5. час

Tuesday, 16 November 2021 16:22

Задаци.

(1) $\displaystyle\int_0^\infty \frac{dx}{x^2+x-2}$, $x^2+x-2=(x-1)(x+2)$

сингуларитети су $x=1$ и $x=\infty$

$\displaystyle\int_0^\infty$ конв. $(\Rightarrow)$ $\displaystyle\int_0^1$ конв. и $\displaystyle\int_1^2$ (нпр.) конв. и $\displaystyle\int_2^\infty$ конв.

I начин - решимо (нађимо примитивну и видимо да ли $\exists \lim$) домаћи

II Начин - помоћу критеријума

$\displaystyle\int_0^1 \frac{dx}{x^2+x-2}$, $\displaystyle\frac{1}{x^2+x-2} \sim \, ?$ $x\to 1$

$$\frac{1}{x^2+x-2} = \frac{1}{(x+2)(x-1)} \sim \frac{1}{3(x-1)}, \quad x\to 1$$

$\displaystyle\int_0^1 \frac{dx}{x-1}$ диверг. јер $\displaystyle\int_0^{1-\varepsilon} \frac{dx}{x-1} = \ln(1-x)\Big|_{x=0}^{1-\varepsilon} = \ln\varepsilon - \ln 1 \xrightarrow{\varepsilon\to 0} -\infty$

$\left(= \displaystyle\int_{-1}^0 \frac{dt}{t} \text{ див. у нули} \right)$

$\Rightarrow \displaystyle\int_0^\infty \frac{dx}{x^2+x-2}$ дивергира.

Домаћи: Да ли конв. $\displaystyle\int_1^2 \frac{dx}{x^2+x-2}$ и $\displaystyle\int_2^\infty \frac{dx}{x^2+x-2}$? Без решавања.

(2.) $\displaystyle\int_0^\infty \frac{x^2}{x^4+x^2+1}\,dx$, једина синг. је $x=\infty$

$$\frac{x^2}{x^4+x^2+1} \sim x^{-2}, \quad x\to\infty$$

а $\displaystyle\int_1^\infty x^{-2}\,dx$ конв. $\Rightarrow$ $\displaystyle\int_0^\infty \frac{x^2}{x^4+x^2+1}\,dx$

Коментар. $\displaystyle\int_0^\infty x^{-2}\,dx$ див., $\displaystyle\int_0^\infty \frac{x^2}{x^4+x^2+1}\,dx$ конв. $(\Leftrightarrow)$ $\displaystyle\int_1^\infty \frac{x^2}{x^4+x^2+1}\,dx$

③ $\displaystyle\int_1^\infty \frac{dx}{\sqrt{1+x^3}}$ сингуларитет $x=\infty$

$$\frac{1}{\sqrt{1+x^3}} \sim x^{-3/2} \qquad x\to\infty$$

$$\int_1^\infty x^{-3/2}\,dx \ \text{ конв.} \quad (-3/2 < -1) \ \Rightarrow \ \int_1^\infty \frac{dx}{\sqrt{1+x^3}}\,dx \ \text{ конв.}$$

④ $\displaystyle\int_0^1 \frac{\ln x}{1-x^2}\,dx$ сингуларитет $x=0$ и $x=1$

$$\int_0^{1/2} \frac{\ln x}{1-x^2}\,dx \qquad \frac{\ln x}{1-x^2} \sim \ln x, \quad x\to 0$$

$$\int_0^{1/2} \ln x\,dx \ \text{ конв. (радили смо)} \ \Rightarrow \ \int_0^{1/2} \frac{\ln x}{1-x^2}\,dx$$

$$\int_{1/2}^1 \frac{\ln x}{1-x^2}\,dx \qquad \frac{\ln x}{1-x^2} \sim \frac{-1}{2}, \quad x\to 1$$

$$\ln x = \ln(1+(x-1)) = (x-1) + o(x-1), \quad x\to 1$$

$$\frac{\ln x}{1-x^2} = \frac{x-1+o(x-1)}{(1-x)(1+x)} = \frac{-1+o(1)}{x+1} \xrightarrow{x\to 1} -\frac{1}{2}$$

$$\int_{1/2}^1 \frac{-1}{2}\,dx \ \text{ конв.} \ \Rightarrow \ \int_{1/2}^1 \frac{\ln x}{1-x^2}\,dx \ \Rightarrow$$

$$\int_0^1 \frac{\ln x}{1-x^2}\,dx \ \text{ конв.}$$

Напомена. Ако f није деф. у $z\in\mathbb{R}$ (коначан), или $\exists$ коначан $\lim\limits_{x\to z^-} f(x)$, онда се z зове отклоњив (ПРИВИДАН) сингуларитет.

Овде се несвојствени, додефинисањем ф-је f у z, своди на Риманов.

⑤ $\displaystyle\int_0^\infty \frac{\operatorname{arctg} ax}{x^p}\,dx$, $a,p\in\mathbb{R}$

$x=0$ и $x=\infty$

$\int_0^1 \frac{\operatorname{arctg}(ax)}{x^p}\,dx$, $\frac{\operatorname{arctg}(ax)}{x^p} \sim \frac{ax}{x^p} = a x^{1-p}$, $x \to 0$

$\operatorname{arctg} t = t + o(t)$, $t \to 0$
$\operatorname{arctg} t \sim t$, $t \to 0$

$\int_0^1 a x^{1-p}\,dx$ конв. $\Leftrightarrow$ $a=0$, $p \in \mathbb{R}$
или $a \neq 0$, $1-p > -1$
$\Leftrightarrow$ $a \neq 0$, $p < 2$

$\int_1^\infty \frac{\operatorname{arctg}(ax)}{x^p}\,dx$ $\frac{\operatorname{arctg}(ax)}{x^p} \sim \frac{\operatorname{sgn} a \cdot \frac{\pi}{2}}{x^p}$, $x \to +\infty$

$\int_1^\infty x^{-p}\,dx$ конв. $\Leftrightarrow$ $-p < -1$ $\Leftrightarrow$ $p > 1$ $\Big\}$ $a \neq 0$

$a=0 \Rightarrow \int_1^\infty \frac{\operatorname{arctg}(ax)}{x^p}$ конв.

Закл. $\int_0^\infty \frac{\operatorname{arctg}(ax)}{x^p}\,dx$ конв. $\Leftrightarrow$ $a=0$ или $a \neq 0$ и $p \in (1,2)$

6. $\int_0^2 \frac{dx}{\ln x}$ $x=0$ (отклоњиви)
$x=1$ (развиј апр. $\ln x$ у околини $x=1$)

Домаћи.

Пример: $\int_2^\infty \frac{dx}{x^p \ln^2 x}$

Доказати.

$p>1$ конв.
$p<1$ див.
$p=1$ испитати

$\boxed{p>1}$ $\exists \alpha \in (1,p)$ $\left(\alpha = \frac{1+p}{2}\right.$ нпр. $\left.\right)$

$\frac{1}{x^p \ln^2 x} \leq \frac{1}{x^\alpha}$, за $x \geq M$

$$\int_2^\infty \frac{dx}{x^\alpha} = \int_2^\infty x^{-\alpha}\,dx \quad \text{конв. јер је } -\alpha < -1$$

$$\frac{\frac{1}{x^p \ln^q x}}{\frac{1}{x^\alpha}} = \frac{1}{x^{p-\alpha}\ln^q x} \xrightarrow{x\to\infty} 0$$

$\boxed{p<1}$ $\exists\, \alpha \in (p,1)$

$$\frac{1}{x^p \ln^q x} \geq \frac{1}{x^\alpha}, \quad x \geq M \quad \text{јер}$$

$$\frac{\frac{1}{x^p \ln^q x}}{\frac{1}{x^\alpha}} = \frac{x^{\alpha-p}}{\ln^q x} \xrightarrow{x\to\infty} \infty$$

$$\int_2^\infty \frac{1}{x^\alpha}\,dx \quad \text{див. јер је } -\alpha > -1 \quad (\alpha < 1)$$

$\boxed{p=1}$

$$\int_2^\infty \frac{dx}{x \ln^q x} = \left\{ \begin{array}{l} t = \ln x \\ \frac{dx}{x} = dt \end{array} \right\} = \int_{\ln 2}^\infty \frac{dt}{t^q}$$

$$= \int_{\ln 2}^\infty t^{-q}\,dt \quad \text{конв.} \iff -q < -1 \iff q > 1$$

ЗАКЉУЧАК. $\int_2^\infty \frac{dx}{x^p \ln^q x}$ конв. $\iff$ $p>1$ или $p=1,\ q>1$ $\square$

Апсолутна конвергенција интеграла

деф. Нека је f деф. на $[a,b)$ и непрекидна. Кажемо да $\int_a^b f(x)\,dx$ апсолутно конв. ако $\int_a^b |f(x)|\,dx$ конвергира.

Тврђење. Ако $\int_a^b f(x)\,dx$ апсолутно конв. онда он конвергира.

Доказ. $\int_a^b f(x)dx$ конв. $\Leftrightarrow \forall \varepsilon\ \exists b_0 \in [a,b)\ \forall b_1, b_2\ \left|\int_{b_1}^{b_2} f(x)dx\right| < \varepsilon$

Знам $\int_a^b |f(x)|dx$ конв. $\Leftrightarrow \forall \varepsilon\ \exists b_0\ \forall b_1, b_2 > b_0\ \left|\int_{b_1}^{b_2} |f(x)|dx\right| < \varepsilon$

али $\left|\int_{b_1}^{b_2} f(x)dx\right| \leq \left|\int_{b_1}^{b_2} |f(x)|dx\right| < \varepsilon$ $\square$

Задатак. $\int_0^\infty \frac{\sin x}{x^{3/2}} dx$

сингуларитети $x=0$ и $x=\infty$

$\int_0^1 \frac{\sin x}{x^{3/2}} dx$, $\frac{\sin x}{x^{3/2}} \sim \frac{x}{x^{3/2}} = x^{-1/2}$, $x \to 0$

$\int_0^1 x^{-1/2} dx$ конв. јер $-1/2 > -1$

$\int_1^\infty \frac{\sin x}{x^{3/2}} dx$ апсолутно конв. $\Rightarrow$ конвергира

$\left|\frac{\sin x}{x^{3/2}}\right| \leq \frac{1}{x^{3/2}}$, $\int_1^\infty \frac{dx}{x^{3/2}}$ конв. $(-3/2 < -1)$

1. пор. кр. $\Rightarrow$ $\int_1^\infty \left|\frac{\sin x}{x^{3/2}}\right| dx$ конв.

Ојлерови интеграли

$B(\alpha, \beta) := \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx$ Бета функција (I Ојлеров интеграл)

$\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x} dx$ Гама функција (II Ојлеров)

Својства Γ - ф-је

1. Домен? За које $\alpha \in \mathbb{R}$ је $\Gamma(\alpha)$ деф. тј. за које α $\int_0^\infty x^{\alpha-1} e^{-x} dx$ конвергира?

$x = 0$

$x^{\alpha-1}e^{-x} \sim x^{\alpha-1}, \quad x \to 0$

$\int_0^1 x^{\alpha-1}dx$ конв. $\Leftrightarrow \alpha - 1 > -1$
$\Leftrightarrow \boxed{\alpha > 0}$

$x = \infty$

$\int_1^\infty$ конв. $\forall \alpha$ jер

$x^{\alpha-1}e^{-x} \le e^{-x/2}, \quad x \ge M$

jер $\dfrac{x^{\alpha-1}e^{-x}}{e^{-x/2}} = x^{\alpha-1}e^{-x/2} \xrightarrow[x\to\infty]{} 0, \quad \forall \alpha$

$\Downarrow$

$\exists M,\ x \ge M \quad \dfrac{x^{\alpha-1}e^{-x}}{e^{-x/2}} \le 1$

$\int_1^\infty e^{-x/2}dx = -2e^{-x/2}\Big|_1^\infty = 0 + 2e^{-1/2}$

<u>Закључак.</u> $D_\Gamma = (0, \infty)$

2. $\alpha > 0 \quad \Gamma(\alpha+1) = \int_0^\infty x^\alpha e^{-x}\,dx = \left\{ \begin{array}{l} x^\alpha = u \\ e^{-x}dx = dv \\ du = \alpha x^{\alpha-1}dv \\ v = -e^{-x} \end{array} \right\} =$

$= -x^\alpha e^{-x}\Big|_0^\infty + \alpha \int_0^\infty x^{\alpha-1}e^{-x}dx = 0 - 0 + \alpha\Gamma(\alpha)$

због e^{-x} због x^α $(\alpha > 0)$

$\boxed{\Gamma(\alpha+1) = \alpha\Gamma(\alpha)}$ $\rightsquigarrow$ (може да се прошири домен Γ ф-је)

Шта је $\Gamma(1)$? Шта је $\Gamma(n)$, $n \in \mathbb{N}$?

$\Gamma(1) = \int_0^\infty e^{-x}dx = -e^{-x}\Big|_0^\infty = 0 + 1 = 1$

$\Gamma(n) = \Gamma(n-1+1) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \ldots = (n-1)!$

Γ ф-ја је проширење на $(0,\infty)$ ф-је факторијел.

3. Формула допуне: $\alpha \in (0,1) \Rightarrow$ $\boxed{\Gamma(\alpha)\Gamma(1-\alpha) = \dfrac{\pi}{\sin \pi\alpha}}$

Без доказа.

Пример. $\Gamma(1/2)\cdot\Gamma(1/2) = \dfrac{\pi}{\sin \pi/2} = \pi \Rightarrow \boxed{\Gamma(1/2) = \sqrt{\pi}}$

Пуасонов интеграл $\displaystyle\int_0^\infty e^{-x^2}dx = \left\{ \begin{array}{l} \text{смена } x^2 = t \\ \begin{array}{c|c|c} x & 0 & \infty \\ \hline t & 0 & \infty \end{array}, \quad \begin{array}{l} x = \sqrt{t} \\ dx = \frac{1}{2\sqrt{t}}dt \end{array} \end{array} \right\}$

$$= \frac{1}{2}\int_0^\infty \frac{1}{\sqrt{t}} e^{-t}dt = \frac{1}{2}\int_0^\infty t^{1/2-1}e^{-t}dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\int_{-\infty}^\infty e^{-x^2}dx = \underbrace{\int_{-\infty}^0 e^{-x^2}dx}_{-x=t} + \int_0^\infty e^{-x^2}dx = 2\int_0^\infty e^{-x^2}dx = \sqrt{\pi}$$

4. Γ ф-ја је непрекидна и ∞ диференцијабилна, $\exists$ k-ти извод $\forall k \in \mathbb{N}$

Γ ф-ја није елементарна ф-ја.

Својства B - ф-је

1. Домен? За које $\alpha, \beta \in \mathbb{R}$ конв. $\displaystyle\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$?

$x=0$ и $x=1$ су синг.

$x \to 0$ $\quad x^{\alpha-1}\underbrace{(1-x)^{\beta-1}}_{\to 1} \sim x^{\alpha-1}$, $\displaystyle\int_0^{1/2} x^{\alpha-1}dx$ конв. $\Leftrightarrow \alpha-1 > -1$ $\Leftrightarrow \alpha > 0$

$x \to 1$ $\quad \underbrace{x^{\alpha-1}}_{\to 1}(1-x)^{\beta-1} \sim (1-x)^{\beta-1}$, $\displaystyle\int_{1/2}^1 (1-x)^{\beta-1}dx$ конв. $\Leftrightarrow \beta-1 > -1$ $\Leftrightarrow \beta > 0$

$$\int_{1/2}^1 (1-x)^{\beta-1}dx = \left\{ \begin{array}{l} t = 1-x \\ dt = -dx \end{array} \right\} = -\int_{1/2}^0 t^{\beta-1}dt = \int_0^{1/2} t^{\beta-1}dt$$

$\boxed{B(\alpha,\beta) \text{ је деф. за } \alpha>0, \beta>0}$

2. $B(\alpha,3) = B(3,\alpha)$

$$\int_0^1 x^{\alpha-1}(1-x)^{3-1}\,dx = \left\{ \begin{array}{l} t = 1-x \\ dx = -dt \\ \begin{array}{c|c|c} x & 0 & 1 \\ \hline t & 1 & 0 \end{array} \end{array} \right\} = -\int_1^0 (1-t)^{\alpha-1} t^{3-1}\,dt$$

$$= \int_0^1 t^{3-1}(1-t)^{\alpha-1}\,dt$$

$$= B(3,\alpha)$$

3. $$B(\alpha+1, 3) = \int_0^1 x^{\alpha}(1-x)^{3-1}\,dx = \left\{ \begin{array}{l} x^{\alpha} = u \\ (1-x)^{3-1}dx = dv \\ du = \alpha x^{\alpha-1}dx \\ v = -\frac{(1-x)^3}{3} \end{array} \right\}$$

$$= -\frac{x^{\alpha}(1-x)^3}{3}\Big|_0^1 + \frac{\alpha}{3}\int_0^1 x^{\alpha-1}\underbrace{(1-x)^{3-1}(1-x)}_{(1-x)^3}\,dx$$

$$= -(0-0) + \frac{\alpha}{3}\left\{ \int_0^1 x^{\alpha-1}(1-x)^{3-1}dx - \int_0^1 x^{\alpha}(1-x)^{3-1}dx \right\}$$

$$B(\alpha+1,3) = \frac{\alpha}{3}\left(B(\alpha,3) - B(\alpha+1,3) \right) \quad / \cdot 3$$

$$(\beta - \alpha)\, B(\alpha+1,3) = \alpha\, B(\alpha,3)$$

$$\boxed{B(\alpha+1,3) = \frac{\alpha}{\alpha+3} B(\alpha,3)}$$

$$B(\alpha,3+1) = B(3+1,\alpha) = \frac{3}{\alpha+3} B(3,\alpha) = \frac{3}{\alpha+3} B(\alpha,3)$$

Домашно. а) Најди $B(1,1)$ $(=1)$

б) Најди $B(m,n)$, за $m,n \in \mathbb{N}$

4. У $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$ направимо смену $x=\frac{t}{1+t}$

$$dx = d\left(\frac{1+t-1}{1+t}\right) = d\left(1-\frac{1}{1+t}\right) = \frac{dt}{(1+t)^2}$$

$x(1+t)=t$, $\quad t(x-1) = -x$

$t = \frac{x}{1-x}$,

x	0	1
t	0	∞

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha-1}}\left(1-\frac{t}{1+t}\right)^{\beta-1}\frac{dt}{(1+t)^2} =$$

$$= \int_0^\infty \frac{t^{\alpha-1}(1+t-t)^{\beta-1}}{(1+t)^{\alpha+\beta}}dt$$

$$\Rightarrow \quad B(\alpha,\beta) = \int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}dt$$

5. Веза између B и Γ ф-је

$$\boxed{B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}}$$ без доказа

Задаци.

1. $\int_0^{\pi/2} \sin^\alpha x \cos^\beta x \, dx$, за које α и β конв. Изразити преко B ф-је

$x=0$ и $x=\pi/2$

$x\to 0 \quad \sin^\alpha x\cos^\beta x \sim x^\alpha$, конв. $\Leftrightarrow \alpha > -1$

$x\to\pi/2 \quad \sin^\alpha x\cos^\beta x \sim 1\cdot \sin(\pi/2-x)^\beta \sim (\pi/2-x)^\beta$ конв. $\Leftrightarrow \beta > -1$

Трик. $\sin^2 x = t$, $\cos^2 x = 1-t$
$2\sin x\cos x = dt$

x	0	$\pi/2$
t	0	1

$$\int_0^{\pi/2} \sin^{\alpha} x \cos^3 x \, dx = \frac{1}{2}\int_0^{\pi/2} \sin^{\alpha-1} x \cos^{3-1} x \; 2 \sin x \cos x \, dx =$$

$$= \frac{1}{2}\int_0^1 t^{\frac{\alpha-1}{2}} (1-t)^{\frac{3-1}{2}} dt = \frac{1}{2}\int_0^1 t^{\frac{\alpha+1}{2}-1} (1-t)^{\frac{3+1}{2}-1} dt$$

$$= \frac{1}{2} B\left(\frac{\alpha+1}{2}, \frac{3+1}{2}\right)$$

Домашн. Намн $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$, $m, n \in \mathbb{N}$

(УПЪТСТВО: сведете ги Γ
$\Gamma(n) = (n-1)!$
$\Gamma(n+\frac{1}{2}) = (n-\frac{1}{2})(n-\frac{3}{2})\cdots\frac{3}{2}\underbrace{\Gamma(\frac{1}{2})}_{\sqrt{\pi}}$)

2. $$\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}} dx = \left\{ \begin{array}{l} \sqrt[3]{x} = t \\ x = t^3 \\ dx = 3t^2 dt \end{array} \right\} = 3\int_0^\infty t^{3/2} e^{-t} t^2 dt =$$

$$= 3\int_0^\infty t^{7/2} e^{-t} dt = 3\,\Gamma(9/2) = 3 \cdot \frac{7}{2}\Gamma\left(\frac{7}{2}\right) = 3 \cdot \frac{7}{2} \cdot \frac{5}{2}\Gamma\left(\frac{5}{2}\right) =$$

$$= 3\,\frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = 3 \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = 3\,\frac{7!!}{16}\sqrt{\pi}$$

$$(2n)!! = 2n \cdot (2n-2)(2n-4) \cdots 2$$

$$(2n+1)!! = (2n+1)(2n-1)(2n-3) \cdots 5 \cdot 3$$

3. $$\int_0^\infty \frac{x^{m-1}}{1+x^n} dx, \qquad 0 < m < n \in \mathbb{N}$$

$$\left(B(\alpha, \beta) = \int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}} dt \right)$$

смени: $x^n = t \qquad x = t^{1/n} \qquad dx = \frac{1}{n} t^{1/n - 1} dt$

$$\int_0^\infty \frac{x^{m-1}}{1+x^n} dx = \frac{1}{n}\int_0^\infty \frac{t^{\frac{m-1}{n}}}{1+t} t^{1/n-1} dt =$$

$$= \frac{1}{m}\int_0^\infty \frac{t^{\frac{m}{n}-1}}{1+t} dt \qquad \frac{m}{n} - 1 = \alpha - 1$$

$$\alpha + \beta = 1$$

$\alpha = \frac{m}{n}$ $\quad \beta = 1-\frac{m}{n}$

$$= \frac{1}{n} B\left(\frac{m}{n}, 1-\frac{m}{n}\right) = \frac{1}{n} \frac{\Gamma\left(\frac{m}{n}\right)\Gamma\left(1-\frac{m}{n}\right)}{\Gamma\left(\frac{m}{n}+1-\frac{m}{n}\right)} = \frac{1}{n} \frac{\frac{\pi}{\sin\frac{m\pi}{n}}}{\Gamma(1)} = \frac{\pi}{n \sin\frac{m\pi}{n}}.$$

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}$$

Домаћи:

4. $\int_0^1 x^2 \left(\ln\frac{1}{x}\right)^3 dx$

5. $\int_0^{\pi} \sin^4 x \cos^6 x \, dx$ (пробати преко B ф-је)

6. $\int_0^a x^2 \sqrt{a^2-x^2}\, dx, \quad a>0$

7. $\int_0^\infty \frac{dx}{1+x^3}$

РЕДОВИ

$a_n \in \mathbb{R}$, $n = 0, 1, \dots$ / $1, 2, \dots$ $\longrightarrow$ $S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \dots + a_N$

низ парцијалних сума

деф. Ако низ S_N конвергира (конвергира у $\mathbb{R}$), кажемо да ред $\sum_{n=1}^{\infty} a_n$ конв.

— || — S_N дивергира — || — дивергира.

Пишемо: $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$.

Пример. $a_n = q^n$ за које q конв. $\sum_{n=0}^{\infty} a_n$

$$S_N = 1+q+\cdots+q^N = \begin{cases} \dfrac{1-q^{N+1}}{1-q}, & q \neq 1 \\ N+1, & q = 1 \end{cases}$$

Кад конв. у $\mathbb{R}$?

$q \neq 1$ S_N конв. $\Leftrightarrow$ q^{N+1} конв. $\Leftrightarrow$ $|q|<1$
$-1<q<1$

$q=1$ S_N дивергира.

$$\boxed{\sum q^n \text{ конвергира } \Leftrightarrow |q|<1}$$

Задаци. Да ли конвергира ред $\sum a_n$?

1. $a_n = \ln\left(1+\frac{1}{n}\right)$

$$S_N = \sum_{n=1}^{N} \ln\left(1+\frac{1}{n}\right) = \sum_{n=1}^{N} \ln\frac{1+n}{n} = \sum_{n=1}^{N} \left[\ln(1+n) - \ln n\right]$$

$$= (\cancel{\ln 2} - \ln 1) + (\cancel{\ln 3} - \cancel{\ln 2}) + (\cancel{\ln 4} - \cancel{\ln 3}) + \cdots + (\cancel{\ln N} - \cancel{\ln (N-1)}) + (\ln(N+1) - \cancel{\ln N})$$

$$= \ln(N+1) \xrightarrow{N\to\infty} \infty \quad \Rightarrow \text{ диверг.}$$

2. $a_n = \dfrac{1}{n(n+1)} = \dfrac{n+1-n}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$

$$S_N = \sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \cdots \cancel{+} \left(\cancel{\tfrac{1}{N-1}} - \cancel{\tfrac{1}{N}}\right) + \left(\cancel{\tfrac{1}{N}} - \tfrac{1}{N+1}\right)$$

$$= 1 - \frac{1}{N+1} \xrightarrow{N\to\infty} 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$