Функционални низови

• $f_n: X \to \mathbb{R}$, $X \subseteq \mathbb{R}^m$, $n \in \mathbb{N}$ конв тачка по тачка ка $f: X \to \mathbb{R}$

$$\forall x \in X \quad \lim_{n\to\infty} f_n(x) = f(x)$$

$$\forall x \in X \quad \forall \varepsilon > 0 \quad \exists \underset{\substack{\downarrow \\ n_0(\varepsilon, x)}}{n_0} \in \mathbb{N} \quad \forall n \geq n_0 \quad |f_n(x) - f(x)| < \varepsilon$$

· ознака за конв. т.п.т.

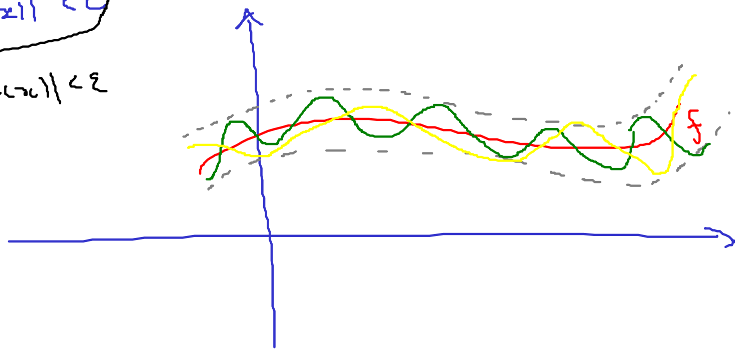
$$f_n \xrightarrow[n\to\infty]{} f$$

• $f_n: X \to \mathbb{R}$, $X \subseteq \mathbb{R}^m$, $n \in \mathbb{N}$ конв равномерно ка $f: X \to \mathbb{R}$

$f_n \to f$ у d_∞ метрици $\left(d_\infty(f,g) = \sup_{x \in X} |f(x) - g(x)| \right)$

$$\forall \varepsilon > 0 \quad \exists \underset{\substack{\downarrow \\ n_0(\varepsilon)}}{n_0} \in \mathbb{N} \quad \forall x \in X \quad \forall n \geq n_0 \quad |f_n(x) - f(x)| < \varepsilon$$

$$\sup_{x \in X} |f_n(x) - f(x)| < \varepsilon$$



· ознака за равн конв.

$$f_n \underset{n\to\infty}{\rightrightarrows} f$$

равн. конв. $\Rightarrow$ конв. т.п.т.
$\not\Leftarrow$

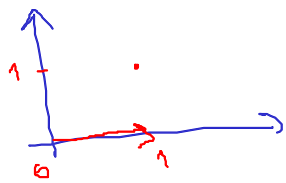
$f_n \underset{n\to\infty}{\rightrightarrows} f$ на X акко $\lim_{n\to\infty} \sup_{x \in X} |f_n(x) - f(x)| = 0$.

(1) Испитати равн конв. $f_n(x) = x^n$, $x \in [0,1]$.

увек кандидата за f тражимо уз помоћ т.п.т. конв.

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

↓
кандидат



$$\sup_{x \in [0,1]} |f_n(x) - f(x)| = ?$$

$$f_n(x) - f(x) = \begin{cases} x^n - 0, & 0 \leq x < 1 \\ \underset{\substack{\| \\ 1}}{1^n} - 1, & x = 1 \end{cases} = \begin{cases} x^n, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| = 1, \quad \lim_{n\to\infty} \sup_{x \in [0,1]} |f_n(x) - f(x)| = \lim_{n\to\infty} 1 \neq 0$$

$\Rightarrow$ f_n не конв. равн ка f!

$\boxed{T1}$ $f_n \in C(X)$, $X \subseteq \mathbb{R}$, $f_n \underset{n\to\infty}{\rightrightarrows} f$ на X $\overset{?}{\Rightarrow}$ $f \in C(X)$

Δ: $f_n \underset{n\to\infty}{\rightrightarrows} f$ на X

$\Rightarrow$ $\forall \varepsilon > 0 \quad \exists n_0(\varepsilon) \in \mathbb{N} \quad \forall n \geq n_0 \quad \forall x \in X \quad |f_n(x) - f(x)| < \varepsilon$

$f_n \in C(X) \Rightarrow \forall x \in X \ \forall \varepsilon > 0 \ \exists \underset{\downarrow \ \delta_n(\varepsilon, x)}{\delta_n > 0} \ \forall y \in X \ |x - y| < \delta_n \Rightarrow |f_n(x) - f_n(y)| < \varepsilon$

? $f \in C(X)$? $\forall x \in X \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall y \in X \quad |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$?

$x \in X$, $\varepsilon > 0$ фикс. ? $\delta = ?$

$$|f(x) - f(y)| = |f(x) - f_n(x) + \underset{-f_n(y)+f_n(y)}{\uparrow} f_n(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f(y)|$$

$$\leq \underbrace{|f(x) - f_n(x)|}_{< \varepsilon/3 \text{ за } n \geq n_0(\frac{\varepsilon}{3})} + \underbrace{|f_n(x) - f_n(y)|}_{< \varepsilon/3} + \underbrace{|f_n(y) - f(y)|}_{< \varepsilon/3} < \varepsilon$$

$\hookrightarrow$ ово важи ако $|x - y| < \delta_n(\varepsilon/3, x)$

$\Rightarrow \delta = \delta_{n_0(\varepsilon/3)}(\varepsilon/3, x) > 0$

$\Rightarrow f \in C(X)$.

- $f_n \underset{n\to\infty}{\rightrightarrows} f$, $f_n \in C(X) \Rightarrow f \in C(X)$

$f_n \rightrightarrows f$, f_n равн непр на X $\Rightarrow$ f равн непр на X $\rightarrow$ слично као и претходно.

- $f_n \underset{n\to\infty}{\longrightarrow} f$ на X, $f_n \in C(X)$, f има прекид $\Rightarrow f_n \underset{n\to\infty}{\not\rightrightarrows} f$ на X

(2) $f_n(x) = x^n - x^{n+1}$, $x \in [0,1]$ равн конв ?

$f(x) = \lim\limits_{n\to\infty} \underbrace{x^n - x^{n+1}}_{f_n(x)} = 0$ $\rightarrow$ непр. на $[0,1]$

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} |\underbrace{x^n - x^{n+1}}_{f_n(x)} - 0|$$

$$f_n'(x) = nx^{n-1} - (n+1)x^n = x^{n-1}(n - (n+1)x) \quad \left(= 0 \text{ за } x_0 = \frac{n}{n+1} \in [0,1] \right)$$

$x < x_0 \quad f_n' > 0$

$x > x_0 \quad f_n' < 0$ $\Rightarrow x_0$ лок. макс.

$$\Rightarrow \sup_{x\in[0,1]} |f_n(x) - f(x)| = |f_n(x_0)| = \frac{n^n}{(n+1)^n}\left(1 - \frac{n}{n+1}\right) = \frac{n^n}{(n+1)^{n+1}} = \frac{e^{n \ln n}}{e^{(n+1)\ln(n+1)}}$$

$$= e^{n\ln n - (n+1)\ln(n+1)} = e^{n(\ln n - \ln(n+1)) - \ln(n+1)} \underset{n\to\infty}{\longrightarrow} 0$$

$n(\ln n - \ln(n+1)) - \ln(n+1) \to -\infty$

$\ln\frac{n}{n+1} = \ln\left(1 - \frac{1}{n+1}\right) \sim -\frac{1}{n+1}$

$\left(\frac{-n}{n+1}\right) - \ln(n+1)$, $\frac{-n}{n+1} \to -1$

$\Rightarrow f_n \underset{n\to\infty}{\rightrightarrows} f$ на $[0,1]$.

(3) $f_n(x) = x^n - x^{2n}$, на $[0,1]$ р.к.?

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} (x^n - x^{2n}) = 0 = f(x)$$

$$\sup_{x\in[0,1]} |f_n(x) - \overset{0}{f(x)}| = \sup_{[0,1]} |f_n(x)| = |f_n(x_0)| = \left(\frac{1}{\sqrt[n]{2}}\right)^n - \left(\frac{1}{\sqrt[n]{2}}\right)^{2n} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \underset{n\to\infty}{\not\to} 0$$

$$f_n'(x) = n x^{n-1} - 2n x^{2n-1} = n x^{n-1}(1 - 2x^n) \quad \left(= 0 \text{ за } x_0 = \frac{1}{\sqrt[n]{2}}\right)$$

$x < x_0$ $f_n' > 0$

$x > x_0$ $f_n' < 0$ $\Rightarrow x_0$ лок. макс.

$$f_n \underset{n\to\infty}{\not\rightrightarrows} f$$

(4) $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$, $x \in \mathbb{R}$ р.к.?

$$f(x) = \lim_{n\to\infty} \sqrt{x^2 + \frac{1}{n^2}} = |x|$$

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} \left| \sqrt{x^2 + 1/n^2} - |x| \right| = \sup_{x\geq 0} (\underbrace{\sqrt{x^2 + 1/n^2} - x}_{g_n(x)})$$

$$x \geq 0 \quad g_n'(x) = \frac{1}{\cancel{2}\sqrt{x^2 + 1/n^2}} \cdot \cancel{2}x - 1 = \frac{x}{\sqrt{x^2 + 1/n^2}} - 1 < 0, \quad \frac{x}{\sqrt{x^2 + \frac{1}{n^2}}} < 1, \quad x \geq 0$$

$$x < \sqrt{x^2 + 1/n^2} \quad {}^2$$

$g_n \downarrow$

$$\sup_{x\geq 0} g_n(x) = g_n(0) = \frac{1}{n} \underset{n\to\infty}{\longrightarrow} 0 \Rightarrow f_n \underset{n\to\infty}{\rightrightarrows} f$$

(5) $f_n(x) = \operatorname{arctg}(nx)$ на $(0, +\infty)$ р.к.?

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \operatorname{arctg}(\underset{+\infty}{\lim nx}) = \frac{\pi}{2}$$

n фиксирано

$$\sup_{x\in(0,+\infty)} |f_n(x) - f(x)| = \sup_{x>0} \left|\operatorname{arctg} nx - \pi/2\right| = \sup_{x>0} (\underbrace{\tfrac{\pi}{2} - \operatorname{arctg} nx}_{g_n(x)})$$

$$g_n'(x) = -\frac{1}{1 + n^2x^2} \cdot n < 0 \Rightarrow g_n \downarrow$$

$$\Rightarrow \sup_{x>0} (\underbrace{\tfrac{\pi}{2} - \operatorname{arctg} nx}_{g_n(x)}) = \lim_{x\to 0} g_n(x) = \frac{\pi}{2} \underset{n\to\infty}{\not\longrightarrow} 0 \Rightarrow f_n \underset{n\to\infty}{\not\rightrightarrows} f$$

(6) $f_n(x) = x \cdot \operatorname{arctg} nx$ на $(0, +\infty)$ р.к.?

$$f(x) = \lim_{n\to\infty} x \cdot \operatorname{arctg} nx = \frac{\pi}{2} x$$

$$\sup_{x\in(0,+\infty)} |f_n(x) - f(x)| = \sup_{x\in(0,+\infty)} \left| x \cdot \operatorname{arctg} nx - \frac{\pi}{2} x \right| = \sup_{x>0} (\underbrace{\tfrac{\pi}{2} x - x \cdot \operatorname{arctg} nx}_{g_n(x)})$$

$$g_n'(x) = \frac{\pi}{2} - \operatorname{arctg} nx - \frac{x}{1+n^2x^2}\cdot n \overset{?}{=} 0$$

$$g_n''(x) = -\frac{n}{1+n^2x^2} - \frac{n(1+n^2x^2) - 2xn^2\cdot xn}{(1+n^2x^2)^2} = -\frac{n}{1+n^2x^2} - \frac{n(1-n^2x^2)}{(1+n^2x^2)^2}$$

$$= -\frac{n}{1+n^2x^2}\left(1+\frac{1-n^2x^2}{1+n^2x^2}\right) = -\frac{n}{1+n^2x^2}\cdot\frac{2}{1+n^2x^2} < 0$$

$\Rightarrow g_n' \searrow$

$g_n'(0) = \frac{\pi}{2} > 0$, $\lim\limits_{x\to\infty} g_n'(x) = \lim\limits_{x\to+\infty}\Big(\underbrace{\frac{\pi}{2}-\operatorname{arctg} nx}_{\downarrow\, 0} - \underbrace{\frac{nx}{1+n^2x^2}}_{\downarrow\, 0}\Big) = 0$

$\Rightarrow g_n' > 0$ на $(0,+\infty)$ $\Rightarrow g_n \nearrow$

$$\Rightarrow \sup_{x\in(0,+\infty)} g_n(x) = \lim_{x\to\infty} g_n(x) = \lim_{x\to\infty} x\cdot\underbrace{\left(\frac{\pi}{2}-\operatorname{arctg} nx\right)} =$$

$$= \lim_{x\to+\infty}\frac{\frac{\pi}{2}-\operatorname{arctg} nx}{\frac{1}{x}} \overset{\text{ЛП}}{=} \lim_{x\to+\infty}\frac{-\frac{n}{1+n^2x^2}}{-\frac{1}{x^2}} = \lim_{x\to+\infty}\frac{nx^2}{1+n^2x^2} = \frac{1}{n}$$

$$\lim_{n\to\infty}\sup_{x>0} g_n(x) = \lim_{n\to\infty}\frac{1}{n} = 0 \Rightarrow f_n \underset{n\to\infty}{\rightrightarrows} f \text{ на } (0,+\infty)$$

(7) $f_n(x) = e^{-(x-n)^2}$ а) на $(-R,R)$, $R>0$

б) на $\mathbb{R}$.

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} e^{-\overbrace{(x-n)^2}^{\to+\infty}} = 0$$

б)

$$\sup_{\mathbb{R}} |f_n(x)-f(x)| = \sup_{\mathbb{R}} f_n(x) = \sup_{\mathbb{R}} \underbrace{e^{-(x-n)^2}}$$

$f_n'(x) = e^{-(x-n)^2}\cdot\big(-2(x-n)\big)$ $(=0$ за $x_0 = n)$

$x < x_0 = n$ $f_n' > 0 \Rightarrow f_n \nearrow$ на $(-\infty, n)$ $\Rightarrow x_0$ макс.

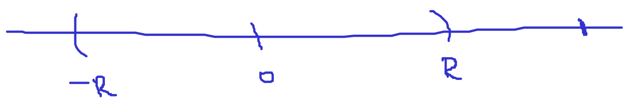
$x > x_0$ $f_n' < 0$

$$f_n(x_0) = f_n(n) = e^{-(n-n)^2} = e^0 = 1 \Rightarrow \sup_{\mathbb{R}}|f_n(x)-f(x)| = 1 \underset{n\to\infty}{\not\to} 0$$

$\Rightarrow f_n \not\rightrightarrows f$ на $\mathbb{R}$

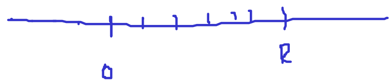
а) $R>0$ $(-R,R)$

$$\sup_{(-R,R)} |f_n(x)-f(x)| = \begin{cases} f_n(n), & n\in(-R,R) \\ f_n(R), & n \geq R \end{cases}$$



$(-R,R) \subseteq (-\infty, n) \Rightarrow f_n \nearrow$ на $(-R,R)$

$$= \begin{cases} 1, & n \in (-R, R) \\ e^{-(R-n)^2}, & n \geq R \end{cases} \xrightarrow[n\to\infty]{}$$



$$\lim_{n\to\infty} \sup_{(-R,R)} |f_n(x)| = \lim_{n\to\infty} e^{-(R-n)^2} = 0 \Rightarrow f_n \underset{n\to\infty}{\rightrightarrows} f \text{ на } (-R,R)$$

$\boxed{T2}$ $f_n \rightrightarrows f$ на X, $Y \subseteq X$ $\Rightarrow$ $f_n \rightrightarrows f$ на Y

(7) $\Rightarrow$ $f_n \rightrightarrows f$ на Y, $Y \subseteq X$ $\not\Rightarrow$ $f_n \rightrightarrows f$ на X

• $f_n \rightrightarrows f$ на $X_1, X_2, \dots, X_k$ $\Leftrightarrow$ $f_n \rightrightarrows f$ на $\bigcup_{i=1}^{k} X_i$

$\Leftarrow$

$\Rightarrow \checkmark$

• $f_n \rightrightarrows f$ на $X_\lambda, \lambda \in \Lambda$ $\not\Rightarrow$ $f_n \rightrightarrows f$ на $\bigcup_{\lambda\in\Lambda} X_\lambda$

беск он.

$\boxed{T3}$ f_n Риман интегр. на $[a,b]$, $f_n \underset{n\to\infty}{\rightrightarrows} f$ на $[a,b]$, f Риман интеграб.

$$\Rightarrow \lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx$$

(8) $f_n(x) = \int_0^1 \sin\frac{xy^2}{n} dy$

а) на $(0,1)$

б) на $(0,+\infty)$.

$x \in (0,+\infty)$ фикс.

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int_0^1 \underbrace{\sin\frac{xy^2}{n}}_{g_n(y) \text{ на } [0,1]} dy = ?$$

? $g_n(y)$ р.к. на $[0,1]$?

$$g_n(y) = \sin\frac{xy^2}{n}$$

$$g(y) = \lim_{n\to\infty} \sin\frac{xy^2}{n} = \sin 0 = 0$$

$$\sup_{y\in[0,1]} |g_n(y)-g(y)| = \sup_{y\in[0,1]} \left|\sin \frac{xy^2}{n}\right|$$

$x>0, \quad y\in[0,1]$
$\Downarrow$
$y^2\in[0,1]$

$\frac{xy^2}{n} \le \frac{x}{n} < \frac{\pi}{2}$ за $n > x$ $\Rightarrow$ $\sin \frac{xy^2}{n} \uparrow$ по y

јер аргумент се налази у $(0,\pi/2)$

$n>x$ $\Rightarrow \sup_{y\in[0,1]} \sin \frac{xy^2}{n} = \sin \frac{x}{n} \xrightarrow[n\to\infty]{} 0$

$\Rightarrow g_n \underset{n\to\infty}{\rightrightarrows} g$ за $y\in[0,1]$

(T3) $\Rightarrow \lim_{n\to\infty}\int_0^1 g_n(y)dy = \int_0^1 \lim_{n\to\infty} g_n(y) dy = \int_0^1 g(y)dy = \int_0^1 0\, dy = 0$

$\Rightarrow f(x)=0 \quad \forall x\in(0,+\infty)$

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Тек сада испитујемо р.к. f_n !

б) $(0,+\infty)$:

$\sup_{x\in(0,+\infty)} |f_n(x)-f(x)| = \sup_{x\in(0,+\infty)} \left| \int_0^1 \sin\frac{xy^2}{n} dy - 0\right| = ?$

$$\left|\int_0^1 \sin\frac{xy^2}{n}dy\right| \le \int_0^1 \left|\sin \frac{xy^2}{n}\right| dy \le \frac{x}{n}\underbrace{\int_0^1 y^2 dy}_{=\frac{1}{3}}$$

$$\left|\sin\frac{xy^2}{n}\right| \le \frac{xy^2}{n}$$

$$\int_0^1 \sin\frac{xy^2}{n} dy = \left[\begin{array}{l} t=\frac{xy^2}{n}>0 \\ y=\sqrt{\frac{nt}{x}} \\ dy=\sqrt{\frac{n}{x}}\cdot\frac{1}{2\sqrt{t}}dt \end{array}\right] = \sqrt{\frac{n}{x}}\cdot\frac{1}{2}\int_0^{x/n} \frac{\sin t}{\sqrt{t}} dt$$

$$\left|\int_0^1 \sin\frac{xy^2}{n}dy\right| = \sqrt{\frac{n}{x}}\cdot\frac{1}{2} \left|\int_0^{x/n}\frac{\sin t}{\sqrt t}dt\right| \quad \Big/ \sup_{x\in(0,+\infty)}$$

$$\sup_{(0,+\infty)} |\text{—}\|\text{—}| = \sup_{x\in(0,+\infty)} \sqrt{\frac{n}{x}}\cdot\frac{1}{2}\left|\int_0^{x/n} \frac{\sin t}{\sqrt t}dt\right| \underset{\text{за } x=n}{\ge} \sqrt{\frac{n}{n}}\cdot\frac{1}{2}\left|\int_0^1\frac{\sin t}{\sqrt t}dt\right| = \frac{1}{2}\underbrace{\left|\int_0^1\frac{\sin t}{\sqrt t}dt\right|}_{=C>0}$$

$\Rightarrow \forall n\in\mathbb{N} \quad \sup_{x\in[0,+\infty)} |f_n(x)| \geq C \quad \underset{n\to\infty}{\not\to} 0$

$\Rightarrow f_n \underset{n\to\infty}{\not\rightrightarrows} f$ на $(0,+\infty)$!

a) $x\in[0,1)$:

$$\sup_{x\in[0,1)} |f_n(x) \overset{\to 0}{-f(x)}| = \sup_{x\in[0,1)} \left| \int_0^1 \sin\frac{xy^2}{n}\,dy \right| \leq \sup_{x\in[0,1)} \int_0^1 \underbrace{\left|\sin\frac{xy^2}{n}\right|}_{\leq \frac{xy^2}{n}} dy \leq \sup_{x\in[0,1)} \frac{x}{n} \int_0^1 y^2\,dy$$

$$= \frac{1}{3n} \underset{n\to\infty}{\longrightarrow} 0$$

$\Rightarrow f_n \underset{n\to\infty}{\rightrightarrows} f$ на $[0,1)$

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• Λ бескон. скуп са неком метриком d_Λ $(\mathbb{R}, |\cdot|)$

$f_\lambda \underset{\lambda\to\lambda_0}{\rightarrow} f_{\lambda_0}$ (т.ч. ш.) на X, $f_\lambda, f_{\lambda_0} : X \to \mathbb{R}$

$\forall x\in X \quad \lim_{\lambda\to\lambda_0} f_\lambda(x) = f_{\lambda_0}(x)$

$\forall x\in X \ \forall \varepsilon>0 \ \underset{\delta(\varepsilon,x)}{\exists \delta>0} \quad d_\Lambda(\lambda,\lambda_0)<\delta \Rightarrow |f_\lambda(x)-f_{\lambda_0}(x)|<\varepsilon$

$f_\lambda \underset{\lambda\to\lambda_0}{\rightrightarrows} f_{\lambda_0}$ (р.к.) на X

$\forall \varepsilon>0 \quad \exists \delta>0 \ \underset{}{\text{''}}\delta(\varepsilon) \quad d(\lambda,\lambda_0)<\delta \Rightarrow \underbrace{\forall x\in X \ |f_\lambda(x)-f_{\lambda_0}(x)|<\varepsilon}$

$\sup_{x\in X} |f_\lambda(x)-f_{\lambda_0}(x)| < \varepsilon$

$\Leftrightarrow \lim_{\lambda\to\lambda_0} \sup_{x\in X} |f_\lambda(x)-f_{\lambda_0}(x)| = 0$

$\Rightarrow$ све теореме важе и у овом случају !

(9) $\Lambda=\mathbb{R}$, $d_\Lambda = |\cdot|$

$f_\lambda(x) = e^{-\left(\frac{x}{\lambda}\right)^2}, \ \lambda\neq 0, \ x\in\mathbb{R}$

Испитати р.к. f_λ ако

а) $\lambda\to 0$ на $x\in [1,+\infty)$

б) $\lambda\to 0$ на $\mathbb{R}$

в) $\lambda\to+\infty$ на $[1,+\infty)$

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б) $x\in\mathbb{R}$ фикс.

$$f_0(x) = \lim_{\lambda\to 0} f_\lambda(x) = \lim_{\lambda\to 0} e^{-\left(\frac{x}{\lambda}\right)^2} = \begin{cases} 0, & x\neq 0 \\ e^0, & x=0 \end{cases} = \begin{cases} 0, & x\neq 0 \\ 1, & x=0 \end{cases}$$

$\left(\frac{x}{\lambda}\right)^2 \to \begin{matrix} +\infty & x\neq 0 \\ 0 & x=0 \end{matrix}$

$f_0(x)$ има прекид, $f_\lambda\in C(\mathbb{R}) \overset{(T1)}{\Rightarrow} f_\lambda \not\rightrightarrows f$ на $\mathbb{R}$

a) $x \in [1,+\infty)$

$$f_0(x) = \lim_{\lambda \to 0} f_\lambda(x) = 0$$

$\lambda \neq 0$ фикс.

$$\sup_{x\in[1,+\infty)} |f_\lambda(x) - \overset{0}{f_0(x)}| = \sup_{x\in[1,+\infty)} e^{-\left(\frac{x}{\lambda}\right)^2} = \left[\begin{array}{l} e^{-t^2} \searrow \text{ по } t \\ t = \frac{x}{\lambda} \end{array}\right] = \underset{x=1}{e^{-\frac{1}{\lambda^2}}}$$

$$\lim_{\lambda\to 0} \sup_{x\in[1,+\infty)} |f_\lambda(x)| = \lim_{\lambda \to 0} e^{-\overbrace{\frac{1}{\lambda^2}}^{\to +\infty}} = 0$$

$$\Rightarrow \quad f_\lambda \underset{\lambda\to 0}{\rightrightarrows} f_0 \text{ на } [1,+\infty)$$

b) $\lambda \to +\infty$, $x \in [1,+\infty)$

$$f_\infty(x) = \lim_{\lambda\to+\infty} f(x) = \lim_{\lambda \to +\infty} e^{-\overbrace{\left(\frac{x}{\lambda}\right)^2}^{\to 0}} = 1 \quad \in C[1,+\infty)$$

λ фикс. $\neq 0$

$$\sup_{x\in[1,+\infty)} |f_\lambda(x) - f_\infty(x)| = \sup_{x\in[1,+\infty)} \left| e^{-\left(x/\lambda\right)^2} - 1 \right| = \sup_{x\in[1,+\infty)} \left(1 - e^{-\left(x/\lambda\right)^2}\right)$$

$$= \left[\begin{array}{l} e^{-t^2} \searrow \text{ по } t \\ 1 - e^{-t^2} \nearrow \text{ по } t \\ t = x/\lambda \end{array}\right] = \lim_{x\to+\infty} \left(1 - \underbrace{e^{-\overbrace{\left(x/\lambda\right)^2}^{\to+\infty}}}_{0}\right) = 1 \underset{\lambda\to+\infty}{\not\to} 0$$

$$\Rightarrow \quad f_\lambda \underset{\lambda\to+\infty}{\not\rightrightarrows} f_\infty \text{ на } [1,+\infty) \; !$$