5. ЧАС

Tuesday, 2 November 2021 16:28

Задатак. $\lim\limits_{n\to\infty} \underbrace{\left(\frac{n}{1^2+n^2} + \frac{n}{2^2+n^2} + \frac{n}{3^2+n^2} + \dots + \frac{n}{n^2+n^2}\right)}_{a_n}$

Нађи f и $[a,b]$ тд. $a_n = S_n(f,[a,b])$

$$\Rightarrow a_n \xrightarrow{n\to\infty} \int_a^b f(x)\,dx$$

$$S_n(f,[a,b]) = \frac{b-a}{n}\sum_{i=1}^{n} f(\xi_i)$$

$$a_n = \frac{n}{1^2+n^2} + \frac{n}{2^2+n^2} + \dots + \frac{n}{n^2+n^2} = \frac{1}{n}\left(\frac{n^2}{1^2+n^2} + \dots + \frac{n^2}{n^2+n^2}\right) =$$

$\boxed{\text{пробам } a=0,\ b=1,\ \xi_i = \frac{i}{n}}$

$$a_n = \frac{1}{n}\left(\frac{1}{\left(\frac{1}{n}\right)^2+1} + \frac{1}{\left(\frac{2}{n}\right)^2+1} + \dots + \frac{1}{\left(\frac{n}{n}\right)^2+1}\right) = \underbrace{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\left(\frac{i}{n}\right)^2+1}}_{= f\left(\frac{i}{n}\right)}$$

$$f(x) := \frac{1}{x^2+1}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \int_0^1 \frac{dx}{x^2+1} = \operatorname{arctg} x \Big|_0^1 = \operatorname{arctg} 1 - \operatorname{arctg} 0 = \frac{\pi}{4} \qquad \square$$

Т са претх. часа: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ (ОСНОВНА Т.)

$\left(\frac{d}{dx} F(x) := F'(x)\right)$

Питање: $\frac{d}{dx}\int_x^b f(t)\,dt = ?$

Одговор: $-f(x)$ јер је $\int_x^b f(t)\,dt = -\int_b^x f(t)\,dt \;\Big/ \frac{d}{dx}$

Последица: $\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} f(t)\,dt = f(\beta(x))\cdot\beta'(x) - f(\alpha(x))\cdot\alpha'(x)$

Δ: $F(x) := \int_a^x f(t)\,dt \Rightarrow F'(x) = f(x)$

$$\frac{d}{dx}\underbrace{\int_a^{\beta(x)} f(t)\,dt} = \frac{d}{dx}F(\beta(x)) = F'(\beta(x))\cdot\beta'(x)$$
$$= f(\beta(x))\cdot\beta'(x)$$

$$\frac{d}{dx}\int_{\alpha(x)}^{a} f(t)\,dt = \frac{d}{dx}(-F(\alpha(x))) = -F'(\alpha(x))\cdot\alpha'(x)$$
$$= -f(\alpha(x))\cdot\alpha'(x)$$

$$\Rightarrow \int_{\alpha(x)}^{\beta(x)} f(t)\,dt = \int_{\alpha(x)}^{a} f(t)\,dt + \int_a^{\beta(x)} f(t)\,dt$$

$\square$

Доказ Њ.-Л. на други начин.

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

фикс. α диф. $F(x) = \int_\alpha^x f'(t)\,dt$

знамо, из осн. Т. $\Rightarrow F'(x) = f'(x)$

Лагранжева т. $\Rightarrow F(x) = f(x) + c$

$$\int_a^b f'(t)\,dt = \int_\alpha^b f'(t)\,dt - \int_\alpha^a f'(t)\,dt$$
$$= F(b) - F(a) = f(b) + c - (f(a) + c)$$
$$= f(b) - f(a)$$

$\square$

Парцијална интеграција и смена променљиве код одређеног интеграла

Т) $u, v \in C^1$ на $[a,b]$ (u' и v' су непрекидне)

Тада је $$\int_a^b u(x)\,v'(x)\,dx = u(b)v(b) - u(a)v(a) - \int_a^b u'(x)\,v(x)\,dx$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Д: $f(x) := u(x)v(x)$

Њ.-Л. $f(b) - f(a) = \int_a^b f'(x)\,dx$

$$u(b)v(b) - u(a)v(a) = \int_a^b u'(x)v(x)\,dx + \int_a^b u(x)v'(x)\,dx$$

□

Задатак: $$\int_0^2 x e^x dx = \left\{ \begin{matrix} x = u & du = dx \\ e^x = dv & v = e^x \end{matrix} \right\} =$$

$$= xe^x\Big|_0^2 - \int_0^2 e^x dx = 2e^2 - 0\cdot e^0 - e^x\Big|_0^2$$

$$= 2e^2 - e^2 + 1 = e^2 + 1$$ □

Т) $f: [a,b] \to \mathbb{R}$ непр. и $\varphi: [\alpha,\beta] \to [a,b]$ класе C^1 (φ' непр.)
$\varphi(\alpha) = a$, $\varphi(\beta) = b$. Тада важи формула

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt.$$

Д: F примитивна за f (нпр. $\int_a^x f(t)\,dt =: F(x)$)

$$\int_a^b f(x)\,dx \overset{\text{Њ.-Л.}}{=} F(b) - F(a) = F(\varphi(\beta)) - F(\varphi(a))$$

$$\left(= G(\beta) - G(\alpha) = \int_\alpha^\beta G'(t)\,dt \right), \quad G := F \circ \varphi$$

$$= \int_\alpha^\beta (F\circ\varphi)'(t)\,dt = \int_\alpha^\beta F'(\varphi(t)) \cdot \varphi'(t)\,dt =$$

$$= \int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t)\,dt \qquad \square$$

Примери: f непр. на $[-a,a]$. Доказати

1. f парне $\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

2. f непарне $\Rightarrow \int_{-a}^{a} f(x)\,dx = 0$

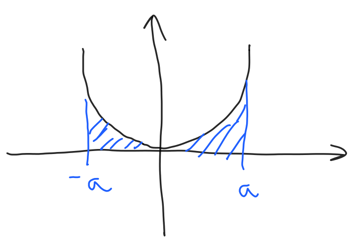
Д. 1. $f(x) = f(-x)$ $\quad \int_{-a}^{a} f(x)\,dx = \underbrace{\int_{-a}^{0} f(x)\,dx}_{\text{смена}} + \int_0^a f(x)\,dx$

смена $t = -x$, $dt = -dx$

x	$-a$	0
t	a	0

$$\int_{-a}^{0} f(x)\,dx = -\int_a^0 f(-t)\,dt = \int_0^a f(-t)\,dt = \int_0^a f(t)\,dt = \int_0^a f(x)\,dx$$

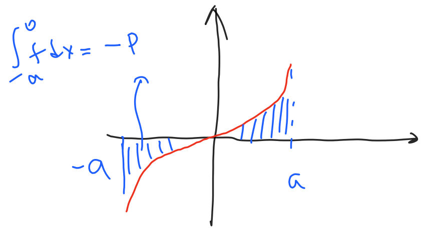
$$\Rightarrow \int_{-a}^{a} = \int_{-a}^{0} + \int_0^a = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

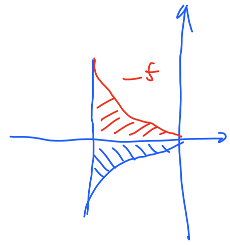


2. $f(-x) = -f(x)$ $\quad$ исто $\int_{-a}^{a} = \int_{-a}^{0} + \int_0^a$

$$\int_{-a}^{0} f(x)\,dx = \left\{\begin{matrix} -x=t \\ dx=-dt\end{matrix}\right\} = -\int_{a}^{0} f(-t)\,dt = \int_{0}^{a} f(-t)\,dt = -\int_{0}^{a} f(t)\,dt$$

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$

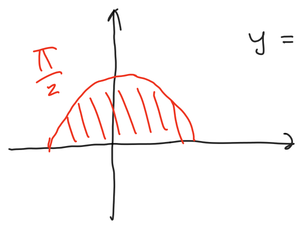




Задатак. $\int_{-1}^{1} \sqrt{1-x^2}\,dx$

$\parallel$

$2\int_{0}^{1} \sqrt{1-x^2}\,dx$



$y = \sqrt{1-x^2}$

$y^2 = 1 - x^2$

$x^2 + y^2 = 1$

смена $x = \sin t = \varphi(t)$ $\quad \varphi(0) = 0$, $\varphi(\pi/2) = 1$

$$\int_{0}^{1} \sqrt{1-x^2}\,dx = \left\{ \begin{matrix} x = \sin t & \\ \begin{array}{c|c|c} t & 0 & \pi/2 \\ \hline x & 0 & 1 \end{array} & dx = \cos t\,dt \end{matrix} \right\} = \int_{0}^{\pi/2} \sqrt{\cos^2 t}\,\cos t\,dt$$

$$\overset{\cos t \geq 0}{=} \int_{0}^{\pi/2} \cos^2 t\,dt = \int_{0}^{\pi/2} \frac{1+\cos 2t}{2}\,dt = \frac{1}{2}\int_{0}^{\pi/2} dt + \int_{0}^{\pi/2} \cos 2t\,dt$$

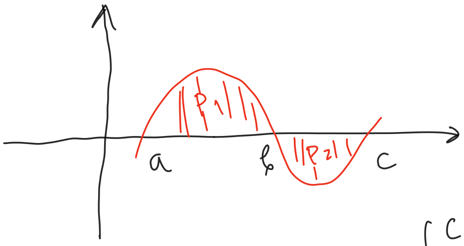
$$= \frac{\pi}{4} + \frac{\sin 2t}{2}\Big|_{0}^{\pi/2} = \frac{\pi}{4} + \frac{1}{2}(\sin \pi - \sin 0) = \frac{\pi}{4}$$

$$\Rightarrow \int_{-1}^{1} \sqrt{1-x^2}\,dx = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}$$

Примене одр. интеграла у геометрији

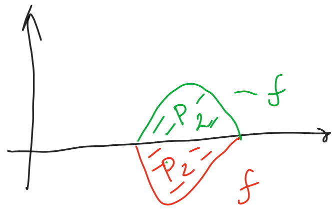
I Површина равног лика

1°



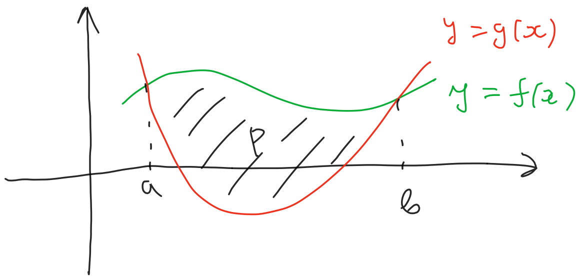
$$\int_a^c f(x)\,dx = \underbrace{\int_a^b f(x)\,dx}_{P_1} + \int_b^c f(x)\,dx$$

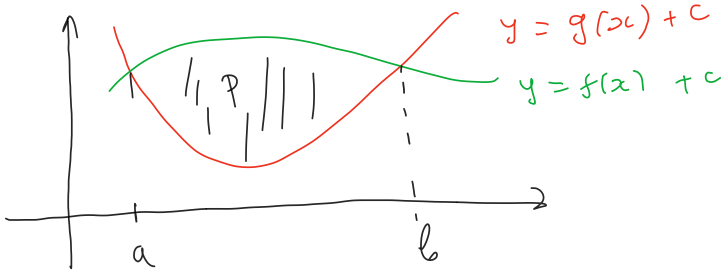
$$\int_b^c f(x)\,dx = -\int_b^c -f(x)\,dx = -P_2$$



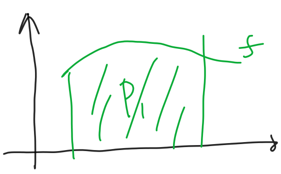
$$\Rightarrow \int_a^c f(x)\,dx = P_1 - P_2$$

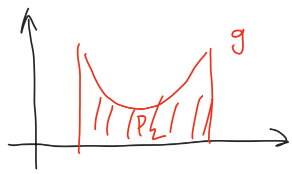
2°



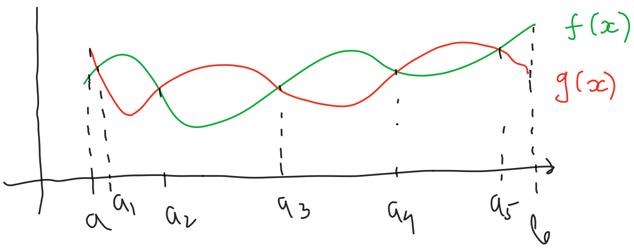


$$P = \underbrace{\int_a^b (f(x)+c)\,dx}_{P_1} - \underbrace{\int_a^b (g(x)+c)\,dx}_{P_2} = \int_a^b \left[f(x) - g(x)\right] dx$$





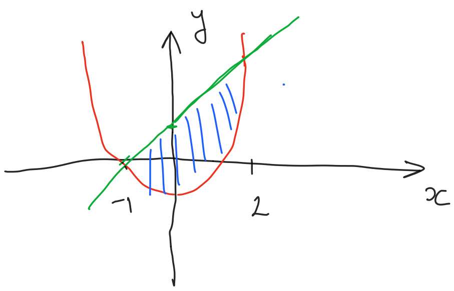
3°



$$P \text{ између } f \text{ и } g = \int_a^{a_1}(g(x)-f(x))\,dx + \int_{a_1}^{a_2}(f(x)-g(x))\,dx + \dots + \int_{a_5}^{b}(f(x)-g(x))\,dx$$

$$\boxed{= \int_a^b |f(x)-g(x)|\,dx}$$

Задатак. Наћи P између кривих $y = x^2-1$ $\quad y = x+1$



$$x^2-1 = x+1$$

$$x^2 - x - 2 = 0$$

$$x^2-1 - x - 1 = 0$$

$$(x-1)(x+1) - (x+1) = 0$$

$$(x+1)(x-2) = 0$$

$$x_1 = -1 \qquad x_2 = 2$$

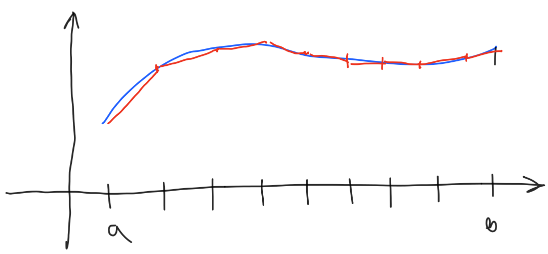
$$P = \int_a^b (\underbrace{f(x)}_{x+1} - \underbrace{g(x)}_{x^2-1})\,dx = \int_{-1}^{2}(x+1-x^2+1)\,dx = \int_{-1}^{2}(-x^2+x+2)\,dx$$

$$= -\frac{x^3}{3}\Big|_{-1}^{2} + \frac{x^2}{2}\Big|_{-1}^{2} + 2\cdot x\Big|_{-1}^{2} =$$

$$= -\frac{1}{3}(8+1) + \frac{1}{2}(4-1) + 2\cdot(2+1) = -3 + \frac{3}{2} + 6 = \frac{9}{2}.$$

II Дужина лука криве

1° $\quad y = f(x) \quad x \in [a,b]$, f је класе C^1



$x_0 = a$
$x_1 = a + \frac{b-a}{n}$
$\vdots$
$x_i = a + \frac{i}{n}(b-a)$
$\vdots$
$x_n = b = a + \frac{n}{n}(b-a)$

l_n – дужина полигоналне линије $= \sum$ дужина n дужи

дужина дужи $A_{i-1}A_i$ $\qquad A_i(x_i, f(x_i))$

$$= \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

$$= (x_i - x_{i-1}) \sqrt{1 + \left[\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right]^2}$$

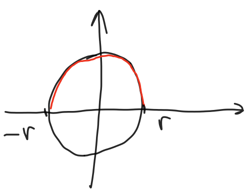
Лагр. т. о ср. вр.

$$= \underset{\overset{\parallel}{\frac{b-a}{n}}}{\Delta x_i} \sqrt{1 + (f'(\xi_i))^2} \quad \text{за неко } \xi_i \in (x_{i-1}, x_i)$$

$$l_n = \frac{b-a}{n} \sum_{i=1}^{n} \underbrace{\sqrt{1 + f'(\xi_i)^2}}_{g(\xi_i)} \boxed{=} S_n\Big(\underbrace{\sqrt{1+f'(x)^2}}_{g(x)}\Big) \longrightarrow \int_a^b \sqrt{1+f'(x)^2}\, dx$$

$$\Rightarrow \quad l = \int_a^b \sqrt{1 + f'(x)^2}\, dx \qquad (1)$$

Задатак. Наћи обим круга полупречника r.



$O = 2$ дужине полукруга

$y = \sqrt{r^2 - x^2} = f(x)$, $\quad f' = \frac{-2x}{2\sqrt{r^2 - x^2}}$

$$\Rightarrow \text{обим} = 2\int_{-r}^{r} \sqrt{1 + f'(x)^2}\, dx = 2\int_{-r}^{r} \sqrt{1 + \left(\frac{x}{\sqrt{r^2 - x^2}}\right)^2}\, dx$$

$$= 2\int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx = 4\int_0^r \sqrt{\frac{r^2 - x^2 + x^2}{r^2 - x^2}}\, dx = 4r \int_0^r \frac{dx}{\sqrt{r^2 - x^2}}$$

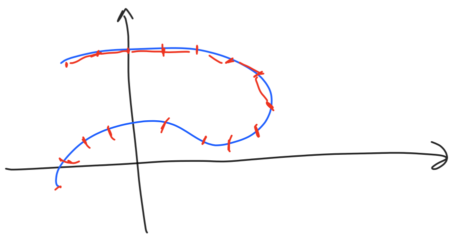
$$= 4\cancel{r} \int_0^r \frac{dx}{\cancel{r}\sqrt{1 - \left(\frac{x}{r}\right)^2}} = \left\{ \begin{array}{l} \frac{x}{r} = t \\ dx = r\, dt \end{array} \right. = 4r \int_0^1 \frac{dt}{\sqrt{1 - t^2}} =$$

$$= 4r \arcsin t \Big|_0^1 = 4r\left(\frac{\pi}{2} - 0\right) = 2r\pi \qquad \square$$

2° параметарски дата крива $\begin{array}{l} x = x(t) \\ y = y(t) \end{array}$ $\quad t \in [a,b]$

(нпр. круг $\begin{array}{l} x = r\cos t \\ y = r\sin t \end{array}$ $\quad t \in [0, 2\pi]$)

радимо исто



поделимо $[a,b]$ на n једнаких делова

$$t_0 = a \qquad t_i = a + i\,\frac{b-a}{n} \qquad t_n = b$$

l_n = дужине полигона $= \sum |A_{i-1}A_i|$

$$A_i\,(x(t_i),\, y(t_i))$$

$$\Rightarrow |A_{i-1}, A_i| = \sqrt{[x(t_i) - x(t_{i-1})]^2 + [y(t_i) - y(t_{i-1})]^2}$$

$$= \underbrace{\Delta t_i}_{t_i - t_{i-1}} \sqrt{\left(\frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}}\right)^2 + \left(\frac{y(t_i) - y(t_{i-1})}{t_i - t_{i-1}}\right)^2}$$

$$\overset{\text{Лагранж}}{=} \frac{b-a}{n}\sqrt{x'(\xi_i)^2 + y'(\eta_i)^2} \qquad \xi_i, \eta_i \in (x_{i-1}, x_i)$$

$$\Rightarrow l_n = \frac{b-a}{n}\sum_{i=1}^{n}\sqrt{x'(\xi_i)^2 + y'(\eta_i)^2} \xrightarrow{n\to\infty} \int_a^b \sqrt{x'(t)^2 + y'(t^2)}\,dt$$

$$\boxed{\begin{array}{l} l = \displaystyle\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt \qquad (2) \\ \quad = \displaystyle\int_a^b \|\vec{r}\,'(t)\|\,dt \qquad \vec{r} = (x, y) \end{array}}$$

ξ_i и η_i не морају бити $=$
али у лимесу се добија
резултат као да јесу $=$
(нисмо доказали)

Задатак са обимом круга

$$O = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$x = r\cos t$
$y = r\sin t$ $\quad t \in [0, 2\pi]$

$$= \int_0^{2\pi} \sqrt{r^2 \sin^2 t + r^2 \cos^2 t}\, dt$$

$$= \int_0^{2\pi} r\, dt = 2\pi r.$$

(2) $\Rightarrow$ (1) јер ако ставимо $x = t$, $y = f(x) = f(t)$

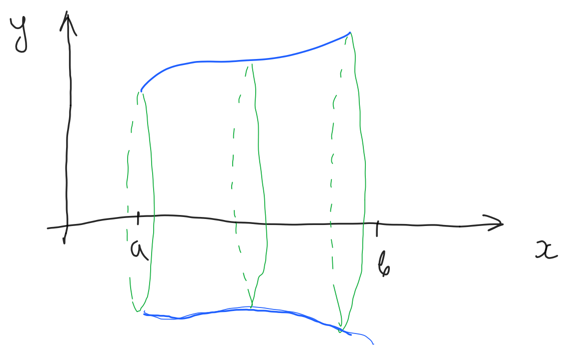
$\Rightarrow x' = 1, \; y' = f'(t)$ $\quad \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{1 + f'(t)^2}$

III Запремина обртних тела

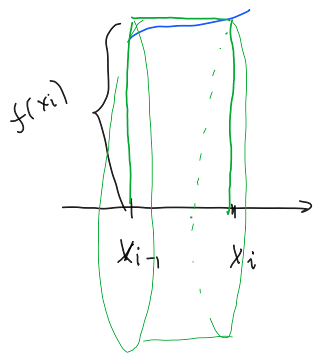
1° око x-осе

$f \geq 0$ непр. на $[a, b]$, график ротира око x-осе

V (тела) $= ?$



$a = x_0$
$x_1 = a + \frac{b-a}{n}$
$\dots$
$x_i = a + i\frac{b-a}{n}$



запремину тела када f на $[x_{i-1}, x_i]$ ротира апроксимирамо запремином ваљка који се добија када дуж висине $f(x_i)$ ротира

$$V(\text{ваљка}) = \Delta x_i\, f(x_i)^2 \cdot \pi$$

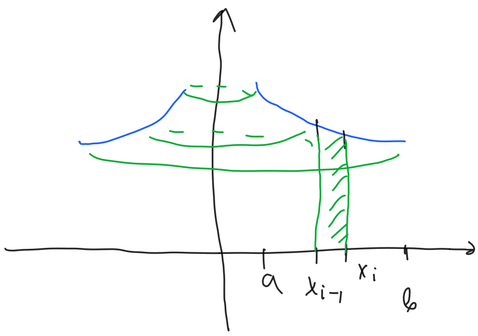
$$V_n = \text{суме запремине } n \text{ ваљака} = \sum_{i=1}^{n} \Delta x_i\, f(x_i)^2 \pi$$

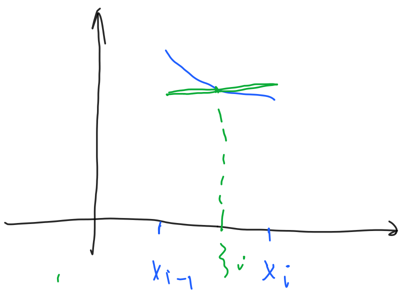
$$= \frac{b-a}{n} \sum_{i=1}^{n} f(x_i)^2 \pi \qquad (\xi_i = x_i)$$

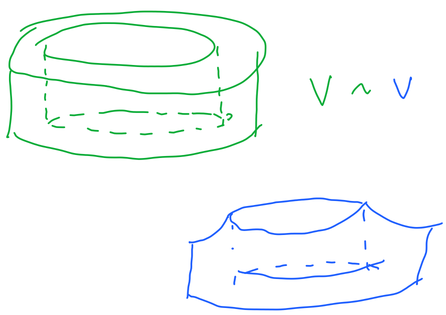
$$\boxed{=} S_n\left(f^2(x)\pi\right)$$

$$n \to \infty: \quad V_n \to \int_a^b f^2(x)\pi\, dx = \boxed{\pi \int_a^b f^2(x)\, dx = V}$$

2° око y-осе







$V \sim V$

$$V = V_1 - V_2 = x_i^2 f(\xi_i)\pi - x_{i-1}^2 f(\xi_i)\pi$$

$$= f(\xi_i)\pi\,(x_i^2 - x_{i-1}^2) = f(\xi_i)\pi(x_i + x_{i-1})\underbrace{(x_i - x_{i-1})}_{\Delta x_i}$$

$$V_n = \text{суме запремине слоја } V = \sum f(\xi_i)\pi(x_i + x_{i-1})\Delta x_i$$

$$= \frac{b-a}{n}\pi \sum f(\xi_i)\frac{(x_i + x_{i-1})}{2} \cdot 2$$

ТРИК: Бирам $\xi_i := \frac{x_i + x_{i-1}}{2} \in [x_{i-1}, x_i]$

$$= \frac{b-a}{n} \pi \cdot 2 \sum f(\xi_i) \cdot \xi_i$$

$$= S_n(2 \cdot \pi \cdot x \cdot f(x)) \xrightarrow{n \to \infty} 2\pi \int_a^b x f(x)\, dx$$

$\square$