НЕОДРЕЂЕНИ ИНТЕГРАЛ

$f : (a,b) \to \mathbb{R}$

деф. $F : (a,b) \to \mathbb{R}$ је примитивна ф-ја ф-је f ако је
$F'(x) = f(x)$.

деф. фамилија примитивних ф-ја ф-је f се зове неодређени интеграл ф-је f, ознака $\boxed{\int f(x)\,dx}$:

дискусија:

F и G примитивне за $f \Rightarrow F' = G'$ на (a,b)

$\Rightarrow (F-G)' = 0 \iff F(x) - G(x) \equiv c$

$\Uparrow \quad \varphi' = 0 \iff \varphi = c$

$\Leftarrow$ из деф извода

$\Rightarrow \varphi(x) - \varphi(y) = \varphi'(\xi)(x-y) = 0$

$\Rightarrow \varphi(x) = \varphi(y) \quad \forall x, y \in (a,b)$

Напомена: $\varphi' = 0 \iff \varphi = c$ важи на интервалима

нпр $\varphi(x) = \begin{cases} 1, & x \in (0,1) \\ 2, & x \in (2,3) \end{cases}$, $\quad \varphi'(x) = 0$, $\varphi(x) \neq c$ на $(0,1) \cup (2,3)$.

закључак: $f : (a,b) \to \mathbb{R}$, F - примитивна за f

$\Rightarrow \int f(x)\,dx = \{F(x) + c \mid c \in \mathbb{R}\}$

пишемо $\boxed{\int f(x)\,dx = F(x) + c}$

Таблица интеграла

$$\int 1\,dx = x + c$$

$$\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} + c, \quad \alpha \neq -1$$

$$\int \frac{dx}{x} = \left.\begin{cases} \ln x + c, & x > 0 \\ \int \frac{-dx}{-x} = \ln(-x) + c, & x < 0 \end{cases}\right\} = \ln|x| + c$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + c$$

$$\int \cos x\,dx = \sin x + c \qquad\qquad \sec x = \frac{1}{\cos x}$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \sec^2 x\,dx = \int \frac{dx}{\cos^2 x} = \operatorname{tg} x + c$$

$$\int \frac{dx}{1+x^2} = \operatorname{arctg} x + c$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$$

Напомена: $\exists$ ф-ја f чији интеграл није елементарна ф-ја
$\left(\int \frac{\sin x}{x}\,dx,\ \int \frac{e^x}{x}\,dx,\ \int e^{-x^2}\,dx,\ \int \sin x^2\,dx,\ \ldots \right)$

Задатак: (1) $\int e^{2x}\,dx = \frac{e^{2x}}{2} + c \qquad \left(\left(\frac{1}{2}(e^{2x})\right)' = \frac{2e^{2x}}{2} = e^{2x} \right)$

(2) $\int e^{2x+1}\,dx = \frac{e^{2x}}{2} + c \qquad \left(\frac{e^{2x+1}}{2}\right) = \frac{1}{2}e^{2x+1}\cdot 2$

Наравоученије: ако је $\int f(x)\,dx = F(x) + c$
онда је $\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + c$

Својства $\int \cdot\, dx$

1 Линеарност. $\int [\lambda f(x) + \mu g(x)]\,dx = \lambda \int f(x)\,dx + \mu \int g(x)\,dx$

јер ако је F примитивна за f $\quad (F' = f)$
а G примитивна за g $\quad (G' = g)$

$\Rightarrow (\lambda F(x) + \mu G(x))' = \lambda f(x) + \mu g(x)$
тј. $\lambda F + \mu G$ је примитивна $\lambda f + \mu g$

$$\int (\lambda f(x) + \mu g(x))\,dx = \lambda \int f(x)\,dx + \mu \int g(x)\,dx$$

$$\overset{\shortparallel}{F(x)} \qquad\qquad \overset{\shortparallel}{G(x)}$$

Задатак. $\int \left(x+\frac{1}{2x}\right)^2 dx = \int \left(x^2 + 2\cdot x \cdot \frac{1}{2x} + \frac{1}{4x^2}\right) dx$

$$= \int x^2 dx + \int dx + \int \frac{1}{4x^2}\, dx = \frac{x^3}{3} + x + \frac{1}{4}\int x^{-2} dx$$

$$= \frac{x^3}{3} + x + \frac{1}{4}\,\frac{x^{-1}}{-1} + c = \frac{x^3}{3} + x - \frac{1}{4x} + c$$

Задатак. $\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}\, dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx$

$$= \frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{2}\sin 2x + c = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

Пример: $F:(a,b)\to\mathbb{C}$ $\qquad F(x) = G(x) + i\,H(x)$

$$F'(x) = G'(x) + i\,H'(x)$$

$\Rightarrow$ $\quad f:(a,b)\to\mathbb{C}$ $\qquad f = g(x) + i\,h(x)$

$\Rightarrow$ $\quad \int f(x)dx = \int g(x)dx + i\int h(x)dx$

$$I = \int e^x \sin x\, dx \qquad R = \int e^x \cos x\, dx$$

$$R + iI = \int (e^x\cos x + i\, e^x \sin x)\, dx$$

$$= \int e^x \underbrace{(\cos x + i\sin x)}_{e^{ix}}\, dx = \int e^{x(1+i)} dx$$

$$= \frac{1}{1+i} e^{x(1+i)} + C \qquad C = C_1 + i\,C_2$$

$\hookrightarrow$ домаћи: доказати да је $\frac{d}{dx} e^{zx} = z e^{zx}$, $z\in\mathbb{C}$

$$\overset{\shortparallel}{\left(e^{zx}\right)'}$$

$$\frac{1}{1+i} e^{x(1+i)} =$$

$$= \frac{1-i}{(1+i)(1-i)}\, e^x e^{ix} = \frac{1-i}{2}\, e^x(\cos x + i\sin x)$$

$$= \frac{1}{2}\left(e^x\cos x + e^x \sin x\right) + i\,\frac{1}{2}\left(-e^x\cos x + e^x\sin x\right)$$

$$= R + iI$$

$$\Rightarrow \quad R = \underline{\frac{e^x(\cos x + \sin x)}{2}} \qquad I = \underline{\frac{e^x(\sin x - \cos x)}{2}}$$

2. ПАРЦИЈАЛНА ИНТЕГРАЦИЈА

(последица ф-ле за извод производа)

$$(u\cdot v)' = u'v + uv' \;/\int \cdot\, dx$$

$$uv = \int u'v\,dx + \int uv'\,dx$$

$$\boxed{\int uv'dx = uv - \int u'v\,dx}$$

$$v'dx = dv \qquad u'dx = du$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Задаци.

1. $\int xe^x\,dx = \left\{ \begin{array}{ll} u = x & du = u'dx = dx \\ e^x dx = dv & v = e^x \end{array} \right\} =$

$$= xe^x - \int e^x\,dx = xe^x - e^x + C$$

2. $\int x\sin x\,dx = \left\{ \begin{array}{ll} u = x & du = dx \\ dv = \sin x\,dx & v = -\cos x \end{array} \right\} =$

$$= -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$$

3. $\int x\cos x\,dx$ домаћи

4. $\int x^2\sin x\,dx = \left\{ \begin{array}{ll} u = x^2 & du = 2x\,dx \\ \sin x\,dx = dv & v = -\cos x \end{array} \right\} =$

$$= -x^2\cos x + 2\underbrace{\int x\cos x\,dx}_{3.\ \text{задатак}}$$

$$x = u$$
$$\cos x\,dx = dv$$

5. $$\int \ln x \, dx = \left\{ \begin{array}{ll} u = \ln x & du = \frac{dx}{x} \\ dv = dx & v = x \end{array} \right\} =$$

$$= x \ln x - \int \cancel{x} \cdot \frac{dx}{\cancel{x}} = x \ln x - \int dx = x \ln x - x + C$$

6. $$\int x \ln x \, dx = \left\{ \begin{array}{ll} u = \ln x & du = \frac{dx}{x} \\ dv = x\, dx & v = \frac{x^2}{2} \end{array} \right\} =$$

$$= \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx = \ln x \cdot \frac{x^2}{2} - \frac{1}{2} \int x \, dx =$$

$$= \ln x \cdot \frac{x^2}{2} - \frac{x^2}{4} + C$$

7. $\int x^2 e^x \, dx$

8. $\int x^2 \ln x \, dx$

домаћи

9. $R = \int e^x \cos x \, dx \qquad I = \int e^x \sin x \, dx$

$$I = \int e^x \sin x \, dx = \left\{ \begin{array}{ll} e^x dx = dv & v = e^x \\ \sin x = u & du = \cos x \, dx \end{array} \right\} =$$

$$= e^x \sin x - \int e^x \cos x \, dx = \left\{ \begin{array}{ll} e^x dx = dv & v = e^x \\ \cos x = u & du = -\sin x \, dx \end{array} \right\}$$

$$= e^x \sin x - \Big(e^x \cos x + \underbrace{\int e^x \sin x \, dx}_{I} \Big) =$$

$$= e^x (\sin x - \cos x) - I$$

$$\Rightarrow \quad 2I = e^x (\sin x - \cos x) \quad \Rightarrow \quad I = \frac{1}{2} e^x (\sin x - \cos x)$$

3. T. $f : (a, b) \to \mathbb{R}$, F примитивна за f, $F' = f$

$\varphi : (\alpha, \beta) \to (a, b)$ диференцијабилна

Тада је $\int f(\varphi(t)) \cdot \varphi'(t) \, dt = F(\varphi(t)) + C$

тј. $F \circ \varphi$ је примитивна за $f \circ \varphi \cdot \varphi'$

Δ: Знам $F' = f \Rightarrow \left(F \circ \varphi(t)\right)' = \left(F(\varphi(t))\right)' =$

$= F'(\varphi(t)) \cdot \varphi'(t) = f(\varphi(t)) \cdot \varphi'(t)$ $\square$

ЛИНЕАРНОСТ $\int \longleftrightarrow (\lambda f + \mu g)'$

ПАРЦИЈАЛНА $\longleftrightarrow (u \cdot v)'$

СМЕНА $\longleftrightarrow (f \circ \varphi)'$

овако размишљамо $\int \underbrace{f(\varphi(t))}_{f(x)} \cdot \underbrace{\varphi'(t)\, dt}_{dx}$, $\quad x = \varphi(t)$, $dx = \varphi'(t)\, dt$

Задаци.

1. $\int e^{t^2} \cdot t\, dt = \left\{ \begin{matrix} x = t^2, & \varphi(t) = t^2, & \varphi'(t) = 2t \\ dx = 2t\, dt, & t\, dt = \frac{dx}{2} & \end{matrix} \right\}$

$= \int e^x \frac{1}{2} dx = \frac{1}{2} \int e^x dx = \frac{1}{2} e^x + c = \frac{1}{2} e^{t^2} + c$

2. $\int \frac{t\, dt}{t^2 + 2} = \left\{ \begin{matrix} t^2 + 2 = x & dx = 2t\, dt \\ & t\, dt = \frac{1}{2} dx \end{matrix} \right\} =$

$= \frac{1}{2} \int \frac{dx}{x} = \frac{1}{2} \ln|x| + c = \frac{1}{2} \ln|t^2 + 2| + c = \frac{1}{2} \ln(t^2 + 2) + c$

3. $\int \operatorname{arctg} x\, dx = \left\{ \begin{matrix} u = \operatorname{arctg} x & du = \frac{1}{1+x^2} dx \\ dv = dx & v = x \end{matrix} \right\}$

$= x \operatorname{arctg} x - \int \frac{x\, dx}{1+x^2}$

$\int \frac{x\, dx}{1+x^2} = \left\{ \begin{matrix} y = 1 + x^2 & x\, dx = \frac{1}{2} dy \\ dy = 2x\, dx & \end{matrix} \right\} = \frac{1}{2} \int \frac{dy}{y} =$

$= \frac{1}{2} \ln|y| + c = \frac{1}{2} \ln(1 + x^2) + c$

$\Rightarrow \int \operatorname{arctg} x\, dx = x \operatorname{arctg} x - \frac{1}{2} \ln(1 + x^2) + c$

4. $\int \arcsin x\, dx = \left\{ \begin{matrix} u = \arcsin x & du = \frac{dx}{\sqrt{1-x^2}} \\ dv = dx & v = x \end{matrix} \right\}$

$$= x \arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}}$$

$$\int \frac{x\,dx}{\sqrt{1-x^2}} = \left\{ \begin{array}{ll} y = 1 - x^2 & x\,dx = -\frac{1}{2}\,dy \\ dy = -2x\,dx & \end{array} \right\} =$$

$$= -\frac{1}{2} \int \frac{dy}{\sqrt{y}} = -\frac{1}{2} \int y^{-1/2}\,dy = -\frac{1}{2}\, \frac{y^{1/2}}{1/2} + c = -\sqrt{y} + c$$

$$= -\sqrt{1-x^2} + c$$

$$\int \arcsin x = x \arcsin x + \sqrt{1-x^2} + c$$

5. $$\int \sqrt{1-x^2}\,dx = \left\{ \begin{array}{ll} x = \sin t & \sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} \\ dx = \cos t\,dt & = |\cos t| = \cos t \end{array} \right\}$$

радимо на интервалу, $x \in [-1, 1]$, $x = \sin t$

$t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \cos t \geq 0$

$$\int \sqrt{1-x^2}\,dx = \int \cos t \cos t\,dt = \int \cos^2 t\,dt = \int \frac{1 + \cos 2t}{2}\,dt$$

$$= \frac{1}{2} \int dt + \frac{1}{2} \int \cos 2t\,dt = \frac{1}{2} t + \frac{1}{4} \sin 2t + c$$

$$= \frac{1}{2} \arcsin x + \frac{1}{4} \sin(2 \arcsin x) + c$$

ТЕХНИКЕ ИНТЕГРАЦИЈЕ

$\boxed{\text{I} \quad \int \sin^n x \cos^m x\,dx}$, $m, n \in \mathbb{N}$

(1) ако је n или m непарно

n - непарно $\quad t = \cos x$, $dt = -\sin x\,dx$

$$\int \sin^n x \cos^m x\,dx = \int \sin^{n-1} x \underbrace{\cos^m x}_{t^m} \underbrace{\sin x\,dx}_{-dt}$$

$$\sin^{\text{парни степен}} = P(\cos x)$$

$= \int (\text{полином по } t)\,dt$

Пример $\int \sin^3 x \cos^7 x\,dx = \int \underset{\underset{1-t^2}{\shortparallel}}{\sin^2 x} \cos^7 x \underbrace{\sin x\,dx} = \left\{ \begin{array}{l} t=\cos x \\ dt = -\sin x\,dx \end{array}\right.$

$$= \int (1-t^2)\cdot t^7(-dt) = -\int (t^7 - t^9)\,dt = -\frac{t^8}{8} + \frac{t^{10}}{10} + C$$

$$= -\frac{\cos^8 x}{8} + \frac{\cos^{10} x}{10} + C$$

$\int \sin^n x \cos^m x\,dx$, m – непарно, смене $\sin x = t$

Пример. $\int \sin^2 x \cos^5 x\,dx = \int \sin^2 x \cos^4 x \underbrace{\cos x\,dx}$

$$= \{ t=\sin x,\ dt = \cos x\,dx \} = \int t^2 (1-t^2)^2\,dt$$

$$= \int t^2(1-2t^2+t^4)\,dt = \int (t^2 - 2t^4 + t^6)\,dt =$$

$$= \frac{t^3}{3} - \frac{2t^5}{5} + \frac{t^7}{7} + C = \frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7} + C$$

(2) ако су и m и n парни, користимо ф-лу

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

Пример: $\int \sin^4 x \cos^2 x\,dx = \int \frac{(\sin^2 x)^2 (1+\cos 2x)}{2}\,dx$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 \frac{(1+\cos 2x)}{2}\,dx =$$

$$= \frac{1}{8} \int (1 - 2\cos 2x + \cos^2 2x)(1+\cos 2x)\,dx = \left\{ \begin{array}{l} 2x = t \\ dx = \frac{1}{2}\,dt \end{array}\right.$$

$$= \frac{1}{16} \int (1 + \cos t - 2\cos t + 2\cos^2 t + \cos^2 t + \cos^3 t)\,dt$$

$$= \frac{1}{16}\left[\int dt - \int \cos t\,dt + 3\int \cos^2 t\,dt + \int \cos^3 t\,dt \right]$$

$$\int \cos^3 t\,dt = \int \cos^2 t \underbrace{\cos t\,dt}_{dy} = \left\{ \begin{array}{l} y = \sin t \\ dy = \cos t\,dt \\ \cos^2 t = 1-\sin^2 t = 1-y^2 \end{array} \right\}$$

$$= \int (1-y^2)\,dy = y - \frac{y^3}{3} + C = \sin t - \frac{\sin^3 t}{3} + C$$

домаћи. 1) $\int \sin^2 x \cos^3 x\,dx$

2) $\int \cos^4 x\,dx$

II интеграли облика

$\int \sin ax \sin bx\,dx$

$\int \sin ax \cos bx\,dx$

$\int \cos ax \cos bx\,dx$

користимо тригон. ф-ле: $\sin\alpha \cdot \sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

$\sin\alpha \cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$

Задатак. $\int \sin 2x \cos 3x\,dx = \frac{1}{2}\int (\sin 5x - \sin x)\,dx =$

$= \frac{1}{2}\left(-\frac{\cos 5x}{5} + \cos x\right) + C$

III ИНТЕГРАЦИЈА РАЦИОНАЛНИХ Ф-ЈА

рационалне ф-ја $R(x) = \frac{P_n(x)}{Q_m(x)}$

(A) $$\int \frac{dx}{(x-a)^n} = \begin{cases} \ln|x-a| + C, & n = 1 \\ \frac{1}{(-n+1)(x-a)^{n-1}}, & n > 1 \end{cases}$$

$n \in \mathbb{N}$

(Б) $$\int \frac{dx}{(x^2+1)^n} = \begin{cases} \operatorname{arctg} x + C, & n = 1 \\ ? & n > 1 \end{cases}$$

$n > 1 \qquad x = \operatorname{tg} t \qquad dx = \frac{1}{\cos^2 t}\, dt$

$$(x^2+1)^n = (\operatorname{tg}^2 t + 1)^n = \left(\frac{\sin^2 t}{\cos^2 t} + 1\right)^n = \left(\frac{\sin^2 t + \cos^2 t}{\cos^2 t}\right)^n$$

$$= \frac{1}{\cos^{2n} t}$$

$$\int \frac{dx}{(x^2+1)^n} = \int \frac{\cos^{2n} t}{\cos^2 t}\, dt = \int \cos^{2n-2} t\, dt \rightsquigarrow \text{patrz } I$$