2.4бс

Tuesday, 12 October 2021 17:11

(в) $\displaystyle\int \frac{x\,dx}{(x^2+1)^n} = \left\{ \begin{array}{l} t = x^2+1 \\ dt = 2x\,dx \\ x\,dx = \frac{1}{2}\,dt \end{array} \right\} = \frac{1}{2}\int \frac{dt}{t^n} =$

$\displaystyle = \frac{1}{2}\,\frac{t^{-n+1}}{-n+1} + C = \frac{(x^2+1)^{-n+1}}{2(-n+1)} + C$

(г) $\displaystyle\int \frac{dx}{(x^2+a^2)^n} = \frac{1}{a^{2n}}\int \frac{dx}{\left(\frac{x^2}{a^2}+1\right)^n} = \{ x/a = t$

скоро исто као (б)

(д) $\displaystyle\int \frac{x\,dx}{(x^2+a^2)^n}$ исто као (в) $t = x^2+a^2$

(ђ) $\displaystyle\int \frac{dx}{(ax^2+bx+c)^n}$ ax^2+bx+c нема нуле у $\mathbb{R}$

$0 > \Delta = b^2-4ac$

ако знам $f(x^2+a^2)$, а имам $g(\underbrace{ax^2+bx+c}_{\text{нема нуле}})$

$\rightsquigarrow$ лин. смена сводi $ax^2+bx+c \rightsquigarrow t^2+\alpha^2$ или t^2+1

$\boxed{a>0}$

$ax^2+bx+c = a\left(x^2+2\frac{b}{2a}x+\frac{b^2}{4a^2}\right)+c-\frac{b^2}{4a}$

$\displaystyle = a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a} =$

$-\Delta > 0$

$\displaystyle = a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a} = a\left(x+\frac{b}{2a}\right)^2+d^2 =$

$\vee$
0

$\left(d=\sqrt{\frac{4ac-b^2}{4a}}\right)$

смена $t=\sqrt{a}\left(x+\frac{b}{2a}\right)$

$ax^2+bx+c = t^2+d^2$

(Е) $\displaystyle\int \frac{x\,dx}{(ax^2+bx+d)^n}$, ax^2+bx+d нема нуле у $\mathbb{R}$

$$\int \frac{x\,dx}{(ax^2+bx+d)^n} = \underbrace{\int \frac{(2ax+b)\,dx}{(ax^2+bx+d)^n}}_{\text{смена } t=ax^2+bx+d} \cdot \frac{1}{2a} - \underbrace{\int \frac{b\,dx}{(ax^2+bx+d)^n}}_{\text{случај (Д)}}$$

(Ж) $\boxed{\text{ОПШТИ СЛУЧАЈ}}$

1. КОРАК: ако је $R(x)=\dfrac{P_n(x)}{Q_m(x)}$ где је $n \geq m$, поделимо

$\rightsquigarrow$ $R(x) = R_{n-m}(x) + \dfrac{S_k(x)}{Q_m(x)}$, $k<m$

2. КОРАК: $R(x)=\dfrac{P_n(x)}{Q_m(x)}$, $n<m$

$\boxed{\text{Т.}}$ Свако $R(x)=\dfrac{P_n(x)}{Q_m(x)}$, $n<m$ може да се напише као збир рац. ф-ја облика (А), (Б), (Д) и (Е).

Како се ово ради? $Q_m(x) = (x-a_1)^{k_1} \cdots (x-a_e)^{k_e}(a_1x^2+b_1x+c_1)^{n_1} \cdots (a_px^2+b_px+c_p)^{n_p}$

нерастављиви

- сваком чиниоцу $(x-a)^k$ придружимо k елемент. рац. ф-ја (А) – (Е)

$$\frac{a_1}{x-a},\ \frac{a_2}{(x-a)^2},\ \ldots,\ \frac{a_k}{(x-a)^k},\quad a_j = ?$$

- сваком чиниоцу $(ax^2+bx+c)^k$ придр. k ел. ф-ја облика

$$\frac{a_1x+b_1}{ax^2+bx+c},\ \frac{a_2x+b_2}{(ax^2+bx+c)^2},\ \ldots,\ \frac{a_kx+b_k}{(ax^2+bx+c)^k}$$

увек пол. ст. 1

ЗАДАЦИ

1. $\displaystyle\int \frac{2x^2+2x+13}{(x-2)(x^2+1)^2}\,dx$

$\deg(2x^2+2x+13) = 2 < 5 = \deg (x-2)(x^2+1)^2$

(прескачемо 1. корак)

$$\underbrace{\frac{2x^2+2x+13}{(x-2)(x^2+1)^2}}_{} \overset{„Л“}{} = \frac{\overset{\text{конст.}}{a}}{x-2} + \frac{bx+c}{x^2+1} + \frac{\overset{\text{полином ст. 1}}{dx+e}}{(x^2+1)^2}$$

$a, b, c, d, e = ?$ – изједначавамо бројиоце на левој и десној страни

ДЕСНА СТРАНА $= \dfrac{a(x^2+1)^2 + (bx+c)(x-2)(x^2+1) + (dx+e)(x-2)}{(x-2)(x^2+1)^2}$ „Д“

$$Д = a(x^4+2x^2+1) + (bx+c)(x^3+x-2x^2-2) + dx^2-2dx+ex-2e$$

$$= (a+b)x^4 + (-2b+c)x^3 + (2a+b-2c+d)x^2 + (-2b+c-2d+e)x + (a-2c-2e)$$

СВИ коеф. су $=$

$Л = 2x^2+2x+13$

$$\left.\begin{array}{lll} \text{уз } x^4: & 0 & = a+b \\ \text{уз } x^3 & 0 & = -2b+c \\ \text{уз } x^2 & 2 & = 2a+b-2c+d \\ \text{уз } x & 2 & = -2b+c-2d+e \\ \text{уз } 1 & 13 & = a-2c-2e \end{array}\right\} \quad \begin{array}{l} a=1 \\ b=-1 \\ c=-2 \\ d=-3 \\ e=-4 \end{array}$$

$$\Rightarrow \int \frac{2x^2+2x+13}{(x-2)(x^2+1)^2}\,dx = \underbrace{\int\frac{dx}{x-2}}_{I_1} + \underbrace{\int \frac{-x-2}{x^2+1}\,dx}_{I_2} + \underbrace{\int\frac{-3x-4}{(x^2+1)^2}\,dx}_{I_3}$$

$$\boxed{I_1 = \ln|x-2| + c_1}$$

$$I_2 = -\int\frac{x+2}{x^2+1}\,dx = -\int\frac{x\,dx}{x^2+1} - 2\int\frac{dx}{x^2+1} = \left\{\begin{array}{l}\text{у првом:} \\ x^2+1 = t \\ x\,dx = \frac{1}{2}\,dt\end{array}\right.$$

$$= -\frac{1}{2}\int\frac{dt}{t} - 2\,\mathrm{arctg}\,x + c_2 = -\frac{1}{2}\ln|t| - 2\,\mathrm{arctg}\,x + c_2$$

$$= \boxed{-\frac{1}{2}\ln(x^2+1) - 2\,\mathrm{arctg}\,x + c_2}$$

$$I_3 = -\int \frac{3x+4}{(x^2+1)^2}\,dx = -3\int \frac{x\,dx}{(x^2+1)^2} - 4\int \frac{dx}{(x^2+1)^2}$$

$$\int \frac{x\,dx}{(x^2+1)^2} = \left\{ \begin{array}{l} x^2+1 = t \\ x\,dx = 1/2\,dt \end{array} \right. = \frac{1}{2}\int \frac{dt}{t^2} = -\frac{1}{2}\,\frac{1}{t} + C_4$$

$$\int \frac{dx}{(x^2+1)^2} = \left\{ \begin{array}{l} x = \mathrm{tg}\,t \\ dx = \dfrac{dt}{\cos^2 t} \end{array} \qquad x^2+1 = \mathrm{tg}^2 t + 1 = \frac{\sin^2 t + \cos^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} \right\}$$

$$= \int \frac{dt}{\cos^2 t \cdot \dfrac{1}{(\cos^2 t)^2}} = \int \cos^2 t\,dt = \int \frac{1+\cos 2t}{2}\,dt =$$

$$= \underbrace{\frac{1}{2}t}_{\frac{1}{2}\mathrm{arctg}\,x} + \frac{1}{4}\sin 2t + C_5$$ треба да вратимо на x

$$\sin 2t = 2\sin t\cos t = 2\,\frac{\sin t}{\cos t}\cdot\cos^2 t = 2\,\overset{x}{\mathrm{tg}\,t}\,\frac{\cos^2 t}{\cos^2 t + \sin^2 t} =$$

$$= 2x\,\frac{1}{1+\mathrm{tg}^2 t} = \frac{2x}{1+x^2}$$

$$\int \frac{dx}{(x^2+1)^2} = \frac{1}{2}\mathrm{arctg}\,x + \frac{1}{2}\,\frac{x}{1+x^2} + C_5$$

$$\boxed{I_3 = \frac{3}{2}\,\frac{1}{x^2+1} - 2\,\mathrm{arctg}\,x + 2\,\frac{x}{1+x^2} + C_3}$$

2. $\displaystyle\int \frac{x\,dx}{x^3+1}$ Нема дељења

- растављамо именилац $\quad x^3+1 = (x+1)^1\underbrace{(x^2-x+1)^1}_{\text{нема нула}}$

- растављамо на ел. рац. ф-је

$$\frac{x}{x^3+1} = \frac{a}{x+1} + \frac{bx+c}{x^2-x+1}$$

$-\ a, b, c = ?$

$$f = \frac{a(x^2-x+1) + (bx+c)(x+1)}{(x+1)(x^2-x+1)} =$$

$$= \frac{ax^2 - ax + a + bx^2 + bx + cx + c}{x^3+1}$$

$$= \frac{(a+b)x^2 + (-a+b+c)x + a+c}{x^3+1}$$

$$f = \frac{x}{x^3+1}$$

$$\Rightarrow \quad x = (a+b)x^2 + (-a+b+c)x + (a+c)$$

y3	x^2	$0 = a+b$	
y3	x	$1 = -a+b+c$	
y3	1	$0 = a+c$	

$$\left.\begin{array}{r} b = -a \\ 1 = -a-a-a \\ c = -a \end{array}\right\} \Rightarrow \begin{array}{l} a = -1/3 \\ b = c = 1/3 \end{array}$$

$$\Rightarrow \int \frac{x}{x^3+1}\,dx = \frac{1}{3}\left\{-\int \frac{dx}{x+1} + \int \frac{x+1}{x^2-x+1}\,dx\right\} =$$

$$= -\frac{1}{3}\ln|x+1| + \frac{1}{3} I_1$$

$(x^2-x+1)' = 2x-1$

$$I_1 = \int \frac{x+1}{x^2-x+1}\,dx = \frac{1}{2}\int \frac{2x-1+3}{x^2-x+1}\,dx =$$

$$= \frac{1}{2}\underbrace{\int \frac{2x-1}{x^2-x+1}\,dx}_{\frac{1}{2}\ln(x^2-x+1)} + \frac{3}{2}\underbrace{\int \frac{dx}{x^2-x+1}}_{I_2}$$

I_2 :

$$x^2-x+1 = x^2 - 2\cdot\frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1 = \left(x-\frac{1}{2}\right)^2 + \frac{3}{4}$$

$$= \frac{3}{4}\left\{\Bigg(\underbrace{\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}}_{t}\Bigg)^2 + 1\right\}$$

$$I_2 = \int \frac{dx}{x^2-x+1} = \frac{4}{3}\int \frac{dx}{\left(\frac{x-1/2}{\sqrt{3}/2}\right)^2 + 1} = \left\{\begin{array}{l} \frac{x-1/2}{\sqrt{3}/2} = t \\ dx = \frac{\sqrt{3}}{2}\,dt \end{array}\right\}$$

$$= \frac{4}{3}\cdot\frac{\sqrt{3}}{2}\int\frac{dt}{t^2+1} = \frac{2\sqrt{3}}{3}\,\mathrm{arctg}\, t + C_2 = \frac{2\sqrt{3}}{3}\,\mathrm{arctg}\left(\frac{2x-1}{\sqrt{3}}\right) + C_2$$

$$\Rightarrow \boxed{I = -\frac{1}{3}\ln|x+1| + \frac{1}{6}\ln(x^2-x+1) + \frac{\sqrt{3}}{3}\,\mathrm{arctg}\,\frac{2x-1}{\sqrt{3}} + C}$$

3. $\displaystyle\int\frac{dx}{x^4+1}$

x^4+1 нема нуле у $\mathbb{R}$ $\Rightarrow$ квадр. · квадр.

$$x^4+1 \overset{\text{ШТОС}}{=} x^4+2x^2+1-2x^2 = (x^2+1)^2-(\sqrt{2}x)^2 =$$
$$= (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$$

$$\frac{1}{x^4+1} = \frac{ax+b}{x^2+\sqrt{2}x+1} + \frac{cx+d}{x^2-\sqrt{2}x+1}$$

$$= \frac{(ax+b)(x^2-\sqrt{2}x+1)+(cx+d)(x^2+\sqrt{2}x+1)}{x^4+1}$$

$$1 = \boxed{ax^3} - \boxed{\sqrt{2}ax^2} + \boxed{ax} + \boxed{bx^2} - \boxed{\sqrt{2}bx} + b + \boxed{cx^3} + \boxed{\sqrt{2}cx^2} + \boxed{cx} + \boxed{dx^2} + \boxed{\sqrt{2}dx} + d$$

уз x^3: $\quad 0 = a+c$

уз x^2: $\quad 0 = -\sqrt{2}a+b+\sqrt{2}c+d$

уз x: $\quad 0 = a-\sqrt{2}b+c+\sqrt{2}d \quad (3)$

уз 1: $\quad 1 = b+d$

$c = -a$

$d = 1-b$

$$0 = -\sqrt{2}a + \cancel{b} - \sqrt{2}a + 1 - \cancel{b}$$
$$\Rightarrow \boxed{a = \frac{1}{2\sqrt{2}}} \quad \boxed{c = -\frac{1}{2\sqrt{2}}}$$

у (3): $\quad 0 = -\sqrt{2}b + \sqrt{2}(1-b) \quad / : \sqrt{2}$

$$0 = -b+1-b \Rightarrow \boxed{b = 1/2 = d}$$

$$I = I_1 + I_2$$

$$I_1 = \int \frac{\frac{x}{2\sqrt{2}} + \frac{1}{2}}{x^2+\sqrt{2}x+1}\,dx \qquad I_2 = \int \frac{-\frac{1}{2\sqrt{2}}x + \frac{1}{2}}{x^2-\sqrt{2}x+1}\,dx$$

продолжуваме I_1, I_2 ја за домашна

$$I_1 = \frac{1}{2\sqrt{2}} \int \frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}\,dx = \frac{1}{4\sqrt{2}} \int \frac{2x+\sqrt{2}+\sqrt{2}}{x^2+\sqrt{2}x+1}\,dx$$

$$= \frac{1}{4\sqrt{2}} \left\{ \underbrace{\int \frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}\,dx}_{\ln(x^2+\sqrt{2}x+1)+c_1} + \underbrace{\int \frac{\sqrt{2}\,dx}{x^2+\sqrt{2}x+1}}_{I_3} \right\}$$

за I_3 :

$$x^2+\sqrt{2}x+1 = x^2+2\frac{\sqrt{2}}{2}x + \frac{1}{2} + \frac{1}{2} = \left(x+\frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2} =$$

$$= \frac{1}{2}\left\{\left(\frac{x+\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}}\right)^2 + 1\right\} = \frac{1}{2}\left\{(\sqrt{2}x+1)^2+1\right\}$$

$$I_3 = \sqrt{2}\int \frac{dx}{\frac{1}{2}\left[(\sqrt{2}x+1)^2+1\right]} = 2\sqrt{2}\int \frac{dx}{(\sqrt{2}x+1)^2+1} = \left\{ \begin{array}{l} t=\sqrt{2}x+1 \\ dx = \frac{1}{\sqrt{2}}dt \end{array} \right.$$

$$= \frac{2\sqrt{2}}{\sqrt{2}}\int \frac{dt}{t^2+1} = 2\,\mathrm{arctg}\, t + c_3 = 2\,\mathrm{arctg}\,(\sqrt{2}x+1) + c_3$$

$$I_1 = \frac{1}{4\sqrt{2}}\left\{\ln(x^2+\sqrt{2}x+1) + 2\,\mathrm{arctg}\,(\sqrt{2}x+1)\right\} + c_1$$

4. $\displaystyle\int \frac{dx}{x^4-1}$

$$x^4-1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$$

$$\frac{1}{x^4-1} = \frac{a}{x-1} + \frac{b}{x+1} + \frac{cx+d}{x^2+1}$$

$$= \frac{a(x+1)(x^2+1) + b(x-1)(x^2+1) + (cx+d)(x^2-1)}{x^4-1}$$

изједначавамо бројиоце:

$$1 = \boxed{ax^3} + \boxed{ax} + \boxed{ax^2} + a + \boxed{bx^3} + \boxed{bx} - \boxed{bx^2} - b + \boxed{cx^3} - \boxed{cx} + \boxed{dx^2} - d$$

уз x^3: $0 = a + b + c \quad (1)$

уз x^2: $0 = a - b + d \quad (2)$

уз x: $0 = a + b - c \quad (3)$

уз 1: $1 = a - b - d \quad (4)$

$(1) + (3)$: $a + b = 0$

$(2) - (4)$: $-1 = 2d \Rightarrow d = -\frac{1}{2}$

$(1) - (3)$: $c = 0$

из (4): $\frac{1}{2} = a - b = 2a \Rightarrow a = \frac{1}{4}$

$b = -\frac{1}{4}$

$$\Rightarrow \frac{1}{x^4-1} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2}\,\frac{1}{x^2+1}$$

$$\Rightarrow \boxed{\int \frac{dx}{x^4-1} = \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\,\mathrm{arctg}\,x + C}$$

домаћи:

1. $\int \frac{dx}{x^2(1+x^2)^2}$ $\left(\frac{1}{x^2(1+x^2)^2} = \frac{a}{x} + \frac{b}{x^2} + \frac{cx+d}{x^2+1} + \frac{ex+f}{(x^2+1)^2} \right)$

2. $\int \frac{x^7 - 2x^6 + 4x^5 - 5x^4 + 4x^3 - 5x^2 - x}{(x-1)^2\,(x^2+1)^2}\,dx$

IV $\boxed{\int R(\sin x, \cos x)\,dx}$

$R(\sin x, \cos x)$ – $\sin x$, $\cos x$ – сабирање, одузимање, множење, дељење

(А) $R(-\sin x, \cos x) = -R(\sin x, \cos x)$ $\boxed{t = \cos x}$ $\longrightarrow$ рац. по t

(Б) $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ $\boxed{t = \sin x}$ $\longrightarrow$ рац. по t

(слично као типови I)

Задаци.

1. $\displaystyle\int \frac{dx}{\sin x}$ $\qquad R(\sin x, \cos x) = \frac{1}{\sin x}$

$$R(-\sin x, \cos x) = \frac{1}{-\sin x} = -R(\sin x, \cos x)$$

$$\int \frac{dx}{\sin x} = \int \frac{\sin x \, dx}{\sin^2 x} = \left\{ \begin{array}{l} \cos x = t \\ -\sin x \, dx = dt \\ \sin^2 x = 1 - \cos^2 x = 1 - t^2 \end{array} \right\}$$

$$= -\int \frac{dt}{1-t^2} = \int \frac{dt}{t^2-1} = \frac{1}{2}\int \frac{t+1-(t-1)}{(t+1)(t-1)} \, dt$$

$$= \frac{1}{2}\left(\int \frac{dt}{t-1} - \int \frac{dt}{t+1}\right) = \frac{1}{2}\left\{\ln|\cos x - 1| - \ln|\cos x + 1|\right\} + C$$

$$= \frac{1}{2}\left\{\ln(1-\cos x) - \ln(\cos x + 1)\right\} + C$$

2. $$\int \frac{\sin^2 x}{\cos^3 x} \, dx = \int \frac{\overset{„\, t^2}{\sin^2 x}\ \overset{„\, dt}{\cos x \, dx}}{\cos^4 x} = \left\{ \begin{array}{l} \sin x = t \\ \cos x \, dx = dt \\ \cos^4 x = (\cos^2 x)^2 = (1-\sin^2 x)^2 \\ \qquad = (1-t^2)^2 \end{array} \right.$$

$$R(\sin x, -\cos x) = -R(\sin x, \cos x)$$

$$= \int \frac{t^2 \, dt}{(1-t^2)^2} = \int \frac{t^2 \, dt}{(1-t)^2(1+t)^2} = \int \frac{a \, dt}{1-t} + \int \frac{b \, dt}{(1-t)^2} + \int \frac{c \, dt}{1+t} + \int \frac{d \cdot dt}{(1+t)^2}$$

$= \dots$ домаћи

(B) $\quad R(-\sin x, \cos x) = R(\sin x, -\cos x) = R(\sin x, \cos x)$

сви степени и уз $\sin x$ и уз $\cos x$ су парни

$\boxed{t = \operatorname{tg} x} \longrightarrow$ рационална

$x = \operatorname{arctg} t \Rightarrow dx = \dfrac{dt}{1+t^2}$ (рационално)

$\sin^2 x$, $\cos^2 x$ – или неки степен од овога

Парни степени sin и cos су рационалне ф-је од $\operatorname{tg} x$

јер $$\sin^2 x = \frac{\sin^2 x}{\sin^2 x + \cos^2 x} = \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{\sin^2}{\cos^2 x} + 1} = \frac{tg^2 x}{tg^2 x + 1} = \frac{t^2}{t^2+1}$$

$\boxed{\sin^2 x = \frac{t^2}{t^2+1}}$ ако је $t = tg\, x$

$$\boxed{\cos^2 x = 1 - \sin^2 x = 1 - \frac{t^2}{1+t^2} = \frac{1}{1+t^2}}$$

Задатак: $$\int \frac{\cos^2 x}{\sin^4 x}\, dx = \left\{ \begin{array}{ll} tg\, x = t & \cos^2 x = \frac{1}{1+t^2} \\ dx = \frac{dt}{1+t^2} & \sin^4 x = \frac{t^4}{(1+t^2)^2} \end{array} \right\}$$

$$= \int \frac{\frac{1}{\cancel{1+t^2}}}{\frac{t^4}{\cancel{(1+t^2)^2}}} \frac{dt}{\cancel{1+t^2}} = \int \frac{dt}{t^4} = -\frac{1}{3} t^{-3} + c$$

$$= -\frac{1}{3(tg\, x)^3} + c$$

(г) УНИВЕРЗАЛНА СМЕНА

$t = tg \frac{x}{2}$

$\sin x$ и $\cos x$ су рационалне ф-је од $tg \frac{x}{2}$

(ово смо, у ствари, већ видели у неком задатку)

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = 2\, tg \frac{x}{2} \overset{\frac{1}{1+t^2}}{\cos^2 \frac{x}{2}} = 2t \cdot \frac{1}{1+t^2} = \frac{2t}{1+t^2}$$

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2}$$

случај (в)