[T] $f: A \to \mathbb{R}$ има лок. екст. у $a \in A$ при условима $\varphi_1, \dots, \varphi_s$ ако
$f \in C^2$

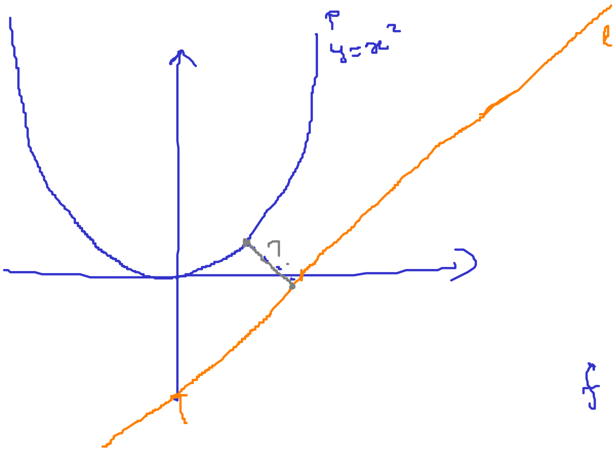
$(a, \lambda_1, \dots, \lambda_s)$ стац. тачка за $F(x, \lambda_1, \dots, \lambda_s) = f(x) + \lambda_1 \varphi_1(x) + \dots + \lambda_s \varphi_s(x)$

и ако $\Phi(a, \lambda_1, \dots, \lambda_s) = \sum_{i,j=1}^{n} \frac{\partial^2 F}{\partial x_i \partial x_j}(a, \lambda_1, \dots, \lambda_s) h_i h_j$ полудефинитна

при условима за $h_1, \dots, h_n$: $\frac{\partial \varphi_t}{\partial x_1}(a) h_1 + \frac{\partial \varphi_t}{\partial x_2}(a) h_2 + \dots + \frac{\partial \varphi_t}{\partial x_n}(a) h_n = 0, \quad 1 \le t \le s$

$d\varphi_t(a)\, h = 0, \quad 1 \le t \le s.$

① Одредити растојање између параболе $y = x^2$ и праве $x - y - 2 = 0$.



$p: y = x^2$

$l: x - y - 2 = 0$

$y = x - 2$

$d = ?$ $\quad d = \inf_{\substack{(x_1, y_1) \in p \\ (x_2, y_2) \in l}} d_2((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$f(x_1, y_1, x_2, y_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$\varphi_1(x_1, y_1, x_2, y_2) = y_1 - x_1^2 = 0$

$\varphi_2(x_1, y_1, x_2, y_2) = x_2 - y_2 - 2 = 0$

$F(x_1, y_1, x_2, y_2, \lambda, \mu) = f(x_1, y_1, x_2, y_2) + \lambda \varphi_1(x_1, y_1, x_2, y_2) + \mu \varphi_2(x_1, y_1, x_2, y_2)$

a1 $\quad \frac{\partial F}{\partial x_1} = 2(x_1 - x_2) + \lambda(-2x_1) = 0$

a2 $\quad \frac{\partial F}{\partial y_1} = \underline{2(y_1 - y_2) + \lambda \cdot 1 = 0}$

a3 $\quad \frac{\partial F}{\partial x_2} = 2(x_1 - x_2) \cdot (-1) + \mu \cdot 1 = 0$

a4 $\quad \frac{\partial F}{\partial y_2} = \underline{-2(y_1 - y_2) - \mu = 0}$

a5 $\quad \frac{\partial F}{\partial \lambda} = y_1 - x_1^2 = 0$

a6 $\quad \frac{\partial F}{\partial \mu} = x_2 - y_2 - 2 = 0$

a2+a4 $\Rightarrow$ $\lambda - \mu = 0 \Rightarrow \lambda = \mu$

a1: $2(x_1 - x_2) - 2\lambda x_1 = 0$

a3: $-2(x_1 - x_2) + \lambda = 0$

a1+a3 $\Rightarrow$ $-2\lambda x_1 + \lambda = 0$

$\lambda(-2x_1 + 1) = 0 \Rightarrow \lambda = 0 \vee x_1 = 1/2$

ако $\lambda = 0 \overset{a3}{\Rightarrow} x_1 = x_2$
$\overset{a4}{\Rightarrow} y_1 = y_2$ $\quad \Rightarrow l \cap p \neq \emptyset$ ↯

$\Rightarrow \lambda \neq 0$

$\Rightarrow x_1 = 1/2 \overset{a5}{\Rightarrow} y_1 = x_1^2 = 1/4$

систем смо свели на

$$x_2 - y_2 - 2 = 0 \quad / \cdot 2$$

$$2\left(\tfrac{1}{4} - y_2\right) + \lambda = 0 \longrightarrow 2y_2 = \tfrac{1}{2} + \lambda$$

$$\lambda - 2\left(\tfrac{1}{2} - x_2\right) = 0 \longrightarrow 2x_2 - 1 + \lambda = 0$$

$$2x_2 = 1 - \lambda$$

$$2x_2 - 2y_2 - 4 = 0$$

$$(1-\lambda) - \left(\tfrac{1}{2} + \lambda\right) - 4 = 0$$

$$-2\lambda + 1 - \tfrac{1}{2} - 4 = 0 \implies -2\lambda = \tfrac{7}{2} \qquad \implies x_2 = \tfrac{1}{2}\left(1 + \tfrac{7}{4}\right) = \tfrac{11}{8}$$

$$\lambda = -\tfrac{7}{4} \qquad y_2 = \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{7}{4}\right) = -\tfrac{5}{8}$$

$$\tfrac{2}{4}$$

стац. тачка: $\left(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{11}{8}, -\tfrac{5}{8}, -\tfrac{7}{4}, -\tfrac{7}{4}\right)$

$$\Phi\left(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{11}{8}, -\tfrac{5}{8}, -\tfrac{7}{4}, -\tfrac{7}{4}\right) h$$

$$\frac{\partial^2 F}{\partial x_1^2} = 2 - 2\lambda = 2 + \tfrac{7}{2} = \tfrac{11}{2} \qquad \frac{\partial^2 F}{\partial x_1 \partial y_1} = 0 \qquad \frac{\partial^2 F}{\partial x_2^2} = 2 \qquad \frac{\partial^2 F}{\partial y_1^2} = 2 \qquad \frac{\partial^2 F}{\partial y_2^2} = 2$$

$$\frac{\partial^2 F}{\partial x_1 \partial x_2} = -2 \qquad \frac{\partial^2 F}{\partial x_1 \partial y_2} = 0 \qquad \frac{\partial^2 F}{\partial x_2 \partial y_1} = \frac{\partial^2 F}{\partial y_2 \partial x_2} = 0 \qquad \frac{\partial^2 F}{\partial y_1 \partial y_2} = -2$$

$$\Phi h = \tfrac{11}{2} \underset{x_1}{h_1^2} - 2 h_1 \underset{x_2}{h_2} - 2h_2 h_1 + 2h_2^2 + 2 \underset{y_1}{h_3^2} - 2h_3 \underset{y_2}{h_4} - 2h_3 h_4 + 2h_4^2$$

$$\varphi_1 = \varphi_1(x_1, y_1) \qquad = 2(h_1 - h_2)^2 + 2(h_3 - h_4)^2 + \tfrac{3}{2} h_1^2 \ \underline{\underline{\geq 0}}$$

$$\frac{\partial \varphi_1}{\partial x_1} h_1 + \frac{\partial \varphi_1}{\partial y_1} h_3 = 0 \longrightarrow -2x_1 h_1 + h_3 = 0 \ \underset{x_1 = \frac{1}{2}}{\longrightarrow} \ -h_1 + h_3 = 0$$

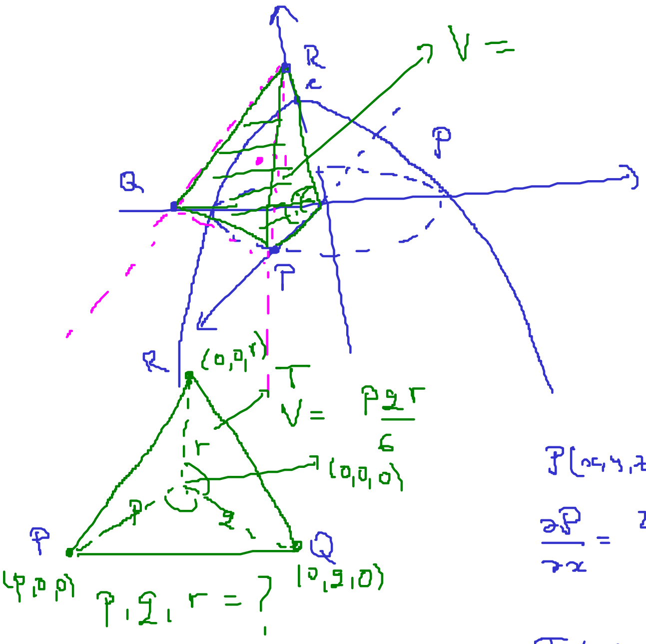
$$\frac{\partial \varphi_2}{\partial x_2} h_2 + \frac{\partial \varphi_2}{\partial y_2} h_4 = 0 \longrightarrow h_2 - h_4 = 0,$$

$\implies \left(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{11}{8}, -\tfrac{5}{8}\right) \rightarrow$ лок. мин. за f $\underline{\text{при условима } \varphi_1 \text{ и } \varphi_2}$

$$d = \sqrt{f\left(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{11}{8}, -\tfrac{5}{8}\right)} = \sqrt{\left(\tfrac{1}{2} - \tfrac{11}{8}\right)^2 + \left(\tfrac{1}{4} + \tfrac{5}{8}\right)^2} = \ldots$$

② P: $z = c - \left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right)$, $c > 0$

Одредити тачку на P у којој тангентна раван на P са коорд. равнима заклапа тетраедар минималне запремине.



V – запремина тетраедра

? min V = ?

$(x_0, y_0, z_0) \in \mathcal{P}$

тангентна раван T у $\underline{(x_0, y_0, z_0)}$

$$\mathcal{P}: \ z = c - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)$$

$$\mathcal{P}(x,y,z) = z + \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) - c = 0$$

$$\frac{\partial \mathcal{P}}{\partial x} = \frac{2x}{a^2}, \quad \frac{\partial \mathcal{P}}{\partial y} = \frac{2y}{b^2}, \quad \frac{\partial \mathcal{P}}{\partial z} = 1$$

$$T(x_0,y_0,z_0): \quad \frac{\partial \mathcal{P}}{\partial x}(A)(x - x_0) + \frac{\partial \mathcal{P}}{\partial y}(A)(y - y_0) + \frac{\partial \mathcal{P}}{\partial z}(A)(z - z_0) = 0$$

$$\overset{A}{\uparrow} \ \mathcal{P}$$

$$\underline{\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) + (z - z_0) = 0}$$

$$P: \ (p,0,0) \in T(x_0,y_0,z_0) \qquad \frac{2x_0}{a^2}(p - x_0) + \frac{2y_0}{b^2}(-y_0) + (-z_0) = 0$$

$$\frac{2x_0}{a^2}p = z_0 + \underbrace{\frac{2y_0^2}{b^2} + \frac{2x_0^2}{a^2}}_{2(c - z_0)}$$

$$= 2c - z_0$$

$$p = \frac{2c - z_0}{2x_0} \cdot a^2$$

$$Q: \ (0,q,0) \in T$$

$$-\frac{2x_0}{a^2} + \frac{2y_0}{b^2}(q - y_0) - z_0 = 0$$

$$\ldots \quad q = \frac{2c - z_0}{2y_0} \cdot b^2$$

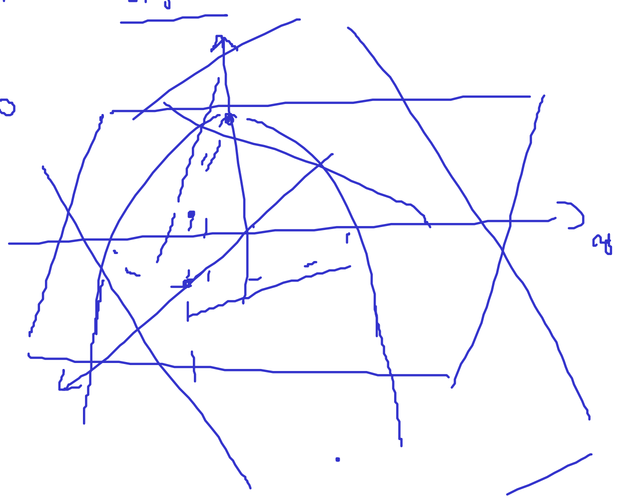
$$R: \ (0,0,r) \in T$$

$$-\frac{2x_0^2}{a^2} - \frac{2y_0^2}{b^2} + r - z_0 = 0$$

$$r = z_0 + 2(c - z_0) = 2c - z_0$$

пп БУО $\underline{x_0, y_0 > 0}$

$\underline{x_0 = 0}$



$$f(x_0,y_0,z_0) = 6V = pgr = \frac{(2c-z_0)a^2}{2x_0}\cdot\frac{(2c-z_0)b^2}{2y_0}(2c-z_0) = \boxed{\frac{a^2b^2}{4}}\frac{(2c-z_0)^3}{x_0y_0}$$

не утиче на мин/макс.

$$\varphi(x_0,y_0,z_0) = z_0 - c + \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 0$$

$$F(x,y,z,\lambda) = \frac{(2c-z)^3}{xy} + \lambda\left(z - c + \frac{x^2}{a^2} + \frac{y^2}{b^2}\right)$$

$$\frac{\partial F}{\partial x} = \frac{(2c-z)^3}{y}\cdot\left(-\frac{1}{x^2}\right) + \frac{2x\lambda}{a^2} = 0$$

$$\frac{\partial F}{\partial y} = \frac{(2c-z)^3}{x}\left(-\frac{1}{y^2}\right) + \frac{2y\lambda}{b^2} = 0$$

$$\frac{\partial F}{\partial z} = \frac{3(2c-z)^2\cdot(-1)}{xy} + \lambda = 0$$

$$\frac{\partial F}{\partial \lambda} = z - c + \frac{x^2}{a^2} + \frac{y^2}{b^2} = 0 \quad (4)$$

$2c - z = t$

$$-\frac{t^3}{yx^2} + \frac{2x\lambda}{a^2} = 0 \;/\cdot x \quad (1)$$

$$-\frac{t^3}{xy^2} + \frac{2y\lambda}{b^2} = 0 \;/\cdot y \quad (2)$$

$$-\frac{3t^2}{xy} + \lambda = 0 \quad (3)$$

$$(1)-(2) \quad \frac{2x^2\lambda}{a^2} - \frac{2y^2\lambda}{b^2} = 0$$

$$2\lambda\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = 0$$

$\lambda = 0$?

$$(3) \Rightarrow \frac{3(2c-z)^2}{xy} = 0 \qquad z \neq 2c,\ \underline{z \leq c}\ \checkmark$$

$$\Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2}, \quad x,y>0$$

$$\boxed{\frac{x}{a} = \frac{y}{b}}, \quad a,b>0$$

$$\boxed{y = \frac{b}{a}x}$$

$$(1) \rightsquigarrow -\frac{t^3}{\frac{b}{a}\cdot x^3} + \frac{2x\lambda}{a^2} = 0 \;/\cdot x^3\cdot b\cdot a^2 \longrightarrow -a^3t^3 + 2x^4b\lambda = 0$$

$$(3) \rightsquigarrow -\frac{3t^2}{\frac{b}{a}x^2} + \lambda = 0 \;/\cdot x^2 b \longrightarrow -3t^2a + \lambda b x^2 = 0 \Rightarrow \lambda b = \frac{3t^2a}{x^2}$$

$$(4) \longrightarrow c - t + 2\frac{x^2}{a^2} = 0$$

$z - c = c - t$

$$c - t + \frac{2x^2}{a^2} = 0$$

$$-a^3t^3 + 2x^4\cdot\frac{3t^2a}{x^2} = 0$$

$$c - t + \frac{2x^2}{a^2} = 0$$

$(2c-z)^2$, $z<c \Rightarrow t\neq 0$, $a\neq 0$

$$t^2a\left(-a^2t + 6x^2\right) = 0 \;/: t^2a$$

$$c - t + \frac{2x^2}{a^2} = 0$$

$$t = \frac{6x^2}{a^2} \to \frac{x^2}{a^2} = \frac{t}{6}$$

$$c - t + \frac{t}{3} = 0 \Rightarrow c = \frac{2t}{3}$$

$\Rightarrow \quad c = \dfrac{2(2c-z)}{3} \quad \Rightarrow \quad 3c = 4c - 2z$

$2z = c$

$z = \dfrac{c}{2} < c \quad \checkmark$

$(4) \Rightarrow \quad z - c + \dfrac{2x^2}{a^2} = 0 \quad \Rightarrow \quad \dfrac{2x^2}{a^2} = \dfrac{c}{2}$

$x^2 = \dfrac{c \cdot a^2}{4}, \quad x > 0$

$x = \dfrac{a}{2}\sqrt{c}$

$y/b = x/a \quad \Rightarrow \quad y = \dfrac{b}{2}\sqrt{c}$

$$\Rightarrow \min f = f\underbrace{\left(\frac{a}{2}\sqrt{c}, \frac{b}{2}\sqrt{c}, \frac{c}{2}\right)}_{\substack{\left(-\frac{a}{2}\sqrt{c}, \frac{b}{2}\sqrt{c}, \frac{c}{2}\right) \\ \left(-\frac{a}{2}\sqrt{c}, -\frac{b}{2}\sqrt{c}, \frac{c}{2}\right) \\ \left(\frac{a}{2}\sqrt{c}, -\frac{b}{2}, \frac{c}{2}\right)}} = \frac{a^2b^2}{4}\,\frac{(2c - c/2)^3}{\frac{a}{2}\sqrt{c}\cdot\frac{b}{2}\sqrt{c}} = \dots$$

③ Материјал за доњу стр. акваријума кошта 2× мање него стакло за преостале стране. Акв. је облика квадра и нема поклопац.

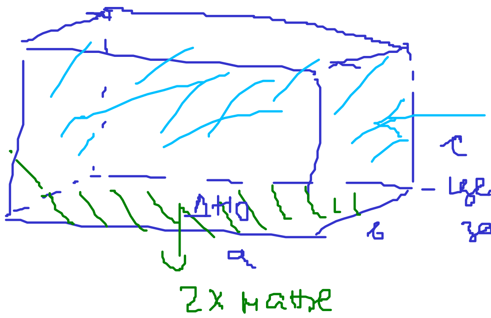
Одредити дим. акв. запремине V који је најјефтинији.

→ф-ја

f → цена акв.

V → запремина

↳услов



цена за $\frac{\text{дно}}{m^2}$ → M

цена за дно дим $a \times b$ је abM

цена стакла за висину c

→стакло дупло скупље

$2M(2bc + 2ac)$

цена акваријума је $abM + 4M(bc + ac)$

$f(a,b,c) = ab + 4(bc + ac), \quad \varphi(a,b,c) = abc - V = 0$ → услов min f ?

$a, b, c = ?$ тд је f мин

$$F(a,b,c,\lambda) = ab + 4(bc+ac) - \lambda(abc - V)$$

$$\frac{\partial F}{\partial a} = b + 4c - \lambda bc = 0 \quad / \cdot a$$

$$\frac{\partial F}{\partial b} = a + 4c - \lambda ac = 0 \quad / \cdot b$$

$$\frac{\partial F}{\partial c} = 4(b+a) - \lambda ab = 0 \quad / \cdot c$$

$$\frac{\partial F}{\partial \lambda} = -(abc - V) = 0.$$

$$ba + 4ca - \lambda V = 0 \quad (1)$$

$$ab + 4bc - \lambda V = 0 \quad (2)$$

$$4bc + 4ac - \lambda V = 0 \quad (3)$$

$(1) - (2) \Rightarrow 4c(a-b) = 0$

$c \neq 0 \rightarrow$ јер је ово дужина ивице квадра

$\Rightarrow a = b$

$(2) \Rightarrow a^2 + 4ac - \lambda V = 0$

$(3) \Rightarrow 8ac = \lambda V$

$a^2 - 4ac = 0$

$a \neq 0 \qquad a = 4c = b$

$abc = V$

$16c^3 = V \Rightarrow c = \sqrt[3]{\frac{V}{16}}$

$a = b = 4\sqrt[3]{\frac{V}{16}}$

$\lambda = \frac{8ac}{abc} = \frac{8}{b} = \frac{2 \cdot \sqrt[3]{16}}{\sqrt[3]{V}}$

$\Phi(\ldots)h = \ldots$

$\rightarrow$ за вежбу.

Одредити лок. екст. ф-је

1. $f(x,y,z) = xyz$, при услову $x^2+y^2+z^2 = 3$
2. $f(x,y,z) = x^2+y^2+z^2$, $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
3. $f(x,y,z) = x+y+z^2$, $\quad z - x = 1$, $\quad y - xz = 1$
4. $f(D) = ?$ $\quad f(x,y) = x^2+y^2-12x+16y$, $\quad D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 25\}$
5. Одредити најудаљенију тачку на елипсоиду $\varepsilon: x^2+2y^2+4z^2 = 8$ од тачке $(0,0,3)$.

I дом. 31. 1. 2020.

① $X = C[0,1]$

$d: X\times X \to \mathbb{R}$ $\quad d(f,g) = \begin{cases} \max\limits_{0\le x\le 1}|f(x)| + \max\limits_{0\le x\le 1}|\alpha g(x)|, & f\ne g \\ 0, & f=g \end{cases}$

а) Одр. α тд је d метрика на X.

1° $d(f,g)\ge 0$ ✓

2° $d(f,g)=0 \iff f=g$

$\Leftarrow$ ✓

$f\ne g$? $d(f,g)=0$

$\underbrace{\max|f(x)|}_{\ge 0} + \underbrace{\max|\alpha g(x)|}_{\ge 0} = 0$

$\Rightarrow$ $\max|f(x)|=0$, $\max|\alpha g(x)|=0$

$\Downarrow$

$\forall x\in[0,1]$ $f(x)=0$, $\forall x\in[0,1]$ $\alpha g(x)=0$

$\Downarrow$ $\qquad\qquad\qquad \Downarrow$

$f\equiv 0$ $\qquad g\equiv 0 \lor \alpha=0$

2° важи $\Rightarrow \alpha\ne 0$.

3° $d(f,g) = d(g,f)$

$f\ne g$

$\max|f(x)| + \max|\alpha g(x)| \overset{?}{=} \max|g(x)| + \max|\alpha f(x)|$

нпр за $f\equiv 0$, $g\equiv 1$

$\Rightarrow d(f,g) = 0+|\alpha| \Rightarrow |\alpha|=1$

$d(g,f) = 1+0$

за $|\alpha|=1$ $\quad$ „$d(f,g)$" $=$ $\max|f(x)| + \max|\alpha g(x)|$

$= \max|\alpha f(x)| + \max|g(x)| = d(g,f)$

$3^\circ \Rightarrow \quad \alpha = 1$ или -1

$$d(f,g) = \begin{cases} \max|f| + \max|g|, & f \neq g \\ 0, & f = g \end{cases}$$

неј. троугла

$f, g, h, \quad f \neq g \neq h \neq f$

$$d(f,g) \leq d(f,h) + d(g,h) \ ?$$

$$\underbrace{\max|f| + \max|g|}_{d(f,g)} \leq \underbrace{\max|f| + \max|h|}_{d(f,h)} + \underbrace{\max|g| + \max|h|}_{d(g,h)} \quad \checkmark$$

d метрика ако $\alpha = 1$ или $\alpha = -1$

б) $f_0(x) = 0, \quad f_1(x) = |6x-4| - 3$

Одредити $B(f_0,1), B(f_1,1), B(f_1,4)$

$$B(f_0,1) = \{ g \in C[0,1] : d(f_0,g) < 1 \}$$

$$= \{ g \in C[0,1] : \underset{g = f_0}{\overset{d(f_0,g)=0}{g = 0}} \vee \max|g(x)| + \max|f_0(x)| < 1 \}$$

$$= \{ g \in C[0,1] : g \equiv 0 \vee \max|g(x)| < 1 \}$$

$$= \{ g \in C[0,1] : \max|g(x)| < 1 \}$$

$$= \{ g : [0,1] \to (-1,1) \text{ непр.} \}$$

$$B(f_1,1) = \{ g \in C[0,1] : d(f_1,g) < 1 \}$$

$$= \{ g \in C[0,1] : g = f_1 \vee \underbrace{\max|g(x)|}_{\geq 0} + \underbrace{\max|f_1(x)|}_{\geq 3} < 1 \}$$

$$\max|f_1(x)| = \max_{x \in [0,1]} \big||6x-4| - 3\big| \geq 3$$

$x = 4/6 \in [0,1]$

$B(f_1, 1) = \{f_1\}$

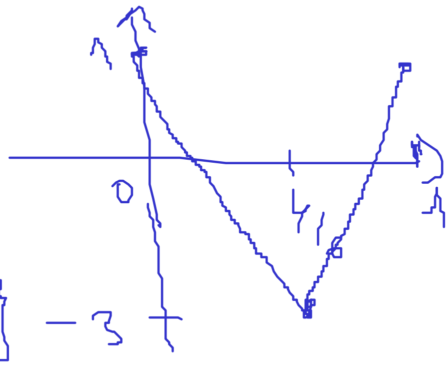
$B(f_1, 4) = \{ g \in C[0,1] : g = f_1 \vee$

$\max|g| + \max|f_1| < 4 \}$

$f_1(x) = |6x-4| - 3 = \begin{cases} 4-6x-3, & x < 4/6 \\ 6x-7, & x \geq 4/6 \end{cases} = \begin{cases} 1-6x, & x < 4/6 \\ 6x-7, & x \geq 4/6 \end{cases}$

$f_1([0,1]) = [-3, 1]$, $|f_1|[0,1] = [0,3]$

$\max|f_1| = 3$



$B(f_1, 4) = \{ g \in C[0,1] : g = f_1 \vee$

$\max|g| < 1 \}$

$= B(f_1, 1) \cup B(f_0, 1)$

$B(f_1, 1) \cap B(f_0, 1) = \emptyset$

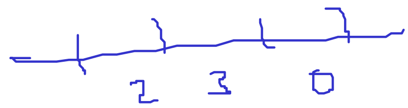
б) Найти $d(B(f_0,1), B(f_1,1))$ и $\operatorname{diam} B(f_1, 4)$

$d(B(f_0,1), B(f_1,1)) = \inf\limits_{g \in B(f_0,1)} d(f_1, g)$

$B(f_1,1) = \{f_1\}$

$= \inf\limits_{\substack{g \in B(f_0,1) \\ \max|g| < 1}} (\max|f_1| + \max|g|) = 3$

$\max|f_1| = 3$



$\operatorname{diam} B(f_1, 4) = \sup\limits_{g,h \in B(f_1,4)} d(g,h) =$

$B(f_1,4) = B(f_0,1) \cup B(f_1,1)$

$$= \sup\Big(\{ d(g,h)^{\,1,3} : h = f_1,\ g \in B(f_0,1)\}$$

$$\cup \Big\{ \overbrace{d(g,h)}^{\le 2} : g,h \in B(f_0,1) \Big\}\Big)$$

$$= \sup\{ d(g,h) \mid h = f_1,\ g \in B(f_0,1)\}$$

$$= \sup\{ \max|g(x)| + \underset{3}{\underset{\shortparallel}{\max|f_1(x)|}} : \underset{\max|g|<1}{\underset{\Updownarrow}{g \in B(f_0,1)}}\}$$

$$= 4$$

$$\downarrow$$

$$g_n = 1 - 1/n$$

"X"

г) $A = \{ f \in C[0,1] : \sin(f(x) - x) = 1 \quad \forall x \in [0,1]\}$

$B = \{ f \in C[0,1] : \exists g \in C[0,1]\ \forall x \in [0,1] \quad 2f(x) = \operatorname{arctg} g(x)\}$

отв./зат./пов. за $A \cup B$.

$$A = \{ f \in X : f(x) - x = \frac{\pi}{2} + 2k\pi,\ x \in [0,1]\} \quad k \in \mathbb{Z}$$

$$f_k(x) = x + \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z},\ x \in [0,1]$$

$$\max|f_k(x)| = |1 + \frac{\pi}{2} + 2k\pi| > 0, \quad \min_k(\max|f_k(x)|) > \varepsilon$$

$$\shortparallel$$

$$\min\{\frac{3\pi}{2} - 1,\ 1 + \frac{\pi}{2}\}$$

$$\bar{A} = A = \bigsqcup_{k \in \mathbb{Z}} (B[f_k, \varepsilon] \cap A)$$

$$d(f_k, f_\ell) > 2\varepsilon$$

f_{k_n} низ у A који конв. $\Rightarrow$ f_{k_n} је конст. почев од неког n

$\Rightarrow$ A зат.

$A = \bigcup_{k \in \mathbb{Z}} B(f_k, \varepsilon)$ $\rightarrow$ отворен

не постоји g довољно ε

$\max|f_k| > \varepsilon \Rightarrow B(f_k, \varepsilon) = \{ g \in X : \max|g(x)| + \max|f_k(x)| < \varepsilon \ \vee\ g = f_k \}$

$= \{ f_k \}$

$A \cup B = \bigsqcup_{k \in \mathbb{Z}} B(f_k, \varepsilon) \cup B$ $\rightarrow$ није повезано

$\exists$ нетривијалан скуп који је у исто време отв. и зат. и није цео простор нпр $B(f_0, \varepsilon)$

$$B = \{ f \in X : \exists g \in X : 2f(x) = \operatorname{arctg} g(x) \} \qquad tg(\operatorname{arctg}\alpha) = \alpha$$

$$\operatorname{arctg}(tg(\beta)) \neq \beta$$

$$= \{ \underset{C[0,1]}{\underset{\shortparallel}{f \in X}} : \underset{g(x)}{\underset{\shortparallel}{tg\, 2f(x)}} \in C[0,1] \}$$

$$\rightsquigarrow f(x) \neq \frac{\pi}{2} + k\pi \quad \forall x$$

$$= \{ f \in X : \overset{\in C[0,1]}{f(x)} \neq \frac{\pi}{2} + k\pi, \forall x \}$$

$$= \{ f \in X : f[0,1] \in (k\pi - \frac{\pi}{2}, \frac{\pi}{2} + k\pi), k \in \mathbb{Z} \}$$

$k \neq 0$, $\underset{\leq \pi/2}{\leq} \min\{|k\pi - \frac{\pi}{2}|, |k\pi + \frac{\pi}{2}|\} \leq \max|f(x)|$, $k = 0 \to f[0,1] \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\max|f_0| = 0$

B затворен ?

$$B = \bigcup_{k \in \mathbb{Z}} \{ f \in X : f \subseteq (k\pi - \frac{\pi}{2}, \pi/2 + k\pi) \}$$

$$= \bigcup_{k \in \mathbb{Z} \setminus \{0\}} \{ f \in X \overset{A_k}{\cdots} \} \cup \{ f \in X : \max f < \frac{\pi}{2} \}$$

$$\frac{\pi}{4} < \max|f|$$

$$\shortparallel \quad B(f_0, \frac{\pi}{2}) \qquad \overset{\shortparallel}{0}$$

$$= \bigcup_{\substack{k \in \mathbb{Z} \\ f \in A_k}} B(f, \frac{\pi}{4}) \cup B(f_0, \frac{\pi}{2})$$

$\to$ отв, затв.

$\shortparallel \{f\}$ и отв и затв

отв, затв.

$B(f_0, \frac{\pi}{2})$ затв ?

f_n^k конв. у (X, d) ка f

$$d(f_n, f) \longrightarrow 0$$

f_n није конст. пошто од неког

$$d(f_n, f) = \underset{\underset{0}{\geq}}{\max|f_n|} + \underset{\underset{0}{\geq}}{\max|f|} \longrightarrow 0$$

$$\Rightarrow \max|f| = 0 \quad \Rightarrow f = 0$$

$$\max|f_n| \longrightarrow 0$$

$$\Rightarrow f \in B(f_0, d)$$

A, B отв / затв $\Rightarrow A \cup B$ отв и затв.

(2) $f(x,y) = \begin{cases} (x-1)y\ln((x-1)^2+y^2), & (x,y)\neq(1,0) \\ a, & (x,y)=(1,0) \end{cases} \in C(\mathbb{R}^2)$

$a \in \mathbb{R}$

а) Испитати диф фје f.

б) Испитати равн непр. f

в) Локални екст. f ?

г) $f(D)$, $D = \{(x,y) : \frac{1}{2} \leq (x-1)^2 + y^2 \leq 2\}$

$a = ?$ $\quad f \in C(\mathbb{R}^2)$

$$\lim_{(x,y)\to(1,0)} \underset{\downarrow\, 0}{(x-1)}\, \underset{\downarrow\, 0}{y}\, \underbrace{\ln((x-1)^2+y^2)}_{\downarrow\, -\infty} = 0 = a$$

$$\ln((x-1)^2+y^2) \leq \ln(2\underset{t^2}{\underbrace{\max\{(x-1)^2, y^2\}}})$$

а) $$\frac{\partial f}{\partial x}(x,y) = y\ln((x-1)^2+y^2) + \frac{(x-1)y}{(x-1)^2+y^2}\cdot 2(x-1)$$
$\cancel{x}$
$(1,0)$

$$\frac{\partial f}{\partial y}(x,y) = (x-1)\ln(\quad) + \frac{(x-1)y}{(x-1)^2+y^2}\cdot 2y$$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ су непр у $(x,y)\neq(1,0)$

$\Rightarrow f$ диф у $(x,y)\neq(1,0)$

у $(1,0)$:

$$\frac{\partial f}{\partial x}(1,0) = \lim_{h\to 0}\frac{f(1+h,0) - \overset{=0}{f(1,0)}}{h} =$$

$$= \lim_{h\to 0}\frac{(1+h-1)\cdot 0 \cdots}{h} = 0$$

$$\frac{\partial f}{\partial y}(1,0) = \lim_{h\to 0}\frac{f(1,h) - f(1,0)}{h} = \lim_{h\to 0}\frac{\overset{=0}{(1-1)\cdot h \cdots}}{h} = 0$$

$$\Delta f(1,0)(h_1,h_2) = \underbrace{h_1 h_2 \ln(h_1^2+h_2^2)}_{\parallel\ f(1+h_1,h_2)} - \underbrace{f(1,0)}_{\parallel\ 0}$$

$$\frac{\Delta f(1,0) - \overbrace{\frac{\partial f}{\partial x}}^{=0} h_1 - \underbrace{\frac{\partial f}{\partial y}}_{\parallel\ 0} h_2}{\sqrt{h_1^2+h_2^2}} \xrightarrow[?]{(h_1,h_2)\to(0,0)} 0$$

$$\frac{h_1 h_2 \ln(h_1^2+h_2^2)}{\sqrt{h_1^2+h_2^2}} \xrightarrow[?]{} 0$$

$h_1 = r\cos\theta$ $\qquad h_1^2+h_2^2 = r^2$

$h_2 = r\sin\theta$

$(h_1,h_2)\to(0,0) \iff r\to 0$

$$\frac{r^2\cos\theta\sin\theta \ln r^2}{r} = \underbrace{2r\ln r}_{\downarrow\ 0}\ \underbrace{\cos\theta\sin\theta}_{\text{ограч}} \xrightarrow{r\to 0} 0$$

$\Rightarrow f$ диф у 0

б) $(x_n,y_n)=(1,n) \to f(1,n) = (1-1)\cdot n\ln((1-1)^2+n^2) = 0$

$(a_n,b_n) = (1+1/n, n) \to f(1+1/n, n) = \frac{1}{n}\cdot n\ln\left(\frac{1}{n^2}+n^2\right) = \ln\left(\frac{1}{n^2}+n^2\right) \to +\infty$

$d_2((x_n,y_n),(a_n,b_n)) = \sqrt{1/n^2 + 0} = \frac{1}{n} \to 0$

$\Rightarrow f$ није равн непр

в) лок. екст.

$(1,0) \to$ стац.? $\qquad \frac{\partial f}{\partial x}(1,0) = 0$

$\qquad \frac{\partial f}{\partial y}(1,0) = 0$

$(x,y)\neq(1,0)$ $\qquad \frac{\partial f}{\partial x}(x,y) = y\ln((x-1)^2+y^2) + \frac{2(x-1)^2 y}{(x-1)^2+y^2} = 0$

$\frac{\partial f}{\partial y}(x,y) = (x-1)\ln((x-1)^2+y^2) + \frac{2y^2(x-1)}{(x-1)^2+y^2} = 0$

$y = 0 \;,\; x \neq 1 \quad \Rightarrow \quad (x-1)\ln\left((x-1)^2\right) = 0 \;\; \lightning$

$x = 1 \;,\; y \neq 0 \quad \Rightarrow \quad y \ln y^2 = 0 \;\; \lightning$

$y \neq 0 \;\wedge\; x \neq 1$

$$\left.\begin{aligned} &\ln\left((x-1)^2 + y^2\right) + \frac{2(x-1)^2}{(x-1)^2 + y^2} = 0 \\ &\ln\left(\qquad\right) + \frac{2y^2}{---} = 0 \end{aligned}\right\} -$$

$$\frac{2(x-1)^2 - 2y^2}{---} = 0$$

$(x - 1 - y) = 0 \quad \vee \quad x - 1 + y = 0$

$x = y + 1 \,,\; y \neq 0 \quad \vee \quad x = -y + 1 \;,\; y \neq 0$

... za lettdy gabe ...