbb{R}^{m+n}$

$(a,b) = (a_1, \dots, a_m, \underbrace{b_1, \dots, b_n}) \in A$

$F : A \to \mathbb{R}^n \qquad F \in C(A)$

$F(\underbrace{x_1, \dots, x_m}_{x}, \underbrace{y_1, \dots, y_n}_{y}) = F(x,y) = (F_1(x,y), \dots, F_n(x,y))$

$F(a,b) = 0$

$$d_y F = \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_n} \\ \frac{\partial F_n}{\partial y_1} & \cdots & \frac{\partial F_n}{\partial y_n} \end{bmatrix}$$

Ako postoje svi parcijalni izvodi u $d_y F$

i $d_y F(a,b)$ invertibilna matrica

tada postoje okoline $U \ni a$ $\quad V \ni b$

i neprekidna funkcija

$f : U \to V$

$f(a) = b$

$F(x, f(x)) = 0, \quad \forall x \in U$

(+) Ako još postoji

$$d_x F = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_m} \end{bmatrix} \quad \text{i} \quad \frac{\partial F_i}{\partial x_j} \in C(A)$$

TADA JE:

$$df(x) = -\left(d_y F(x, f(x))\right)^{-1} d_x F(x, f(x))$$

---

(1)

$$\left.\begin{aligned} x^2 + y^2 &= \frac{1}{2} z^2 \\ x + y + z &= 2 \end{aligned}\right\}$$

POKAZATI DA SISTEM ZADAJE IMPLICITNU FUNKCIJU $(x, y) = (x(z), y(z))$

U OKOLINI TAČKE $(1, -1, 2)$

NAĆI $x'(z)$, $y'(z)$, $x''(z)$, $y''(z)$.

$$F(x,y,z) = (F_1(x,y,z), F_2(x,y,z)) = (x^2+y^2-\frac{1}{2}z^2, \; x+y+z-2)$$

$$F(1,-1,2) = (0,0)$$

$$d_{x,y} F(1,-1,2) \overset{?}{=}$$

$$d_{x,y} F = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 1 & 1 \end{bmatrix}$$

$$d_{x,y} F = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$$

$$\det d_{x,y} F = \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} = 4 \neq 0$$

$\Rightarrow$ $d_{x,y} F > 0$ — INVERTIBILNA.

$\Rightarrow$ POSTOJI IMPLICITNA FUNKCIJA:

$$f : \underset{\mathbb{R}}{\overset{\cap}{U}} \to \underset{\mathbb{R}^2}{\overset{\cap}{V}}$$

$$f(z) = \begin{bmatrix} x(z) \\ y(z) \end{bmatrix}$$

V HEKOJ OKOLINI TAČKE 2

$2 \in U \qquad (1,-1) \in V$

$$d_z F = \begin{bmatrix} \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial z} \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$d f(2) = \begin{bmatrix} x'(2) \\ y'(2) \end{bmatrix} = - \left[d_{x,y} F \right]^{-1} d_z F \qquad x(t), y(t), 2$$

$$\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$$

$$d f(2) = -\frac{1}{4} \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$x'(2) = 0 \qquad y'(2) = -1$

II NAČIN

$x^2 + y^2 = \frac{1}{2} z^2 \quad / \frac{d}{dz}$

$x + y + z = 2 \quad / \frac{d}{dz}$

$2xx' + 2yy' = z$

$x' + y' + 1 = 0$

$x = 1 \quad y = -1 \quad z = 2$

$2x' - 2y' = 2$

$x' + y' = 1 \quad /2$

$4x' = 4 \qquad x' = 1 \qquad y' = 0$

(2) NAĆI LOKALNE EKSTREMUME IMPLICITNO ZADATE FUNKCIJE $z = z(x,y)$

$$x^2 + y^2 + z^2 - 2x + 2y - 4z - 10 = 0$$

$$F(x,y,z) = x^2 + y^2 + z^2 - 2x + 2y - 4z - 10 \in C$$

$$\frac{\partial F}{\partial z} = 2z - 4$$

$$\frac{\partial F}{\partial z} \neq 0 \iff z \neq 2$$

ZA $z \neq 2$ DEFINISANA JE IMPLICITNA FUNKCIJA.

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{2x-2}{2z-4} = -\frac{x-1}{z-2}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{2y+2}{2z-4} = -\frac{y+1}{z-2}$$

$\frac{\partial z}{\partial x} = 0 \qquad \frac{\partial z}{\partial y} = 0$ – USLOVI ZA LOKALNE EKSTREMUME.

$x - 1 = 0 \quad ; \quad y + 1 = 0$

$x = 1 \qquad y = -1$

$$F(1,-1,z) = 1 + 1 + z^2 - 2 - 2 - 4z - 10 = z^2 - 4z - 12$$

$$z^2 - 4z - 12 = 0$$

$$(z-6)(z+2) = 0$$

$z_1 = 6 \qquad z_2 = -2$

POSTOJI JEDNA IMPLICITNA FUNKCIJA $z_1(x,y)$ U OKOLINI TAČKE $(1,-1,6)$ I DRUGA $z_2(x,y)$ U OKOLINI TAČKE $(1,-1,-2)$ I JEDINE SU

KOJE MOGU IMATI LOKALNE EKSTREMUME.

$$d^2 z = \begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix}$$

$$\frac{\partial z}{\partial x} = \frac{1-x}{z-2} \qquad \frac{\partial z}{\partial y} = -\frac{y+1}{z-2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{1-x}{z-2}\right) = \frac{-1(z-2) - (1-x)\frac{\partial z}{\partial x}}{(z-2)^2} = \frac{2-z-(1-x)\frac{1-x}{z-2}}{(z-2)^2}$$

$$= \frac{(2-z)(z-2) - (1-x)^2}{(z-2)^3}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{y+1}{z-2}\right) = -\frac{z-2-(y+1)\frac{\partial z}{\partial y}}{(z-2)^2} = -\frac{z-2-(y+1)\left(-\frac{y+1}{z-2}\right)}{(z-2)^2}$$

$$= -\frac{(z-2)^2 + (y+1)^2}{(z-2)^3}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(-\frac{y+1}{z-2}\right) = -\frac{-(y+1)\frac{\partial z}{\partial x}}{(z-2)^2} = \frac{(y+1)\frac{1-x}{z-2}}{(z-2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(y+1)(1-x)}{(z-2)^3}$$

$$d^2 z_1(1,-1) = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}$$

$$-\frac{1}{4} < 0 \qquad \frac{1}{16} > 0$$

$\Rightarrow$ FORMA JE NEGATIVNO DEFINITNA

$\Rightarrow$ z_1 IMA LOKALNI MAKSIMUM U $(1,-1)$

$$d^2 z_2(1,-1) = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

$\frac{1}{4} > 0$ $\quad \frac{1}{16} > 0$.

$\Rightarrow$ FORMA JE POZITIVNO DEFINITNA

$z_2(x,y)$ IMA LOKALNI MINIMUM U $(1,-1)$.

USLOVNI EKSTREMUMI

$$f : \underset{\mathbb{R}^n}{\overset{\cap}{A}} \to \mathbb{R}$$

IMAMO FUNKCIJE USLOVA

$\varphi_1, \varphi_2, \ldots, \varphi_s$

PODSKUP DOMENA GDE SU ISPUNJENI USLOVI

$B \subseteq A$ $\quad \underbrace{B = \{ x \in A \mid \varphi_1(x) = \varphi_2(x) = \ldots = \varphi_s(x) = 0 \}}$

$$F(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_s) = f(x_1, \ldots, x_n) - \lambda_1 \varphi_1(x_1, \ldots, x_n)$$

$$- \ldots - \lambda_s \varphi_s(x_1, \ldots, x_n)$$

F - LAGRANŽOV MNOŽIOC

$$\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = \ldots = \frac{\partial F}{\partial x_n} = \frac{\partial F}{\partial \lambda_1} = \ldots = \frac{\partial F}{\partial \lambda_s} = 0$$

$\downarrow$

OVAKO DOBIJAMO STACIONARNE TAČKE U B

$$\phi(a)(h_1, \ldots, h_n) = \sum_{i,j=1}^{n} \frac{\partial^2 F}{\partial x_i \partial x_j}(a) \, h_i h_j$$

$$\left.\begin{array}{l} \frac{\partial \varphi_1}{\partial x_1}(a) h_1 + \ldots + \frac{\partial \varphi_1}{\partial x_n}(a) h_n = 0 \\ \vdots \\ \frac{\partial \varphi_s}{\partial x_1}(a) h_1 + \ldots + \frac{\partial \varphi_s}{\partial x_n}(a) h_n = 0 \end{array}\right|$$

II NAČIN DA ODREDIMO STACIONARNE TAČKE

$$\nabla f = \lambda_1 \nabla \varphi_1 + \ldots + \lambda_s \nabla \varphi_s$$

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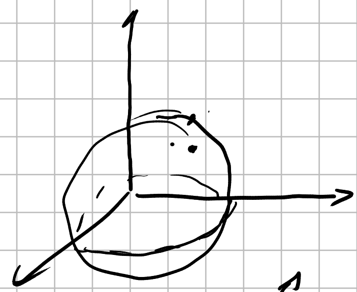
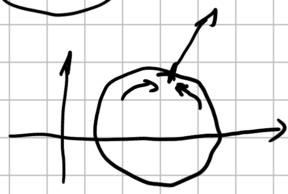
(1)

$$f : \mathbb{R}^3 \to \mathbb{R}$$

$$f(x,y,z) = xyz$$



UZ USLOV $x^2 + y^2 + z^2 = 3$



ODREDITI LOKALNE EKSTREMUME

$$F(x,y,z) = xyz - \lambda\,(x^2+y^2+z^2-3)$$

$$\frac{\partial F}{\partial x} = yz - 2\lambda x = 0 \qquad \frac{\partial^2 F}{\partial x^2} = -2\lambda = \frac{\partial^2 F}{\partial y^2} = \frac{\partial^2 F}{\partial z^2}$$

$$\frac{\partial F}{\partial y} = xz - 2\lambda y = 0 \qquad \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = z$$

$$\frac{\partial F}{\partial z} = xy - 2\lambda z = 0 \qquad \frac{\partial^2 F}{\partial x \partial z} = \frac{\partial^2 F}{\partial z \partial x} = y$$

$$\frac{\partial F}{\partial \lambda} = x^2+y^2+z^2-3 = 0 \qquad \frac{\partial^2 F}{\partial y \partial z} = \frac{\partial^2 F}{\partial z \partial y} = x$$

1) $\lambda = 0$

$yz = 0 \to y = 0$ ili $z = 0$

$xy = 0 \to x = 0$ ili $y = 0$

$xz = 0 \to x = 0$ ili $z = 0$

$x^2+y^2+z^2 = 3 \to$

$\Rightarrow$ BAR 2 OD x, y, z MORAJU BITI 0.

DOBIJAMO 6 STACIONARNIH TAČAKA

$(0, 0, \sqrt{3})$ $\quad$ $(0, 0, -\sqrt{3})$

$(0, \sqrt{3}, 0)$ $\quad$ $(0, -\sqrt{3}, 0)$

$(\sqrt{3}, 0, 0)$ $\quad$ $(-\sqrt{3}, 0, 0)$

2) $\lambda \neq 0$

$$yz = 2\lambda x \quad / x$$
$$xz = 2\lambda y \quad / y$$
$$xy = 2\lambda z \quad / z$$

$$xyz = 2\lambda x^2$$
$$\quad\quad\quad \shortparallel$$
$$xyz = 2\lambda y^2$$
$$\quad\quad\quad \shortparallel$$
$$xyz = 2\lambda z^2$$

$2\lambda x^2 = 2\lambda y^2 \Rightarrow x^2 = y^2$

$2\lambda y^2 = 2\lambda z^2 \Rightarrow y^2 = z^2$

$$x^2 = y^2 = z^2$$

$$x^2 + y^2 + z^2 = 3 \qquad 3x^2 = 3 \qquad x^2 = 1 = y^2 = z^2$$

$$\Downarrow$$

$$|x| = |y| = |z| = 1$$

$$yz = 2\lambda x$$

$$|\lambda| = \frac{1}{2}\left|\frac{\overset{=1}{yz}}{x}\right|_{=1} \qquad |\lambda| = \frac{1}{2} \qquad \lambda_1 = \frac{1}{2} \qquad \lambda_2 = -\frac{1}{2}$$

a) $\lambda = \frac{1}{2}$ $\qquad\qquad$ $\lambda = -\frac{1}{2}$

$A_1 = (1, 1, 1)$ $\qquad$ $B_1 = (1, 1, -1)$

$A_2 = (1, -1, -1)$ $\qquad$ $B_2 = (1, -1, 1)$

$A_3 = (-1, 1, -1)$ $\qquad$ $B_3 = (-1, 1, 1)$

$A_4 = (-1, -1, 1)$ $\qquad$ $B_4 = (-1, -1, -1)$

$f(x,y,z) = xyz$ AKO UBACIMO DVA MINUSA NE MENJA SE

$$f(A_1) = f(A_2) = f(A_3) = f(A_4)$$

$$f(B_1) = f(B_2) = f(B_3) = f(B_4)$$

$$\phi(h_1, h_2, h_3) = \frac{\partial^2 F}{\partial x^2} h_1^2 + \frac{\partial^2 F}{\partial y^2} h_2^2 + \frac{\partial^2 F}{\partial z^2} h_3^2 + \ldots$$

$$\phi(x,y,z)(h_1, h_2, h_3) = -2\lambda h_1^2 - 2\lambda h_2^2 - 2\lambda h_3^2 + 2z h_1 h_2$$
$$+ 2y h_1 h_3 + 2x h_2 h_3$$

$$= -2\lambda \left(h_1^2 + h_2^2 + h_3^2 + 2z\, h_1 h_2 + 2y\, h_1 h_3 + 2x\, h_2 h_3 \right)$$

$$\frac{\partial \varphi}{\partial x} h_1 + \frac{\partial \varphi}{\partial y} h_2 + \frac{\partial \varphi}{\partial z} h_3 = 0$$

$\varphi: \quad x^2 + y^2 + z^2 - 3 = 0$ $\qquad 2x h_1 + 2y h_2 + 2z h_3 = 0$

$x = y = z = 1$

$$2h_1 + 2h_2 + 2h_3 = 0 \longrightarrow h_1 = -h_2 - h_3$$

1) $A(1, 1, 1) \qquad \lambda = \frac{1}{2}$

$$\phi(A)(h_1, h_2, h_3) = -h_1^2 - h_2^2 - h_3^3 + 2h_1 h_2 + 2h_2 h_3 + 2h_1 h_3$$

SVODI SE NA FORMU 2 PROMENLJIVE

$$\phi(A)(h_2, h_3) = -(-h_2 - h_3)^2 - h_2^2 - h_3^2 + 2(-h_2 - h_3) - \ldots$$

$$\phi(A)(h_2, h_3) = -2\left(h_2^2 + h_3^2 + (h_2 + h_3)^2\right) < 0$$

FORMA JE NEGATIVNO DEFINITNA

PA JE A TAČKA LOKALNOG MAKSIMUMA

(ODNOSNO TAČKE A_1, A_2, A_3, A_4)

2) $B = (-1, 1, 1) \qquad \lambda = -\frac{1}{2}$

$$\phi(B)(h_1, h_2, h_3) = h_1^2 + h_2^2 + h_3^2 + 2h_1 h_2 + 2h_2 h_3 + 2h_1 h_3$$

$$2x h_1 + 2y h_2 + 2z h_3 = 0$$

$$-2h_1 + 2h_2 + 2h_3 = 0 \qquad h_1 = h_2 + h_3$$

$$\phi(B)(h_2, h_3) = 2h_2^2 + 2h_3^2 + 2(h_2 + h_3)^2 > 0$$

FORMA JE POZITIVNO DEFINITNA

$\Rightarrow$ B JE LOKALNI MINIMUM (B_1, B_2, B_3, B_4)

3) $C_1(0,0,\pm\sqrt{3})$ $\quad C_2(0,\pm\sqrt{3},0)$ $\quad C_3(\pm\sqrt{3},0,0)$

OVE TAČKE NISU LOKALNI EKSTREMUMI.

AKO UZMEMO PROIZVOLJNO MALU OKOLINU NPR. C_1 POSTOJE U NJOJ TAČKE OBLIKA $(\varepsilon,\varepsilon,\sqrt{3})$ i $(-\varepsilon,\varepsilon,\sqrt{3})$

$$f(\varepsilon,\varepsilon,\sqrt{3}) = \varepsilon^2\sqrt{3} > 0 \qquad f(-\varepsilon,\varepsilon,\sqrt{3}) = -\varepsilon^2\sqrt{3} < 0$$

$\Rightarrow$ U SVAKOJ OKOLINI OVIH TAČAKA FUNKCIJA UZIMA I POZITIVNE I NEGATIVNE VREDNOSTI, A VREDNOST FUNKCIJE U TAČKI JE 0.

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(2)

$$f(x,y) = xy$$

NAĆI EKSTREMNE VREDNOSTI NA ELIPSI

$$\frac{x^2}{8} + \frac{y^2}{2} = 1$$

I SLIKU ELIPSE FUNKCIJOM f

ODREDIMO STACIONARNE TAČKE

$$f(x,y) = xy \qquad g = \frac{x^2}{8} + \frac{y^2}{2} - 1$$

$$\underline{\nabla f = \lambda \nabla g}$$

$$\nabla f = (y, x) \qquad \nabla g = \left(\frac{x}{4}, y\right)$$

$$(y, x) = \lambda\left(\frac{x}{4}, y\right)$$

$$y = \lambda\frac{x}{4} \qquad \frac{x^2}{8} + \frac{y^2}{2} = 1$$

$$x = \lambda y$$

$y \neq 0$ $\qquad \cancel{y} = \lambda^2\frac{\cancel{y}}{4}$

$$\lambda^2 = 4$$

$$\lambda_1 = 2 \qquad \lambda_2 = -2$$

$$y = \frac{x}{2}$$

$$x = 2y \qquad\qquad x = -2y$$

$$\frac{x^2}{8} + \frac{y^2}{2} = 1$$

$$\frac{4y^2}{8} + \frac{y^2}{2} = 1$$

$$y^2 = 1$$

$y = 1$ ILI $y = -1$

$x = 2$ ILI $x = -2$

$A = (2,1) \qquad B = (-2,-1)$

$C(-2,1) \qquad D(2,-1)$

A, B, C, D SU STACIONARNE TAČKE I KANDIDATI ZA EKSTREMUME.

ELIPSA JE KOMPAKTAN SKUP, PA NA NJEMU NEPREKIDNA FUNKCIJA DOSTIŽE MINIMUM I MAKSIMUM.

$f(A) = f(B) = 2$ MAKSIMUM

$f(C) = f(D) = -2$ MINIMUM

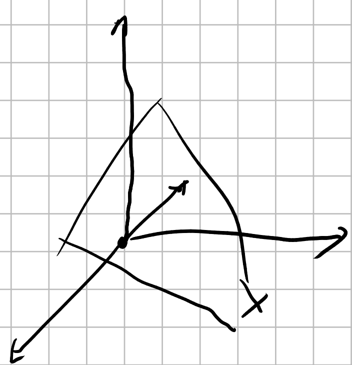
SLIKA ELIPSE JE $[-2, 2]$

---

③ DATA JE RAVAN

$\pi : \quad 2x + y - z + 5 = 0$

NAĆI TAČKU NA NJOJ BLIŽU KOORDINATNOM POČETKU



RASTOJANJE OD KOORDINATNOG POČETKA JE.

$$d\left((x,y,z),(0,0,0)\right) = \sqrt{x^2+y^2+z^2}$$

$\sqrt{}$ - RASTUĆA FUNKCIJA PA ČUVA NEJEDNAKOSTI

$$f(x,y,z) = x^2 + y^2 + z^2$$

$$g(x,y,z) = 2x + y - z - 5$$

$$\nabla f = (2x,\ 2y,\ 2z)$$

$$\nabla g = (2,\ 1,\ -1)$$

$$\nabla f = \lambda \nabla g$$

$$\cancel{2}x = \lambda \cdot \cancel{2}$$

$$2x + y - z + 5 = 0$$

$$\cancel{2}y = \lambda$$

$$2z = -\lambda$$

$$x = \lambda \qquad y = \frac{1}{2}\lambda \qquad z = -\frac{1}{2}\lambda$$

$$2\lambda + \frac{1}{2}\lambda + \frac{1}{2}\lambda + 5 = 0$$

$$3\lambda = 5 \qquad \lambda = -\frac{5}{3}$$

$$x = -\frac{5}{3} \qquad y = -\frac{5}{6} \qquad z = \frac{5}{6}$$

$$A\left(-\frac{5}{3},\ -\frac{5}{6},\ \frac{5}{6}\right)$$

BIĆE NAJBLIŽA KOORDINATNOM POČETKU

II NAČIN

$$f(x,y,z) = x^2 + y^2 + z^2$$

$$2x + y - z + 5 = 0 \quad \Rightarrow \quad z = 2x + y + 5$$

$$\tilde{f}(x,y) = x^2 + y^2 + (2x + y + 5)^2$$

SVELI SMO NA JEDNU PROMENLJIVU MANJE

$$\frac{\partial \tilde{f}}{\partial x} = 0 \qquad \frac{\partial \tilde{f}}{\partial y} = 0$$

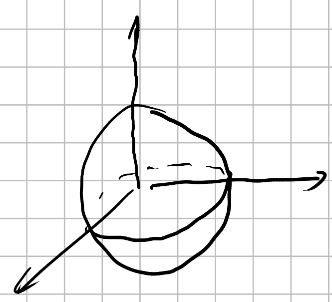
---

(4)

$$f(x,y,z) = x^2 + y^2 + 2z^2 + 2x + 4y$$

NAĆI MAKSIMUM I MINIMUM f NA LOPTI

$$S: \ x^2 + y^2 + z^2 \leq 9$$



PRVO ISPITUJEMO $\text{int}\, S$

ZATIM ISPITUJEMO ∂S (USLOVNI EKSTREMUM)

1) $\text{int}\, S$

$$x^2 + y^2 + z^2 < 9$$

$$f(x,y,z) = x^2 + y^2 + 2z^2 + 2x + 4y$$

$$\frac{\partial f}{\partial x} = 2x + 2$$

$$\frac{\partial f}{\partial y} = 2y + 4$$

$$\frac{\partial f}{\partial z} = 4z$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 2x + 2 = 0 \qquad x = -1$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 2y + 4 = 0 \qquad y = -2$$

$$\frac{\partial f}{\partial z} = 0 \Rightarrow 4z = 0 \qquad z = 0$$

DOBILI SMO STACIONARNU TAČKU $A(-1,-2,0)$

$A \overset{?}{\in} \operatorname{int} S$

$(-1)^2 + (-2)^2 + 0^2 = 5 < 9$

$\Rightarrow A \in \operatorname{int} S$

2) ∂S - USLOVNI EKSTREMUM

$$f(x,y,z) = x^2 + y^2 + 2z^2 + 2x + 4y$$

$$g(x,y,z) = x^2 + y^2 + z^2 - 9$$

$$\nabla f = \lambda \nabla g$$

$$\nabla f = (2x+2,\ 2y+4,\ 4z)$$

$$\nabla g = (2x,\ 2y,\ 2z)$$

$$2x + 2 = 2\lambda x$$

$$2y + 4 = 2\lambda y$$

$$4z = 2\lambda z$$

$z = 0$ ili $\lambda = 2$

a) $\lambda = 2$

$$2x + 2 = 4x \Rightarrow x = 1$$

$$2y + 4 = 4y \Rightarrow y = 2$$

$z^2 + x^2 + y^2 = 9$

$z^2 = 9 - 1^2 - 2^2 = 4$

$z = 2$ ili $z = -2$

$B(1, 2, -2)$ $C(1, 2, 2)$

b) $z = 0$

$x^2 + y^2 = 9$

$x + 1 = \lambda x$

$y + 2 = \lambda y$

$x = \dfrac{1}{\lambda - 1}$

$1 = \lambda x - x = x(\lambda - 1)$

$2 = \lambda y - y = y(\lambda - 1) \Rightarrow y = \dfrac{2}{\lambda - 1}$

$z = 0$ i $\lambda = 1$

$x + 1 = x$ ↯

$$\frac{1}{(\lambda-1)^2} + \frac{4}{(\lambda-1)^2} = 9$$

$$\frac{5}{(\lambda-1)^2} = 9$$

$$(\lambda - 1)^2 = \frac{5}{9}$$

$\lambda - 1 = \dfrac{\sqrt{5}}{3}$ ili $\lambda - 1 = \dfrac{-\sqrt{5}}{3}$

$\lambda_1 = 1 + \dfrac{\sqrt{5}}{3}$ $\lambda_2 = 1 - \dfrac{\sqrt{5}}{3}$

$x = \dfrac{1}{\lambda_1 - 1} = \dfrac{1}{1 + \frac{\sqrt{5}}{3} - 1} = \dfrac{3}{\sqrt{5}}$ $x = \dfrac{1}{\lambda_2 - 1} = \dfrac{-3}{\sqrt{5}}$

$y = \dfrac{2}{\lambda_1 - \lambda} = \dfrac{6}{\sqrt{5}}$

$D\left(\dfrac{3}{\sqrt{5}}, \dfrac{6}{\sqrt{5}}, 0\right)$ $E\left(-\dfrac{3}{\sqrt{5}}, -\dfrac{6}{\sqrt{5}}, 0\right)$

DOBILI SMO 5 STACIONARNIH TAČAKA.

DA BISMO ODREDILI MINIMUM I MAKSIMUM

DOVOLJNO JE IZRAČUNATI VREDNOSTI U NJIMA.

$f = x^2 + y^2 + 2z^2 + 2x + 4y$

$f(A) = f(-1,-2,0) = 1+4-2-8 = -5$

$f(B) = f(1,2,-2) = 1+4+8+2+8 = 23$

$f(C) = f(1,2,2) = 23$

$f(D) = f\left(\frac{3}{\sqrt5}, \frac{6}{\sqrt5}, 0\right) = \frac{9}{5} + \frac{36}{5} + \frac{6}{\sqrt5} + \frac{24}{\sqrt5} = 9 + \frac{30}{\sqrt5} = 9 + 6\sqrt5$

$f(E) = f\left(-\frac{3}{\sqrt5}, -\frac{6}{\sqrt5}, 0\right) = \frac{9}{5} + \frac{36}{5} - \frac{6}{\sqrt5} - \frac{24}{\sqrt5} = 9 - 6\sqrt5$

$9 + 6\sqrt5 \gtrless 23$

$6\sqrt5 \gtrless 14 \quad /^2$

$36 \cdot 5 \gtrless 14^2$

$180 \gtrless 196$

$\Rightarrow$ 23 JE MAKSIMUM FUNKCIJE

$9 - 6\sqrt5 \gtrless -5$

$14 \gtrless 6\sqrt5$

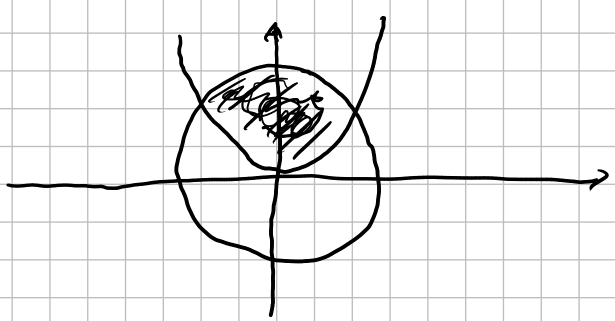
$\Rightarrow -5$ JE MINIMUM FUNKCIJE

5

$f(x,y) = x^2 + xy + y^2$

USLOVI : $x^2 + y^2 \leq 1$ i $y \geq x^2$: D

D - DOMEN



1) int D

$x^2+y^2<1 \qquad y>x^2$

$f(x,y)=x^2+xy+y^2$

$\frac{\partial f}{\partial x}=2x+y$

$\frac{\partial f}{\partial y}=x+2y$

$\frac{\partial f}{\partial x}=0 \Rightarrow 2x+y=0 \quad / \cdot -2$

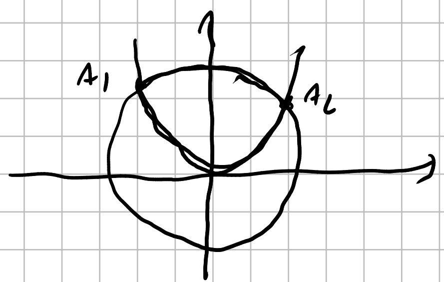
$\frac{\partial f}{\partial y}=0 \Rightarrow x+2y=0$

$-3x=0$

$x=0 \quad y=0 \qquad (0,0) \notin \text{int } D$

$\Rightarrow$ NEMAMO STACIONARNE TAČKE U int D

2) ISPITUJEMO ∂D



$x^2+y^2=1 \quad ; \quad y=x^2$

$x^2+x^4=1$

$x^2=t\geq 0 \qquad t^2+t-1=0$

$t_{1/2}=\frac{-1\pm\sqrt{1+4}}{2}=\frac{-1\pm\sqrt{5}}{2}$

$t=\frac{-1+\sqrt{5}}{2}$

$x_2=\sqrt{\frac{\sqrt{5}-1}{2}} \qquad x_1=-\sqrt{\frac{\sqrt{5}-1}{2}} \qquad y_0=\frac{\sqrt{5}-1}{2}$

$\partial D = S_1 \cup S_2$ $\qquad A_1 = (x_1, y_0) \quad A_2 = (x_2, y_0)$

$S_1 : \quad x^2 + y^2 = 1 \qquad i \qquad x_1 \leq x \leq x_2$

$S_2 : \quad y = x^2 \qquad i \qquad x_1 \leq x \leq x_2$

ŠTA SE DEŠAVA NA S_1?

$f(x,y) = x^2 + xy + y^2 \qquad S_1 : x^2 + y^2 = 1$

$g(x,y) = x^2 + y^2 - 1$

$\nabla f = \lambda \nabla g$

$(2x + y, \; 2y + x) = \lambda (2x, 2y)$

$2x + y = 2\lambda x$

$2y + x = 2\lambda y \quad / \cdot (-1)$

$2x + y - 2y - x = 2\lambda x - 2\lambda y$

$x - y = 2\lambda (x - y)$

$x = y \qquad\qquad \lambda = \frac{1}{2}$

$2x^2 = 1$

$2x + y = x \quad \Rightarrow \quad x + y = 0$

$x + 2y = y$

$x = \pm \frac{1}{\sqrt{2}} \qquad\qquad y = -x$

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \quad \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \quad \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

S_2 :

$y = x^2$

$f(x,y) = x^2 + xy + y^2$

$f|_{S_2} = x^2 + x^3 + x^4 = x^2 (1 + x + x^2)$

$$f(x) = x^4 + x^3 + x^2$$

$$f'(x) = 4x^3 + 3x^2 + 2x = 0$$

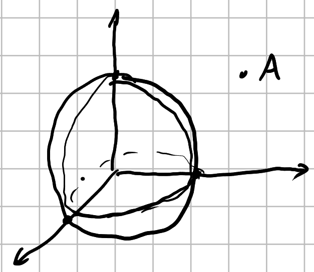
$$= x(4x^2 + 3x + 2) = 0$$

$$x = 0 \quad \text{u} \quad x_{1/2} = \frac{-3 \pm \sqrt{9 - 4 \cdot 4 \cdot 2}}{2} \notin \mathbb{R}$$

$\boxed{x = 0}$ $\boxed{y = 0}$

(1) NAĆI TAČKU NAJBLIŽU TAČKI $A(0,3,3)$ SA KRUŽNICE:

$K: \quad x^2+y^2+z^2=1 \qquad x+y+z=1$



$A \notin K$

K JE KOMPAKTAN SKUP

$K \subseteq \mathbb{R}^3$ PA JE DOVOLJNO DA JE OGRANIČEN I ZATVOREN

$K \subseteq B((0,0,0), 3) \Rightarrow$ OGRANIČEN

$g_1(x,y,z) = x^2+y^2+z^2-1 \qquad g_2(x,y,z) = x+y+z-1$

$K = g_1^{-1}(\{0\}) \cap g_2^{-1}(\{0\})$

$\Rightarrow$ PRESEK DVA ZATVORENA

KAKO JE K KOMPAKTAN FUNKCIJA NEPREKIDNA DOSTIŽE MINIMUM NA KRUGU K.

$$d((x,y,z), A) = \sqrt{x^2 + (y-3)^2 + (z-3)^2}$$

$\sqrt{}$ je rastući, pa možemo posmatrati funkciju

$$f(x,y,z) = x^2 + (y-3)^2 + (z-3)^2$$

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \qquad \lambda, \mu \in \mathbb{R}$$

$$\nabla f = (2x, 2(y-3), 2(z-3))$$

$$\nabla g_1 = (2x, 2y, 2z)$$

$$\nabla g_2 = (1, 1, 1)$$

$$\left.\begin{aligned} 2x &= \lambda 2x + \mu \\ 2(y-3) &= \lambda 2y + \mu \\ 2(z-3) &= \lambda 2z + \mu \end{aligned}\right\} \quad (-1) \qquad \begin{aligned} x^2 + y^2 + z^2 &= 1 \\ x + y + z &= 1 \end{aligned}$$

$$\cancel{2}(z-y) = \cancel{2}\lambda(z-y)$$

$$z = y \qquad \lambda = 1$$

1° $z = y$

$$x^2 + 2y^2 = 1$$

$$x + 2y = 1 \Rightarrow x = 1 - 2y$$

$$(1-2y)^2 + 2y^2 = 1$$

$$4y^2 - 4y + \cancel{1} + 2y^2 = \cancel{1}$$

$$6y^2 - 4y = 0$$

$$2y(3y-2) = 0$$

$y = 0$ ili $y = \frac{2}{3}$

$x = 1$ $\qquad x = -\frac{1}{3}$

$z = 0$ $\qquad z = \frac{2}{3}$

2° $\lambda = 1$

$$2x = 2x + \mu \Rightarrow \mu = 0$$
$$2(y-3) = 2y + \mu$$
$$2y - 6 = 2y + \mu \Rightarrow \mu = -6$$

$$\Rightarrow \lambda \neq 1$$

Imamo 2 stacionarne tačke

$B(1, 0, 0)$ $\qquad$ $C\left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

$$f(B) = 1^2 + (0-3)^2 + (0-3)^2 = 19$$
$$f(C) = \left(-\frac{1}{3}\right)^2 + 2\left(\frac{2}{3} - 3\right)^2 = \frac{1}{9} + 2\frac{49}{9} = \frac{99}{9} = 11$$

Minimalno rastojanje je 11

Domaći:

Naći tačke najbliže i najdalje od koordinatnog početka koje pripadaju ravni

$$L: x + y + z = 0$$

i nalaze se na cilindru

$$C: x^2 + y^2 = 1$$

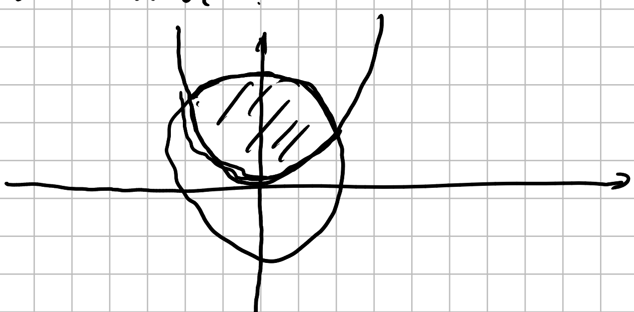
Mogli smo raditi i na II način

$$x + y + z = 1 \Rightarrow z = 1 - x - y$$

$$\tilde{f} = x^2 + (y-3)^2 + (1 - x - y - 3)^2$$

$$\tilde{g}: x^2 + y^2 + (1 - x - y)^2 - 1 = 0$$

Od juče:



Nisu tačke isto uvećane na krugu i paraboli

② $f : D \to \mathbb{R}$

$f(x,y,z) = x^2 + y^2 + z^2$

$D: \quad -1 \le x \le 2 \qquad 0 \le y \le 3 \qquad -2 \le z \le 4$

Kako da imamo zbir 3 nezavisne funkcije.

f max je kad je $|x|$ najveći i slično za min

minimum je za $x = y = z = 0$ $\quad f(0,0,0) = 0$

maksimum je za $x = 2 \quad y = 3 \quad z = 4$ $\quad f(2,3,4) = 29$

$f[D] = [0, 29]$

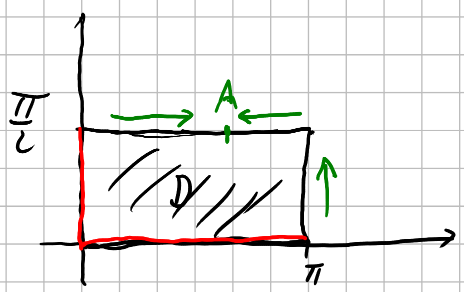
---

③ $g : D \to \mathbb{R}$

$D = [0, \pi] \times [0, \frac{\pi}{2}]$

$g(x,y) = \sin x + \sin y - \sin(x+y)$

Naći $g[D]$.



1) int D

$\frac{\partial g}{\partial x} = \cos x - \cos(x+y) = 0$

$\frac{\partial g}{\partial y} = \cos y - \cos(x+y) = 0 \quad (-1)$

$$\cos x - \cos y = 0$$

$$\cos x = \cos y$$

Na $[0, \pi]$ je $\cos$ 1-1 funkcija

$$\Rightarrow \quad x = y$$

$$\cos x - \cos 2x = 0$$

$$\cos x - 2\cos^2 x + 1 = 0$$

$$\cos x_{1,2} = \frac{-1 \pm \sqrt{1+8}}{-4} = \frac{-1 \pm 3}{-4} \quad \begin{cases} \nearrow 1 \in \partial D \\ \searrow -\frac{1}{2} \notin D \end{cases}$$

$$\cos x = -\frac{1}{2} \qquad \cos y = -\frac{1}{2} \quad \Rightarrow \quad y \notin \left[0, \frac{\pi}{2}\right] \ \lightning$$

$\Rightarrow$ NEMA STACIONARNE TOČKE U Int D

2) ∂D

$$\partial D = D_1 \cup D_2 \cup D_3 \cup D_4$$

D_1 : $y = 0 \qquad 0 \leq x \leq \pi$ $\qquad$ D_2 : $y = \frac{\pi}{2} \qquad 0 \leq x \leq \pi$

D_3 : $x = 0 \qquad 0 \leq y \leq \frac{\pi}{2}$ $\qquad$ D_4 : $x = \pi \qquad 0 \leq y \leq \frac{\pi}{2}$

$\underline{D_1:}$

$$g_1(x) = \sin x - \sin x \equiv 0$$

$\underline{D_2:}$

$$g_2(x) = \sin x + 1 - \sin\left(x + \frac{\pi}{2}\right) = 1 + \sin x - \sin x \cdot \overset{0}{\cancel{\cos\frac{\pi}{2}}} - \cos x \cdot \overset{1}{\cancel{\sin\frac{\pi}{2}}}$$

$$g_2(x) = \sin x - \cos x + 1$$

$$g_2'(x) = \cos x + \sin x = 0 \quad \Rightarrow \quad x = \frac{3\pi}{4}$$

$$A = \left(\frac{3\pi}{4}, \frac{\pi}{2}\right)$$

$\underline{D_3:}$

$$x = 0 \qquad 0 \leq y \leq \frac{\pi}{2}$$

$g_3(y) = \sin y - \sin y \equiv 0$

D_4:

$g_4(y) = \sin y - \sin(y+\pi) = \sin y - \sin y \overbrace{\cos\pi}^{-1} - \cos y \overbrace{\cancel{\sin\pi}}^{0}$

$= 2\sin y$

$0 \le y \le \frac{\pi}{2}$ $\quad g_4'(y) = 2\cos y > 0 \quad g_4(y) \nearrow$

minimum funkce je 0

maximum funkce je $g\left(\frac{3\pi}{4}, \frac{\pi}{2}\right) =$

$= \sin\frac{3\pi}{4} + \sin\frac{\pi}{2} - \sin\left(\frac{3\pi}{4} + \frac{\pi}{2}\right) = \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} = 1 + \sqrt{2}$

$g[D] = [0, 1+\sqrt{2}]$

---

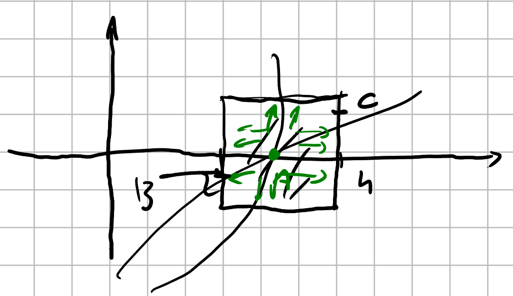
(4)

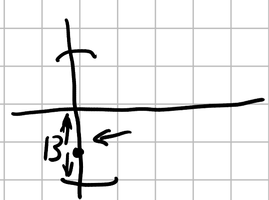
$f: D \to \mathbb{R}$

$D: \quad 2 \le x \le 4 \qquad -1 \le y \le 1$

$f(x,y) = \frac{3x}{y+3} + y + \frac{9}{x}$

$f[D] = ?$





1) int D

$\frac{\partial f}{\partial x} = \frac{3}{y+3} - \frac{9}{x^2} = 0 \qquad \cancel{3}x^2 = \cancel{9}(y+3)$

$y = \frac{x^2}{3} - 3$ – PARABOLA

$\frac{\partial f}{\partial y} = \frac{-3x}{(y+3)^2} + 1 = 0 \qquad 3x = (y+3)^2$ – PARABOLA

$$x^2 = 3(y+3) \qquad y+3 = \frac{x^2}{3}$$

$$3x = (y+3)^2$$

$$3x = \left(\frac{x^2}{3}\right)^2$$

$$3x = \frac{x^4}{9}$$

$$x^3 = 27 \Rightarrow x = 3 \qquad y = 0$$

DOBILI SMO STACIONARNU TAČKU U $\operatorname{int} D$:

$$A = (3, 0)$$

GRAFIČKI MOŽEMO ZAKLJUČITI DA JE A TAČKA LOKALNOG MINIMUMA.

2) $\partial D = D_1 \cup D_2 \cup D_3 \cup D_4$

$\underline{D_1}$: $x = 2 \quad -1 \leq y \leq 1$

$$g_1(y) = \frac{6}{y+3} + y + \frac{9}{2}$$

$$g_1'(y) = \frac{-6}{(y+3)^2} + 1 = 0$$

$$\frac{6}{(y+3)^2} = 1$$

$$(y+3)^2 = 6$$

$$y+3 = \pm\sqrt{6}$$

$$y_1 = \cancel{-\sqrt{6}-3} \notin D$$

$$y_2 = -3+\sqrt{6} \in D$$

$$B\,(2, -3+\sqrt{6})$$

$\underline{D_2}$ $\quad x = 4 \quad -1 \leq y \leq 1$

$$g_2(y) = \frac{12}{(y+3)} + 1 + \frac{9}{4}$$

$$g_2(y)' = \frac{-12}{(y+3)^2} + 1 = 0$$

$$(y+3)^2 = 12$$

$$y+3 = \pm\sqrt{12}$$

$$y = -3 \cancel{-} \sqrt{12} \notin D \qquad y = -3 + \sqrt{12}$$

Treba ispitati i

$D_3: \; y = -1 \qquad 2 \le x \le 4$

$D_4: \; y = 1 \qquad 2 \le x \le 4$

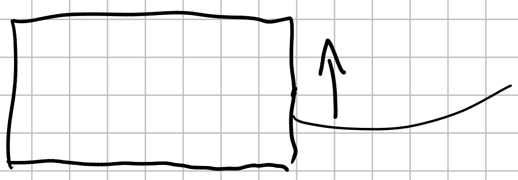
Dobiti sve stacionarne tačke

Izračunati vrednost funkcije u svim ovim tačkama

+ u temenima $(2,-1)$, $(2,1)$, $(4,-1)$ i $(4,1)$

U nekoj od tih tačaka je minimum, a u nekoj maksimum

!



$g(y) \nearrow$

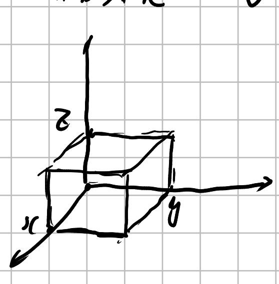
i maksimum je na temenu, a $g' \neq 0$

---

6 Dat je kvadar površine $6a^2$

Kolika je maksimalna zapremina kvadra?

Ubacimo kvadar u koordinatni sistem



Ubacimo ga da je $(0,0,0)$ jedno teme i u I oktantu je

$V = xyz$ — funkcija

$P = 2(xy + yz + xz) \longrightarrow$ uslov

$$2(xy + yz + xz) - 6a^2 = 0$$

$$\nabla V = (yz,\ xz,\ xy)$$

$$\nabla P = (2(y+z),\ 2(x+z),\ 2(x+y))$$

$$\nabla V = \lambda \nabla P$$

$$\begin{aligned} yz &= \lambda 2(y+z) \\ xz &= \lambda 2(x+z) \quad (-1) \\ xy &= \lambda 2(x+y) \quad (-1) \end{aligned}$$

+ važi uslov

$$z(y-x) = 2\lambda(y-x)$$
$$x(y-z) = 2\lambda(y-z)$$
$$y(x-z) = 2\lambda(x-z)$$

1° pretpostavimo $x \neq y \neq z \neq x$

$z = 2\lambda \qquad x = 2\lambda \qquad y = 2\lambda$ ↯

$\Rightarrow$ bar dva su jednaka

2° $x = y \neq z$

$x = 2\lambda \qquad y = 2\lambda$

$$yz = 2\lambda(y+z)$$
$$2\lambda z = 2\lambda(2\lambda + z)$$
$$2\lambda z = 4\lambda^2 + 2\lambda z$$
$$\Rightarrow \lambda = 0$$

Ako je $\lambda = 0$ bar neka koordinata je nula, pa nemamo kvadar

$\Rightarrow$ zapremina je nula

3° Mora biti $x = y = z$

$$P = 2(xy + xz + yz) = 6x^2 = 6a^2$$

$$x = |a| = y = z$$

$$V = |a|^3$$

7. DATA JE FUNKCIJA:

$$f : \mathbb{R}^3 \to \mathbb{R}$$

$$f(x,y,z) = x^2 + \frac{1}{2}y^2 + z^2$$

POVRŠ S JE DEFINISANA SA:

$$S = f^{-1}(\{22\})$$

DATA JE I FUNKCIJA g

$$g : S \to \mathbb{R}$$

$$g(x,y,z) = 2x + y + 4z.$$

RAVAN ξ JE TANGENTNA NA POVRŠ S U TAČKI U KOJOJ g DOSTIŽE MAKSIMUM.

ODREDITI TAČKE SA ELIPSOIDA

$$\varepsilon : \frac{1}{2}x^2 + \frac{1}{4}y^2 + z^2 = 81$$

KOJE SU NAJBLIŽE I NAJDALJE OD RAVNI ξ.

$$f = x^2 + \frac{1}{2}y^2 + z^2$$

$$S = f^{-1}(\{22\})$$

$$S : x^2 + \frac{1}{2}y^2 + z^2 = 22$$

$$g = 2x + y + 4z$$

$$\tilde{f} = x^2 + \frac{1}{2}y^2 + z^2 - 22 = 0$$

$$\nabla g = \lambda \nabla \tilde{f}$$

$$\nabla g = (2, 1, 4)$$

$$\nabla \tilde{f} = (2x, y, 2z)$$

$$\cancel{2} = \lambda \cancel{2} x \Rightarrow x = \frac{1}{\lambda}$$

$$1 = \lambda y \Rightarrow y = \frac{1}{\lambda}$$

$$\cancel{4} = \cancel{2}\lambda z \Rightarrow z = \frac{2}{\lambda}$$

$$x^2 + \frac{1}{2}y^2 + z^2 = \frac{1}{\lambda^2} + \frac{1}{2\lambda^2} + \frac{4}{\lambda^2} = 22$$

$$\frac{11}{2\lambda^2} = 22$$

$$\lambda^2 = \frac{1}{4} \qquad \lambda_1 = \frac{1}{2} \qquad \lambda_1 = -\frac{1}{2}$$

Dobijamo 2 stacionarne tačke

$A(2,2,4) \qquad B(-2,-2,-4)$

$g(A) = 2\cdot 2 + 2 + 4\cdot 4 = 22 \qquad g(B) = -22$

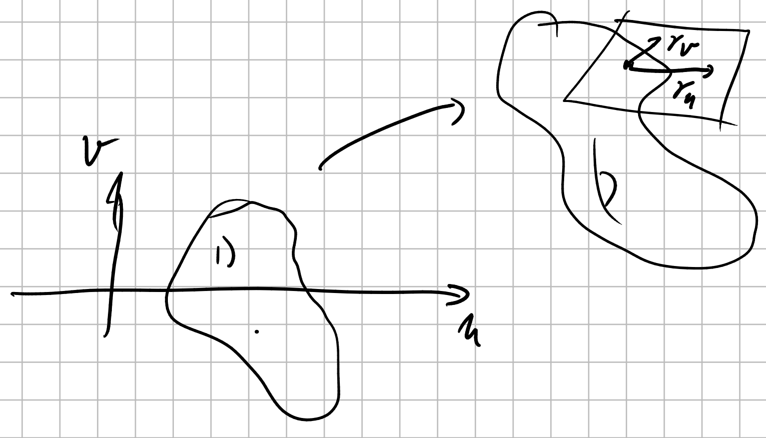
S je kompaktan pa funkcija g dostiže maksimum koji mora biti u tački A

$A(2,2,4)$ - tačka maksimuma

$$x^2 + \frac{1}{2}y^2 + z^2 = 22$$

Tangentna ravan površi

Površ je preslikavanje otvorenog skupa iz ravni u neki prostor ($\mathbb{R}^3$)



r - vektor položaja

$$r(u,v) = (x(u,v), y(u,v), z(u,v))$$

r_u и r_v [illegible] ако

је диференцијабилно и су линеарно не зависни

и одређују раван

$$r_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right)$$

$$r_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$$

$$\vec{n} = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

специјални случај ако је $u = x$ $v = y$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{\partial z}{\partial x} \\ 0 & 1 & \frac{\partial z}{\partial y} \end{vmatrix}$$

ако је површ задата имплицитно

$F(x, y, z) = 0 \rightsquigarrow$ површ S

$\nabla F \perp S$

$\vec{n} = \nabla F$

$S: F = x^2 + \frac{1}{2} y^2 + z^2 - 22 = 0$

$\nabla F = (2x, y, 2z)$ $\qquad A = (2, 2, 4)$

$\nabla F(A) = (4, 2, 8)$

$\vec{n} = (2, 1, 4)$

$\xi: \quad 2x + y + 4z + D = 0$

$A \in \xi \qquad 2 \cdot 2 + 2 + 4 \cdot 4 + D = 0$

$D = -22$

$\xi: \quad 2x + y + 4z - 22 = 0$

$\varepsilon: \quad \frac{1}{2}x^2 + \frac{1}{4}y^2 + z^2 = 81$

ŠINAX NASTOONNU,I OD RAVNI

$$d((x,y,z),\xi) = \frac{|2x+y+4z-22|}{\cancel{\sqrt{2^2+1^2+4^2}}}$$

$$h(x,y,z) = |2x+y+4z-\overset{\neq 0}{22}|$$

USLOV: $\varphi = \frac{1}{2}x^2 + \frac{1}{4}y^2 + z^2 - 81 = 0$

$\nabla h = \lambda \nabla \varphi \qquad\qquad 2x+y+4z-22 \geq 0$

$(2,1,4) = \lambda\,(2x, \frac{1}{2}y, 2z)$

$2 = \lambda 2x \qquad x = \frac{1}{\lambda}$

$1 = \frac{1}{2}\lambda y \qquad y = \frac{2}{\lambda}$

$4 = 2\lambda z \qquad z = \frac{2}{\lambda}$

$\frac{1}{2}\frac{1}{\lambda^2} + \cancel{\frac{1}{4}}\frac{\cancel{4}}{\lambda^2} + \frac{4}{\lambda^2} = 81$

$\frac{9}{2}\frac{1}{\lambda^2} = 81$

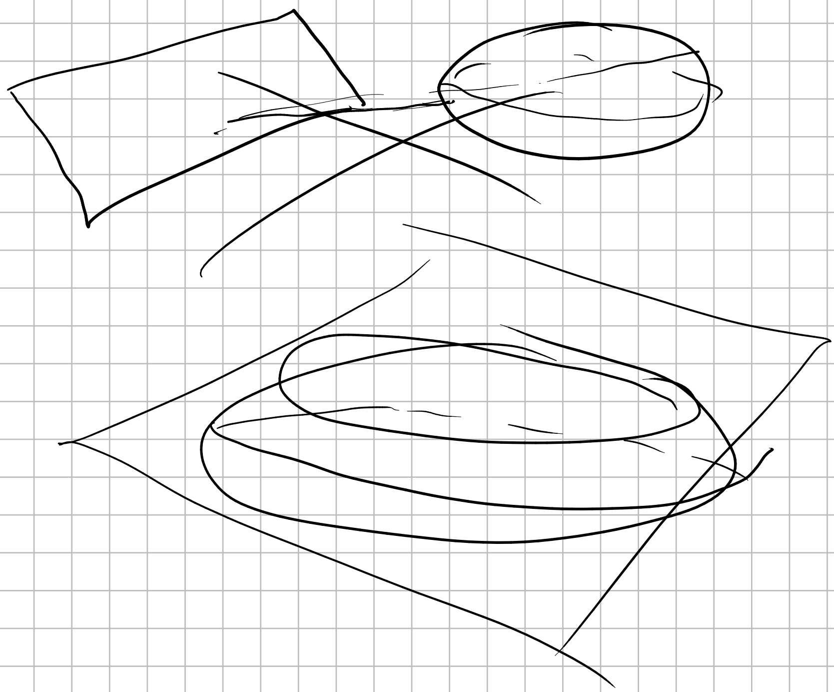
$\lambda^2 = \frac{9}{2} \qquad \lambda_1 = \frac{3}{\sqrt{2}} \qquad \lambda_2 = -\frac{3}{\sqrt{2}}$

$B = \left(\frac{\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}\right) \qquad C = \left(-\frac{\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}\right)$

RAZMISLITI ŠTA $2x+y+4z-22<0$

$\nabla h_1 = -\nabla h$

DOBIJAMO SE ISTE TAČKE



Da li je prazan presek elipsoida i ravni?