Вишеструки интеграли

f огран реална ф-ја $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$
„мерљив"

$I = \prod_{i=1}^{n} [a_i, b_i] \supseteq D$, $g: I \to \mathbb{R}$

$$g(x) = \begin{cases} f(x), & x \in D \\ 0, & x \in I \setminus D \end{cases}$$

$\int_I g(x)\, dx$ ($dx = dx_1 dx_2 \dots dx_n$, $x = (x_1, \dots, x_n)$) $\overset{*}{=} \lim_{\lambda(P) \to 0} \sigma(f, P, \xi)$

(*) → уколико десна стр. постоји и не зависи од

$P = \{ I_k : I_k \subseteq I \}$, $I_k = \prod (a_i^k, b_i^k)$, $\bigcup_{I_k \in P} I_k = I$, $I_k \cap I_j = \emptyset$

$\lambda(P) = \sup_{I_k \in P} \operatorname{diam} I_k$

$\xi = \{ \xi_k \in I_k \}$ → истакнуте тачке

$\sigma(f, P, \xi) = \sum_{I_k \in P} f(\xi_k)\, m(I_k)$, $m(I_k) = \prod (b_i^k - a_i^k)$

$\int_D f(x)\, dx = \int_I g(x)\, dx$ (уколико десна стр. постоји и онда је f Риман интеграб. на D, $f \in \mathcal{R}(D)$)

Својства:

1° $f, g \in \mathcal{R}(D)$, $f \pm g \in \mathcal{R}(D)$, $\int_D f(x) \pm g(x)\, dx = \int_D f(x)\, dx \pm \int_D g(x)\, dx$

2° $f \in \mathcal{R}(D)$, $\alpha \in \mathbb{R}$, $\int_D \alpha f(x)\, dx = \alpha \int_D f(x)\, dx$

3° $A \cap B = \emptyset$, $D = A \cup B$, $f \in \mathcal{R}(D) \Leftrightarrow f \in \mathcal{R}(A), f \in \mathcal{R}(B)$

$$\int_D f(x)\, dx = \int_A f(x)\, dx + \int_B f(x)\, dx$$

4° $f, g \in \mathcal{R}(D) \Rightarrow |f| \in \mathcal{R}(D)$, $f \cdot g \in \mathcal{R}(D)$

$|f(x)| \geq d > 0$, $x \in D$ $\Rightarrow$ $\frac{1}{f} \in \mathcal{R}(D)$

5° $f(x) \leq g(x)$ $\forall x \in D \Rightarrow \int_D f(x)\, dx \leq \int_D g(x)\, dx$

7° $m \leq f(x) \leq M$ $\forall x \in D \Rightarrow m \int_D g(x)\, dx \leq \int_D f(x) g(x)\, dx \leq M \int_D g(x)\, dx$

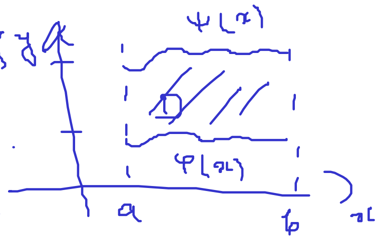
8° $V(D) = \int_D dx$

9° Фубинијева теорема: $f(x,y) \in \mathcal{R}(I)$, $I = [a,b] \times [c,d]$, $x \mapsto f(x,y) \in \mathcal{R}([a,b])$ $\forall y$, $y \mapsto f(x,y) \in \mathcal{R}([c,d])$ $\forall x$

$$\iint_I f(x,y)\, dx\, dy = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy.$$

$\cdot\, D = \{(x,y)\in\mathbb{R}^2 : a\le x\le b,\ \varphi(x)\le y\le \psi(x)\}$

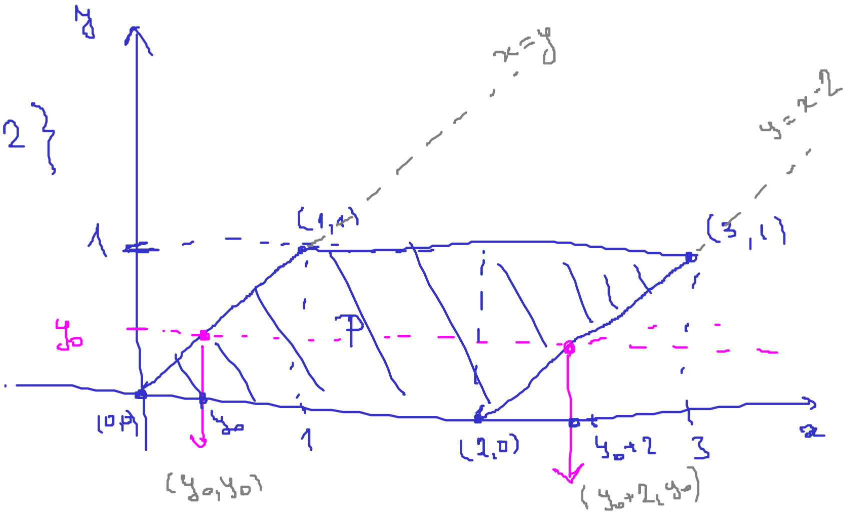
$$\int_D f(x,y)\,dx\,dy = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy \right) dx .$$



① P паралелограм са теменима $(0,0), (2,0), (3,1), (1,1)$

Израчунати $\iint_P \frac{dx\,dy}{1+x+y}$.

$P = \{(x,y)\in\mathbb{R}^2 : 0\le y\le 1,\ y\le x\le y+2\}$



$$I = \iint_P \frac{dx\,dy}{1+x+y} = \int_0^1 \left(\int_y^{y+2} \frac{dx}{1+x+y} \right) dy$$

$$\int_y^{y+2} \frac{dx}{1+x+y} = \left[\begin{array}{l} t = 1+x+y \\ dt = dx \end{array} \right] =$$

$$= \int_{1+2y}^{3+2y} \frac{dt}{t} = \ln t \Big|_{1+2y}^{3+2y} = \ln(3+2y) - \ln(1+2y)$$

$$I = \int_0^1 \big(\ln(3+2y) - \ln(1+2y) \big)\, dy = \int_0^1 \big(\ln(3+2y) - \ln(1+2y) \big)\, dy =$$

$$= \left[\begin{array}{l} u = \ln(3+2y) - \ln(1+2y) \to du = \frac{2}{3+2y} - \frac{2}{1+2y} \\ dv = dy \to v = y \end{array} \right]$$

$$= \underbrace{y\big(\ln(3+2y) - \ln(1+2y)\big)\Big|_0^1}_{\ln 5 - \ln 3} - \underbrace{\int_0^1 \left(\frac{2y+3-3}{3+2y} - \frac{2y}{1+2y} \right) dy}$$

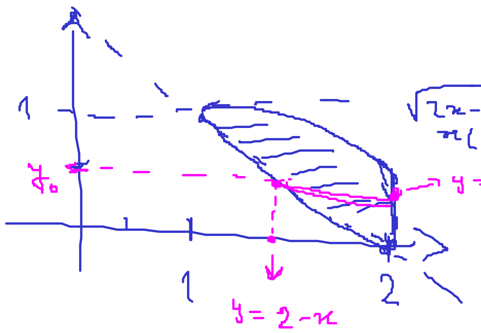
$$= \int_0^1 \left(\cancel{1} - \frac{1}{1+2y} \right) dy - \int_0^1 \left(\cancel{1} - \frac{3}{3+2y} \right) dy + \ln \frac{5}{3}$$

$$= -\frac{1}{2} \ln(1+2y) \Big|_0^1 + \frac{3}{2} \ln(3+2y) \Big|_0^1 + \ln \frac{5}{3} =$$

$$= -\frac{1}{2} \ln 3 + \left(\frac{3}{2} \ln 5 - \frac{3}{2} \ln 3 \right) + \ln \frac{5}{3} = \ldots$$

2) Изменити поредак интеграције

а) $$I = \int_1^2 \int_{2-x}^{\sqrt{2x-x^2}} f(x,y)\,dy\,dx = \underbrace{\int_c^d \int_{\varphi(y)}^{\psi(y)} f(x,y)\,dx\,dy}_{?}$$



$\sqrt{2x-x^2}$ $\quad x(2-x)$ $\quad \sqrt{2x-x^2}$

$\to y = \sqrt{2x-x^2}$ $\quad \to x(y) = ?$

$0 \le y \le 1$

$2-y \le x$

$y = 2-x$

$x = 2-y$

$y = \sqrt{2x-x^2}$ $\quad /^2$

$x^2 - 2x + y^2 = 0$

$x_{1/2} = \dfrac{2 \pm \sqrt{4-4y^2}}{2}$

$x > 1$

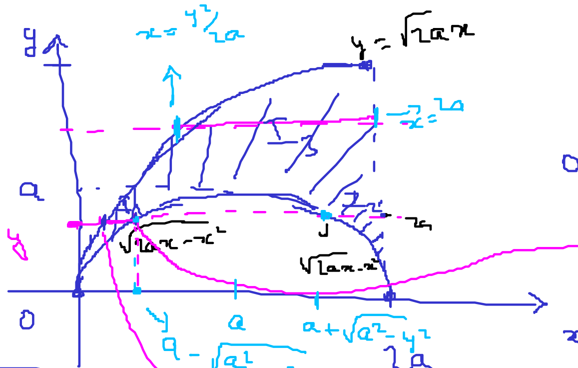
$x = 1 + \sqrt{1-y^2}$

$$\Rightarrow I = \int_0^1 \left(\int_{2-y}^{1+\sqrt{1-y^2}} f(x,y)\,dx \right) dy$$

б) $$\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dy\,dx \,,$$

в) $$\int_0^{2\pi} \int_0^{\sin x} f(x,y)\,dy\,dx \longrightarrow \text{за вежбу}$$

б)



$\varphi(x) = 2ax - x^2 = x(2a-x)$

$\max \varphi = \varphi(a) = a^2$

$\sqrt{\varphi(x)} = a$

$0 \le y \le 2a$

$y = \sqrt{2ax - x^2}$ $\quad /^2 \Rightarrow x =$

$y = \sqrt{2ax}$ $\quad /^2$

$x = \dfrac{y^2}{2a}$

$y^2 = 2ax - x^2$

$x^2 - 2ax + a^2 = a^2 - y^2$

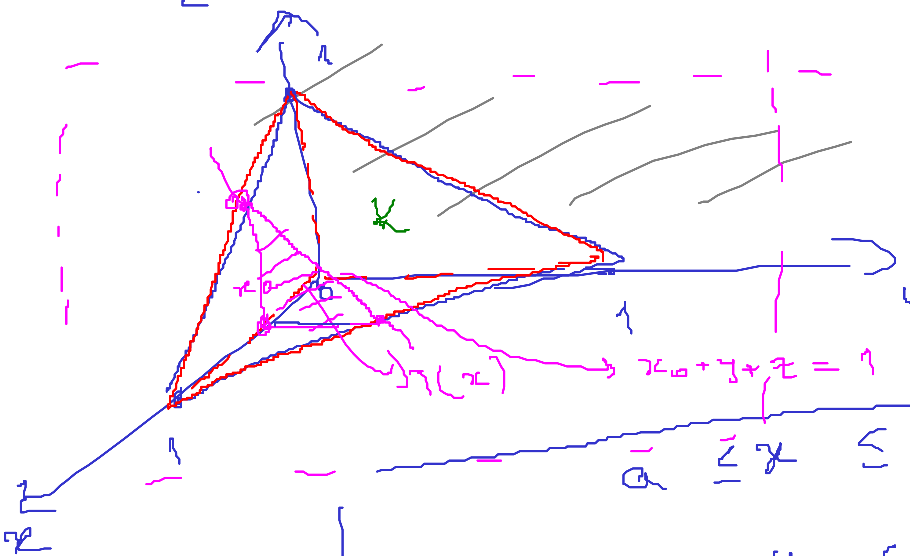
$(x-a)^2 = a^2 - y^2$

$x = a \pm \sqrt{a^2 - y^2}$

$$I = \underbrace{\int_0^a \int_{\frac{y^2}{2a}}^{a-\sqrt{a^2-y^2}} f(x,y)\,dx\,dy}_{I_1} + \overbrace{\int_0^a \int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y)\,dx\,dy}^{I_2} + \int_a^{2a} \int_{y^2/2a}^{2a} f(x,y)\,dx\,dy$$

③ $I=\iiint\limits_K \frac{dx\,dy\,dz}{(1+x+y+z)^3}$

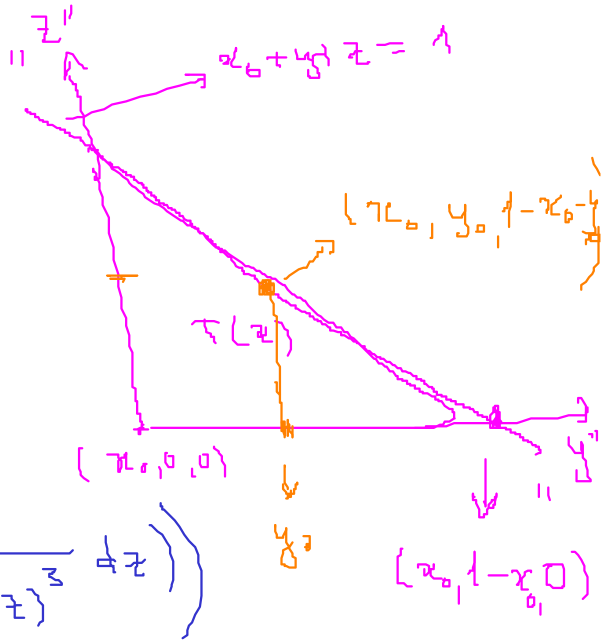
K - тетраедр огран
равнина
$x=0,\ y=0,\ z=0,\ x+y+z=1$



$$\boxed{\begin{aligned} a &\le x \le b \\ \varphi(x) &\le y \le \psi(x) \\ u(x,y) &\le z \le v(x,y) \end{aligned}}\ K$$

$0\le x\le 1$

$$I = \int_0^1 \left(\iint\limits_{T(x)} f(x,y,z)\,dy\,dz \right) dx$$



$0\le y\le 1-x$

$0\le z\le 1-x-y$

$$I = \int_0^1 dx \left(\int_0^{1-x} dy \left(\underbrace{\int_0^{1-x-y} \frac{1}{(1+x+y+z)^3}\,dz}_{||} \right) \right)$$

$$\int_0^{1-x-y} \frac{dz}{(1+x+y+z)^3} = \left[t = 1+x+y+z,\ t(z) \right] = \int_{1+x+y}^{2} \frac{dt}{t^3} =$$

$2 = t(1-x-y)$

$1+x+y = t(0)$

$$= -\frac{1}{2t^2}\Big|_{1+x+y}^{2} = -$$

$$I = \int_0^1 dx \left(\int_0^{1-x} \left(-\frac{1}{2(1+x+y)^2} + \frac{1}{8} \right) dy \right)$$

$$= \underbrace{\quad \cdots \quad}_{3x}$$

Lettoy