3. ЧАС

Tuesday, 19 October 2021 16:23

смене $t = \operatorname{tg}\frac{x}{2}$

извели смо $\boxed{\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}}$

$dx = ?$ $\quad x = 2\operatorname{arctg} t \Rightarrow \boxed{dx = \frac{2\,dt}{1+t^2}}$

Задаци:

1. $$\int \frac{dx}{\sin x} = \left\{ \begin{array}{ll} t = \operatorname{tg}\frac{x}{2} & dx = \frac{2\,dt}{1+t^2} \\ \sin x = \frac{2t}{1+t^2} & \end{array} \right\} = \int \frac{2\,dt}{(1+t^2)\frac{2t}{1+t^2}} =$$
$$= \int \frac{dt}{t} = \ln|t| + C = \ln\left|\operatorname{tg}\frac{x}{2}\right| + C$$

2. $$\int \frac{\sin^2 x}{\cos^3 x}\,dx = \left\{ t = \operatorname{tg}\frac{x}{2} \right\} = \int \frac{\left(\frac{2t}{1+t^2}\right)^2}{\left(\frac{1-t^2}{1+t^2}\right)^3} \frac{2\,dt}{1+t^2} =$$
$$\int \frac{8t^2 \cdot (1+t^2)^3}{(1+t^2)^2(1+t^2)(1-t^2)^3} = \int \frac{8t^2}{\underset{\overset{\parallel}{(1-t)^3(1+t)^3}}{(1-t^2)^3}}\,dt = \int \left\{ \frac{a}{1-t} + \frac{b}{(1-t)^2} + \frac{c}{(1-t)^3} + \dots \right.$$
$$\left. \dots + \frac{d}{1+t} + \frac{e}{(1+t)^2} + \frac{f}{(1+t)^3} \right\} dt = \dots$$

(може брже са сменом $\sin x = t$ (непарна по $\cos$))

3. $$\int \frac{\cos^2 x}{\sin^4 x}\,dx = \left\{ \operatorname{tg}\frac{x}{2} = t \right\} = \int \frac{\frac{(1-t^2)^2}{(1+t^2)^2}}{\frac{(2t)^4}{(1+t^2)^4}} \frac{2\,dt}{1+t^2} =$$
$$= 2 \int \frac{(1-t^2)^2(1+t^2)}{16t^4}\,dt = \dots$$ поделити итд.

(лакше $\operatorname{tg} x = t$)

4. $$\int \frac{dx}{3+\sin x} = \left\{ t = \operatorname{tg}\frac{x}{2} \right\} = \int \frac{2\,dt}{(1+t^2)\left(3 + \frac{2t}{1+t^2}\right)} =$$
$$= \int \frac{2\,dt}{3+3t^2+2t}$$

$$3t^2+2t+3 = 3\left[\left(t^2 + \frac{2}{3}t\right) + 1\right] = 3\left[t^2 + 2\cdot\frac{1}{3}t + \frac{1}{9} + \frac{8}{9}\right]$$
$$\Delta = 4 - 4\cdot 9 < 0$$
$$= 3\left[\left(t+\frac{1}{3}\right)^2 + \frac{8}{9}\right] = 3\cdot\frac{8}{9}\left[\left(\frac{t+\frac{1}{3}}{2\sqrt{2}}\right)^2 + 1\right]$$

$, y$

инт.

V $\int R(x, \sqrt{ax^2+bx+c})\,dx$ $\quad$ R - рационална ф-ја

(а) $\int R(x, \sqrt{1-x^2})\,dx = \left\{ \begin{array}{l} x = \sin t \\ dx = \cos t\,dt \\ \sqrt{1-x^2} = \sqrt{\cos^2 t} = |\cos t| = \cos t \\ x \in [-1,1] \\ t \in [-\pi/2, \pi/2] \Rightarrow \cos t \geq 0 \end{array} \right\}$

$\rightsquigarrow \int R_1(\sin t, \cos t)\,dt \rightsquigarrow$ случај $\underline{IV}$

(б) $\int R(x, \sqrt{1+x^2})\,dx$ $\quad$ (као $\int \frac{dx}{(x^2+1)^n}$)

$= \left\{ \begin{array}{l} x = \operatorname{tg} t, \quad t \in (-\pi/2, \pi/2) \\ \sqrt{1+x^2} = \sqrt{1+\frac{\sin^2 t}{\cos^2 t}} = \sqrt{\frac{\cos^2 t + \sin^2 t}{\cos^2 t}} = \frac{1}{|\cos t|} = \frac{1}{\cos t} \\ dx = \frac{dt}{\cos^2 t} \end{array} \right.$

$= \int R_1(\sin t, \cos t)\,dt \rightsquigarrow$ случај $\underline{IV}$

(в) $\int R(x, \sqrt{x^2-1})\,dx = \left\{ \begin{array}{l} x = \sec t = \frac{1}{\cos t} \quad t \in [0, \pi/2) \\ \sqrt{x^2-1} = \sqrt{\frac{1}{\cos^2 t} - 1} = \sqrt{\frac{1-\cos^2 t}{\cos^2 t}} = \\ = \frac{\sqrt{\sin^2 t}}{\sqrt{\cos^2 t}} = \frac{\sin t}{\cos t} \\ dx = -\frac{1}{\cos^2 t} \cdot (-\sin t)\,dt = \frac{\sin t}{\cos^2 t}\,dt \end{array} \right.$

$= \int R_1(\sin t, \cos t)\,dt \rightsquigarrow$ случај $\underline{IV}$

(г) $\int R(x, \sqrt{ax^2+bx+c})\,dx$

допуном до потп. квадрата, линеарном сменом сводимо на (а), (б), (в), и то

- ако је $ax^2+bx+c \geq 0$ $\rightsquigarrow$ (б)
- ако нема нула, $a<0$ $\rightsquigarrow$ (а)
- ако има нула, $a>0$ $\rightsquigarrow$ (в)

Задаци.

1. $$\int \frac{x}{\sqrt{x^2-2x+5}}\,dx = \left\{ \begin{array}{ll} \text{смене} & x = 2t+1 \\ \frac{x-1}{2} = t & dx = 2dt \end{array} \right\}$$

$$x^2-2x+5 = x^2-2x+1+4 = (x-1)^2+4 = 4\left\{\left(\frac{x-1}{2}\right)^2+1\right\}$$

$$= \int \frac{x\,dx}{2\sqrt{\left(\frac{x-1}{2}\right)^2+1}} = \int \frac{(2t+1)\cdot 2\,dt}{2\sqrt{t^2+1}} = \int \frac{2t+1}{\sqrt{t^2+1}}\,dt$$

може на два начина: 1. одмах смене $t = tg\, y$

2. $$\int \frac{2t+1}{\sqrt{t^2+1}}\,dt = \int \frac{2t}{\sqrt{t^2+1}}\,dt + \int \frac{dt}{\sqrt{t^2+1}}$$

$$\int \frac{2t}{\sqrt{1+t^2}}\,dt = 2\sqrt{1+t^2} + C_1 = 2\sqrt{1+\left(\frac{x-1}{2}\right)^2} + C_1$$

$$\boxed{\int \frac{dt}{\sqrt{t^2+1}}} = \left\{ \begin{array}{ll} t = tg\, y & dt = \frac{dy}{\cos^2 y} \\ \sqrt{t^2+1} = \frac{1}{\cos y} & \end{array} \right\} =$$

$$= \int \frac{dy}{\cos^2 y \cdot \frac{1}{\cos y}} = \int \frac{dy}{\cos y} = \int \frac{\cos y\,dy}{\cos^2 y} = \left\{ \begin{array}{l} u = \sin y \\ du = \cos y\,dy \end{array} \right\} =$$

$$= \int \frac{du}{1-u^2} = \frac{1}{2}\int \frac{1+u+1-u}{(1-u)(1+u)}\,du = \frac{1}{2}\left(\int \frac{du}{1-u} + \int \frac{du}{1+u}\right) =$$

$$= \frac{1}{2}\left(\int \frac{du}{u+1} - \int \frac{du}{u-1}\right) = \frac{1}{2}\left\{\ln|u+1| - \ln|u-1|\right\} + C$$

$$= \frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| + C$$

треба да вратимо на t!

$$\left|\frac{u+1}{u-1}\right| = \left|\frac{\sin y+1}{\sin y-1}\right| = \frac{\sin y+1}{1-\sin y}\cdot\frac{1+\sin y}{1+\sin y} = \frac{(1+\sin y)^2}{1-\sin^2 y} =$$

$t = tg\, y$

$$= \left(\frac{1+\sin y}{\cos y}\right)^2 = \Big(\frac{1}{\cos y} + \underbrace{tg\, y}_{=t}\Big)^2$$

$$\frac{1}{\cos y} = \sqrt{\frac{1}{\cos^2 y}} = \sqrt{\frac{\sin^2 y + \cos^2 y}{\cos^2 y}} = \sqrt{t^2+1}$$

$$\left|\frac{u+1}{u-1}\right| = \left(\sqrt{t^2+1} + t\right)^2$$

$$\boxed{\int \frac{dt}{\sqrt{1+t^2}} = \frac{1}{2} \ln\left|\frac{u+1}{u-1}\right| + C = \ln\left(\sqrt{t^2+1} + t\right) + C}$$

2. $$\int \frac{1-x}{\sqrt{8+2x-x^2}}\,dx = \left\{\begin{array}{l} 8+2x-x^2 = t \\ (2-2x)\,dx = dt \\ (1-x)\,dx = 1/2\,dt \end{array}\right\} = \int \frac{dt}{2\sqrt{t}} = \sqrt{t} + C$$

$$= \sqrt{8+2x-x^2} + C$$

3. $$\int \frac{x+1}{\sqrt{2x-x^2}}\,dx = \int \frac{x+1}{\sqrt{1-(x-1)^2}}\,dx = \left\{\begin{array}{l} x-1 = t \\ dx = dt \\ x = t+1 \end{array}\right. = \ldots$$

$$2x - x^2 = -1 + 2x - x^2 + 1 = 1 - (x-1)^2$$

$$\ldots = \int \frac{t+2}{\sqrt{1-t^2}}\,dt = \int \frac{t\,dt}{\sqrt{1-t^2}} + 2\arcsin t + C$$

$$\int \frac{t\,dt}{\sqrt{1-t^2}} = \left\{\begin{array}{l} y = \sqrt{1-t^2} \\ dy = \dfrac{-2t}{2\sqrt{1-t^2}}\,dt = -\dfrac{t\,dt}{\sqrt{1-t^2}} \end{array}\right\} = -\int dy =$$

$$= -y + C_1 = -\sqrt{1-t^2} + C_1$$

$$\int \frac{x+1}{\sqrt{2x-x^2}}\,dx = -\sqrt{1-(x-1)^2} + 2\arcsin(x-1) + C$$

4. $$\int \frac{x-1}{\sqrt{x^2-4x+3}}\,dx = \frac{1}{2}\int \frac{2x-4+2}{\sqrt{x^2-4x+3}}\,dx = I_1 + I_2$$

$$I_1 = \frac{1}{2}\int \frac{2x-4}{\sqrt{x^2-4x+3}}\,dx = \sqrt{x^2-4x+3} + C_1$$

$$I_2 = \int \frac{dx}{\sqrt{x^2-4x+3}} = \left\{\begin{array}{l} x-2 = t \\ dx = dt \end{array}\right\} = \int \frac{dt}{\sqrt{t^2-1}}$$

$$x^2-4x+3 = x^2-4x+4-1 = (x-2)^2-1 \qquad x-2=t$$

$$\int \frac{dt}{\sqrt{t^2-1}} = \left\{ \begin{array}{l} t = \sec y = \dfrac{1}{\cos y} \\[2mm] dt = \dfrac{\sin y}{\cos^2 y}\,dy \end{array} \qquad \begin{array}{l} \sqrt{t^2-1} = \sqrt{\dfrac{1}{\cos^2 y} - 1} = \\[2mm] = \dfrac{\sqrt{1-\cos^2 y}}{\cos y} = \dfrac{\sin y}{\cos y} \end{array} \right\}$$

$\left(t>1 \quad \cos y \in (0,1) \quad y \in (0, \pi/2) \quad \sin y > 0 \right)$

$$= \int \frac{\frac{\sin y}{\cos^2 y}}{\frac{\sin y}{\cos y}}\,dy = \int \frac{dy}{\cos y} = \frac{1}{2} \ln \frac{1+\sin y}{1-\sin y} + C_2$$

радили данас

$$\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-\frac{1}{t^2}} = \frac{\sqrt{t^2-1}}{t}$$

$$I_2 = \frac{1}{2} \ln \frac{1+\frac{\sqrt{t^2-1}}{t}}{1-\frac{\sqrt{t^2-1}}{t}} + C_2 = \frac{1}{2} \ln \frac{t+\sqrt{t^2-1}}{t-\sqrt{t^2-1}} + C_2$$

$$= \frac{1}{2} \ln \frac{t+\sqrt{t^2-1}}{t-\sqrt{t^2-1}} \cdot \frac{t+\sqrt{t^2-1}}{t-\sqrt{t^2-1}} + C_2 =$$

$$= \frac{1}{2} \ln \frac{\left(t+\sqrt{t^2-1}\right)^2}{t^2-(t^2-1)} + C_2 = \ln\left(t+\sqrt{t^2-1}\right) + C_2$$

$$= \ln\left(x-2+\sqrt{x^2-4x+3}\right) + C_2$$

Хиперболичке смене. - може у типовима (б) и (в)

тј. када имамо израз $\sqrt{x^2+1}$ или $\sqrt{x^2-1}$

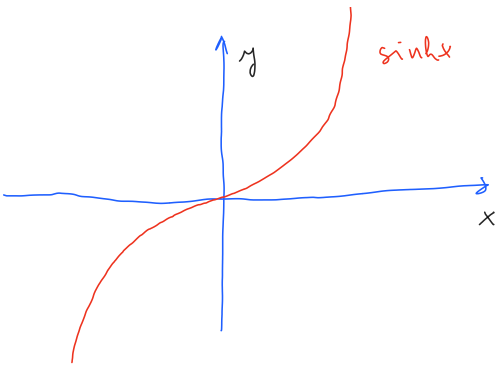
Хиперболичке ф-је $\quad \sinh x = \dfrac{e^x - e^{-x}}{2}$ (sh x) синус хиперболички

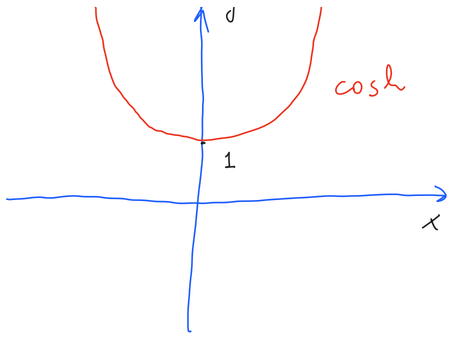
$\cosh x = \dfrac{e^x + e^{-x}}{2}$ (ch x) косинус хиперболички

sinh непарна

cosh парна

$$\cosh x = \frac{\overbrace{\left(e^{x/2}\right)^2 - 2e^{x/2}e^{-x/2} + \left(e^{-x/2}\right)^2}^{\geq 0} + 2}{2} \geq 1$$





$$\cosh^2 x - \sinh^2 x = \frac{1}{4}\left\{ (e^x + e^{-x})^2 - (e^x - e^{-x})^2 \right\} =$$

$$= \frac{1}{4}\left\{ \cancel{e^{2x}} + 2 + e^{-2x} - \cancel{e^{2x}} + 2 - e^{-2x} \right\}$$

$$= 1$$

$\boxed{\cosh^2 x - \sinh^2 x = 1}$ $\rightsquigarrow$ нам даје хиперболичке смене:

$$\sqrt{1+x^2} \underset{x=\sinh t}{=} \sqrt{\cosh^2 t} = \cosh t$$

$$\sqrt{x^2-1} \underset{x=\cosh t}{=} \sqrt{\sinh^2 t} = |\sinh t| \underset{t\in[0,\infty)}{=} \sinh t$$

$$(\sinh t)' = \left(\frac{e^t - e^{-t}}{2}\right)' = \frac{e^t + e^{-t}}{2} = \cosh t$$

$$(\cosh t)' = \left(\frac{e^t + e^{-t}}{2}\right)' = \frac{e^t - e^{-t}}{2} = \sinh t$$

Задаци:

1. $\displaystyle\int \frac{dx}{\sqrt{x^2+1}}$ $\quad\left(= \ln(x + \sqrt{x^2+1}) + c\right)$

2. $\displaystyle\int \frac{dx}{\sqrt{x^2-1}}$ $\quad\left(= \ln(x + \sqrt{x^2-1}) + c\right)$ домаћи

1. смена $x = \sinh t$ $\quad x^2+1 = \cosh^2 t$

$\sqrt{x^2+1} = \cosh t$

$dx = \cosh t \, dt$

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\cosh t \, dt}{\cosh t} = \int dt = t + C$$

$t = \operatorname{arcsinh} x = ?$ sinh је 1-1 и НА $\Rightarrow$ $\exists$ инверз

$$x = \frac{e^t - e^{-t}}{2} \Rightarrow e^t - e^{-t} = 2x \qquad y := e^t > 0 \,!$$

$$y - \frac{1}{y} = 2x \qquad y^2 - 2xy - 1 = 0$$

$$y_{1,2} = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

$$x + \sqrt{x^2+1} > 0 \qquad x - \sqrt{x^2+1} < 0$$

$$y = x + \sqrt{x^2+1} = e^t \Rightarrow t = \ln\left(x + \sqrt{x^2+1}\right)$$

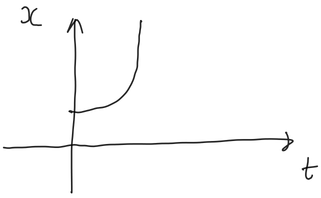
$$\Rightarrow \boxed{\int \frac{dx}{\sqrt{x^2+1}} = \ln\left(x + \sqrt{x^2+1}\right) + C}$$

2. упутство $\sqrt{x^2-1} = \sqrt{\sinh^2 t} = |\sinh t|$

због $x = \cosh t$

$x \in [1, \infty)$ или $x \in (-\infty, -1]$

бирам, $t \in [0, \infty)$ бирамо $\Rightarrow |\sinh t| = \sinh t$



$\cosh\big|_{[0,\infty)} : [0, \infty) \to [1, \infty)$ бијекција

$x \pm \sqrt{x^2-1} \geq 1 \quad \left(t = \ln\left(x \pm \sqrt{x^2-1}\right) \right)$

$\Updownarrow$

Кад тражимо $\operatorname{arccosh} x = t \geq 0 \rightsquigarrow$ изаберемо $+\sqrt{x^2-1}$

$$\frac{e^t + e^{-t}}{2} = x \qquad y = e^t > 0$$

VI

$$\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx$$

$t = \sqrt[n]{\frac{ax+b}{cx+d}} \Rightarrow x =$ рационална ф-ја по t

$$t^n (cx+d) = ax + b$$

$$x\,(c \cdot t^n - a) = b - d \cdot t^n$$

$$x = \frac{b - d t^n}{c \cdot t^n - a} \quad , \quad dx = (\text{рац. ф-ја од } t) \cdot dt$$

$$\rightsquigarrow \int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx = \int R_1(t)\, dt$$

Задаци. 1. $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\, dx = \left\{ \begin{array}{l} t = \sqrt{\dfrac{1+x}{1-x}} \\ t^2(1-x) = 1+x \\ t^2 - 1 = x(1+t^2) \\ x = \dfrac{t^2-1}{t^2+1} = \dfrac{t^2+1-2}{t^2+1} = 1 - \dfrac{2}{t^2+1} \\ dx = \dfrac{2}{(t^2+1)^2} \cdot 2t\, dt \end{array} \right\} =$

$$= \int t \cdot \frac{4t}{(t^2+1)^2}\, dt = 4 \int \frac{t^2+1-1}{(t^2+1)^2}\, dt = 4\int \frac{dt}{t^2+1} - 4 \underbrace{\int \frac{dt}{(t^2+1)^2}}_{t = \operatorname{tg} y} = \dots$$

2. $\displaystyle\int \sqrt[4]{2x+3}\, dx = \left\{ \begin{array}{l} \sqrt[4]{2x+3} = t \\ 2x = t^4 - 3 \\ dx = \frac{1}{2} \cdot 4t^3\, dt = 2t^3\, dt \end{array} \right\}$

$$= \int t \cdot 2t^3\, dt = 2\frac{t^5}{5} + C = \frac{2}{5}\sqrt[4]{(2x+3)^5} + C$$

НАПОМЕНА. $\int R(x, \sqrt[n]{ax+b})\, dx$ је специјални случај, $c=0$, $d=1$

$$\sqrt[n]{ax+b} = t$$

VII $\displaystyle\int x^p (ax^q + b)^r\, dx$ $\quad p, q, r \in \mathbb{Q}$ (Чебишевљев интеграл)

1. случај $r \in \mathbb{Z}$ смене $x = t^n$ где је

$$n = \text{НЗС}(p_2, q_2) \in \mathbb{N}$$

$$p = \frac{p_1}{p_2}, \quad q = \frac{q_1}{q_2}$$

$$x^p = t^{\text{цпр. број}} \quad , \quad x^q = t^{\text{цпр. број}}$$

$$dx = n \cdot t^{n-1}\, dt$$

$\rightsquigarrow$ добије $\int R(t)\,dt$ рац. ф-је

2. случај. $r \notin \mathbb{Z}$ смена $x^2 = t$, $x = t^{1/2}$, $dx = \frac{1}{2} t^{1/2-1}\, dt$

$$\int x^p (ax^2+b)^r\, dx = \int t^{p/2} (at+b)^r \frac{1}{2} t^{1/2-1}\, dt$$

$$= \frac{1}{2} \int (at+b)^r\, t^{\frac{p+1}{2}-1}\, dt$$

ако је $\underbrace{\frac{p+1}{2}-1}_{=:\, m} \in \mathbb{Z}$ $\left(\Leftrightarrow \frac{p+1}{2} \in \mathbb{Z}\right)$

$$\rightsquigarrow \int (at+b)^r\, t^m\, dt = \int R\left(t, \sqrt[r_2]{at+b}\right) dt$$

$r = \frac{r_1}{r_2}$ $\rightarrow$ случај VI

случај 3. ако $\frac{p+1}{2} \notin \mathbb{Z}$

$$\frac{1}{2} \int (at+b)^r\, t^{\frac{p+1}{2}-1}\, dt = \frac{1}{2} \int \left(a + \frac{b}{t}\right)^r t^{\frac{p+1}{2}+r-1}\, dt$$

ако је $\underbrace{\frac{p+1}{2}+r-1}_{=:\, m} \in \mathbb{Z}$ $\left(\Leftrightarrow \frac{p+1}{2}+r \in \mathbb{Z}\right)$

$$\rightsquigarrow \frac{1}{2} \int \underbrace{\left(\frac{at+b}{t}\right)^r t^m}_{=\, R\left(t, \sqrt[r_2]{\frac{at+b}{t}}\right)} dt \longrightarrow \text{случај VI}$$

$r = \frac{r_1}{r_2}$

Доказали смо: ако је $r \in \mathbb{Z}$ или $\frac{p+1}{2} \in \mathbb{Z}$ или $\frac{p+1}{2}+r \in \mathbb{Z}$

тада се $\int$ своди на случај VI

T (ЧЕБИШЕВ) Ако није $r \in \mathbb{Z}$ ни $\frac{p+1}{2} \in \mathbb{Z}$ ни $\frac{p+1}{2}+r \in \mathbb{Z}$

тада $\int (ax^2+b)^r x^p \, dx$ није елементарна ф-ја.

Задаци.

1. $\int \frac{dx}{x \sqrt[3]{1+x^5}} = \int x^{-1}(1+x^5)^{-1/3} dx =$

$\left\{ t = x^5 \quad x = t^{1/5} \quad dx = \frac{1}{5} t^{-4/5} dt \right\}$

$= \frac{1}{5} \int t^{-1/5} (1+t)^{-1/3} t^{-4/5} dt = \frac{1}{5} \int \frac{dt}{t \sqrt[3]{1+t}}$

$= \left\{ \begin{array}{ll} y = \sqrt[3]{1+t} & \\ t = y^3 - 1 & dt = 3y^2 dy \end{array} \right\} = \frac{1}{5} \int \frac{3y^2 dy}{(y^3-1)y} = \frac{3}{5} \int \frac{y \, dy}{y^3 - 1}$

$= \dots \quad \left(\frac{y}{y^3-1} = \frac{y}{(y-1)(y^2+y+1)} = \frac{a}{y-1} + \frac{by+c}{y^2+y+1} \right)$

2. $\int \frac{dx}{\sqrt[4]{1+x^4}} = \int (1+x^4)^{-1/4} dx = \left\{ \begin{array}{ll} x^4 = t & dx = \frac{1}{4} t^{-3/4} dt \\ x = t^{1/4} & \end{array} \right\}$

$= \int (1+t)^{-1/4} \cdot \frac{1}{4} t^{-3/4} dt = \frac{1}{4} \int \left(1+\frac{1}{t}\right)^{-1/4} t^{-1/4} \, t^{-3/4} dt$

$= \frac{1}{4} \int \sqrt[4]{\frac{t}{t+1}} \cdot \frac{1}{t} \, dt$ тип VI, $y = \sqrt[4]{\frac{t}{t+1}}$