Услови екстремуми

$\boxed{T}$ $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$, $f \in C^2$

$\varphi_1, \ldots, \varphi_s \in C^2(A)$, $s < n$

$B = \{ x \in A,\ \varphi_1(x) = \cdots \varphi_s(x) = 0 \}$

$F: A \times \mathbb{R}^s \to \mathbb{R}$, $F(\underbrace{x_1, \ldots, x_n}_{x}, \lambda_1, \ldots, \lambda_s) = f(x_1, \ldots, x_n) + \lambda_1\varphi_1(x) + \lambda_2\varphi_2(x) + \cdots + \lambda_s\varphi_s(x)$

Лагранжев множилац

a услови лок. екст. f при условима $\varphi_1, \varphi_2, \ldots, \varphi_s$ ако $a \in B$

$\Rightarrow \exists \lambda_1, \ldots, \lambda_s \in \mathbb{R}:$ $\dfrac{\partial F}{\partial x_k}(a, \lambda_1, \ldots, \lambda_s) = 0$, $1 \le k \le n$

$$\frac{\partial F}{\partial \lambda_t}(a, \lambda_1, \ldots, \lambda_s) = 0$$

$= \varphi_t(a)$

$$\Phi(\overset{= h}{h_1, \ldots, h_n}) = \sum_{i,j=1}^{n} \frac{\partial^2 F(a)}{\partial x_i \partial x_j} h_i h_j$$

$$d\varphi_t h = 0 \to \frac{\partial \varphi_t}{\partial x_1} h_1 + \frac{\partial \varphi_t}{\partial x_2} h_2 + \cdots + \frac{\partial \varphi_t}{\partial x_n} h_n = 0, \quad 1 \le t \le s$$

$\Phi h \geq 0 \Rightarrow$ лок. мин. у a

$\Phi h \le 0 \Rightarrow$ лок. макс. у a

① Одредити локалне екст. ф-је $\underbrace{f(x_1, \ldots, x_n) = x_1^m + x_2^m + \cdots + x_n^m}$, $a > 0$, $m > 0$.
при условима $x_1 + x_2 + \cdots + x_n = n \cdot a$, $x_k > 0$, $1 \le k \le n$,

$f(x_1, \ldots, x_n) = x_1^m + x_2^m + \cdots + x_n^m$

$\varphi(x_1, \ldots, x_n) = x_1 + \cdots + x_n - na$

$F: \mathbb{R}^{n+1} \to \mathbb{R}$

$$F(x_1, \ldots, x_n, \lambda) = f(x_1, \ldots, x_n) + \lambda\varphi(x_1, \ldots, x_n)$$
$$= x_1^m + \cdots + x_n^m + \lambda(x_1 + \cdots + x_n - na)$$

$1 \le k \le n$, $\dfrac{\partial F}{\partial x_k}(x_1, \ldots, x_n, \lambda) = m x_k^{m-1} + \lambda = 0 \longrightarrow \underbrace{x_k^{m-1} = -\dfrac{\lambda}{m}}$, $\underbrace{x_k > 0}$ $\Longrightarrow$ $m \neq 1$ $\Downarrow$ $x_k = x_i$

$m = 1$?

$$\frac{\partial F}{\partial \lambda} = \varphi = x_1 + \cdots + x_n - na = 0$$

1° $m \neq 1 \to x_k = x_i$, $1 \le k, i \le n$

$n \cdot x_1 - na = 0 \Rightarrow x_1 = a > 0 \Rightarrow x_k = a$

$(a, \ldots, a) \to$ кандидат за лок. екст., $a^{m-1} = -\lambda/m$

$\lambda = -m a^{m-1}$

$$\Phi(h_1,\dots,h_n) = \sum_{i,j=1}^{n} \frac{\partial^2 F}{\partial x_i \partial x_j} h_i h_j$$

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial x_i}\right) = \frac{\partial}{\partial x_j}\left(m x_i^{m-1} + \lambda\right) = \begin{cases} 0 & , i \neq j \\ m(m-1) x_i^{m-2} & , i=j \end{cases}$$

$$\Phi^{(a)}(h_1,\dots,h_n) = \sum_{i=1}^{n} m(m-1) \underbrace{x_i^{m-2}}_{a} \cdot h_i^2 = m(m-1) a^{m-2} \sum_{i=1}^{n} h_i^2$$

$$\frac{\partial \varphi}{\partial x_1} h_1 + \dots + \frac{\partial \varphi}{\partial x_n} h_n = 0 \quad , \varphi = x_1 + \dots + x_n - na$$

$$\left\{ \frac{\partial \varphi}{\partial x_k} = 1 \right.$$

$h_1 + \dots + h_n = 0 \rightarrow$ постоје овакве n-торке

$m > 1 \Rightarrow \Phi(h_1,\dots,h_n) > 0 \Rightarrow (a,\dots,a)$ лок. мин

$m < 1 \Rightarrow \Phi(h_1,\dots,h_n) < 0 \Rightarrow (a,\dots,a)$ лок. макс.

2° $m = 1$:

$f(x_1,\dots,x_n) = x_1^1 + x_2^1 + \dots + x_n^1 = x_1 + \dots + x_n = na$

$\varphi(x_1,\dots,x_n) = x_1 + \dots + x_n - na = 0$

$\Rightarrow f$ је константна фја при услову φ.

(2) Одредити лок. екст. фје $f(x_1,\dots,x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$

ако $x_1 + \dots + x_n = a$, $x_k > 0$, $\alpha_k > 0$, $a > 0$.

$f > 0$
$g = \ln f$ исте је „монотоности" као f
понаша се

а лок. екст. фје $f \Rightarrow a$ је лок. екст. фје g исте врсте као за f

$$g(x_1,\dots,x_n) = \ln f(x_1,\dots,x_n) = \ln\left(x_1^{\alpha_1} \cdots x_n^{\alpha_n}\right)$$
$$= \alpha_1 \ln x_1 + \alpha_2 \ln x_2 + \dots + \alpha_n \ln x_n$$

$\varphi(x_1,\dots,x_n) = x_1 + \dots + x_n - a = 0 \rightarrow$ услов.

$$F(x_1,\dots,x_n,\lambda) = \sum_{k=1}^{n} \alpha_k \ln x_k + \lambda(x_1 + \dots + x_n - a)$$

$$1 \le k \le n \quad \frac{\partial F}{\partial x_k} = \frac{\alpha_k}{x_k} + \lambda = 0 \longrightarrow \frac{\alpha_k}{x_k} = -\lambda \rightarrow \underbrace{x_k = -\frac{\alpha_k}{\lambda}}$$

$$\frac{\partial F}{\partial \lambda} = x_1 + \dots + x_n - a = 0$$

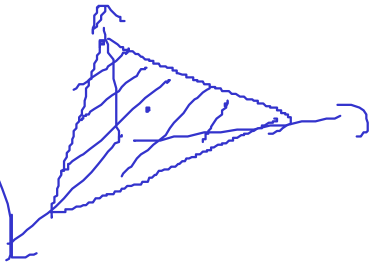
$$-\sum_{k=1}^{n} \frac{\alpha_k}{\lambda} - a = 0$$

$$\frac{1}{\lambda} \sum_{k=1}^{n} \alpha_k = -a$$

$$\lambda = -\frac{\sum_{k=1}^{n} \alpha_k}{a} \Longrightarrow x_k = \frac{\alpha_k}{\sum_{k=1}^{n} \alpha_k} \cdot a$$

$$x = (x_1, \dots, x_n)$$

$$A = \left(\frac{\alpha_1}{\sum_{1}^{n} \alpha_k} a, \frac{\alpha_2}{\sum \alpha_k} a, \dots, \frac{\alpha_n}{\sum \alpha_k} a \right)$$



$$\frac{\partial^2 F}{\partial x_k \partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial x_k}\right) = \frac{\partial}{\partial x_j}\left(\frac{\alpha_k}{x_k} + \lambda\right) = \begin{cases} -\frac{\alpha_k}{x_k^2}, & k = j \\ 0, & k \neq j \end{cases}$$

$$\frac{\partial^2 F}{\partial x_k \partial x_j}(A) = \begin{cases} -\frac{\alpha_k}{\frac{\alpha_k^2}{(\sum \alpha_i)^2} a^2}, & k = j \\ 0, & k \neq j \end{cases} = \begin{cases} -\frac{\left(\sum_{1}^{n} \alpha_i\right)^2}{\alpha_k a^2}, & k = j \\ 0, & k \neq j \end{cases}$$

$$\Phi(h_1, \dots, h_n) = \sum_{i,j=1}^{n} \frac{\partial^2 F}{\partial x_i \partial x_j}(A) h_i h_j$$

$$= \sum_{i=1}^{n} -\frac{\left(\sum_{k=1}^{n} \alpha_k\right)^2}{\alpha_i a^2} \cdot h_i^2$$

$$= -\frac{\left(\sum_{k=1}^{n} \alpha_k\right)^2}{a^2} \sum_{i=1}^{n} \frac{h_i^2}{\alpha_i}$$

$$\frac{\partial \varphi}{\partial x_1} h_1 + \dots + \frac{\partial \varphi}{\partial x_n} h_n = 0 \longrightarrow h_1 + \dots + h_n = 0 \quad \checkmark$$

$\Rightarrow \Phi < 0 \Rightarrow A$ је лок. макс.

③ $D = \overline{B(0,100)}$, $f(x,y,z) = x^2 + 2y^2 + 3z^2$, $f: \mathbb{R}^3 \to \mathbb{R}$

$f(D) = ?$

$\overline{D} = \overline{B(0,100)} \to$ затворен + огран у $\mathbb{R}^3$ $\to$ компакт
$\to$ повезан

$f(D) \to$ компакт + повезан у $\mathbb{R}$ $\Rightarrow f(D) = [a,b]$

$D = \{ (x,y,z): \quad x^2+y^2+z^2 \le 100^2 \}$

1° $B(0,100)$:

стац. тачке $f: \mathbb{R}^3 \to \mathbb{R}$:

$$\frac{\partial f}{\partial x} = 2x = 0$$
$$\frac{\partial f}{\partial y} = 4y = 0 \quad \Rightarrow \quad x=0,\ y=0,\ z=0$$
$$\frac{\partial f}{\partial z} = 6z = 0$$

$\Rightarrow$ имамо једну стац. $(0,0,0) \in B(0,100)$

$f(x,y,z) = x^2 + 2y^2 + 3z^2 > f(0,0,0)$, $(x,y,z) \neq (0,0,0)$

$(0,0,0) \to$ лок. мин.

2° $\partial B(0,100) \longrightarrow \varphi(x,y,z) = x^2+y^2+z^2 - 100^2 = 0$

$F(x,y,z,\lambda) = x^2+2y^2+3z^2 + \lambda(x^2+y^2+z^2-100^2)$

$$\frac{\partial F}{\partial x} = 2x + 2\lambda x = 0 \xrightarrow{2x(1+\lambda)=0} x=0 \quad \vee \quad \lambda=-1$$
$$\frac{\partial F}{\partial y} = 4y + 2\lambda y = 0 \xrightarrow{2y(2+\lambda)} y=0 \quad \vee \quad \lambda=-2$$
$$\frac{\partial F}{\partial z} = 6z + 2\lambda z = 0 \longrightarrow z=0 \quad \vee \quad \lambda=-3$$
$$\frac{\partial F}{\partial \lambda} = x^2+y^2+z^2-100 = 0$$

$x=0, y=0, z=0$ не важи у исто време јер онда
$x^2+y^2+z^2-100^2 \neq 0$

1° $\lambda=-1 \Rightarrow y=0, z=0 \Rightarrow x^2+0^2+0^2=100^2 \Rightarrow x^2=100^2$

$A_\pm(\pm 100, 0, 0)$

2° $\lambda=-2 \Rightarrow x=0, z=0 \Rightarrow 0^2+y^2+0^2=100^2 \Rightarrow y=\pm 100$

$B_\pm(0, \pm 100, 0)$

3° $\lambda=-3 \Rightarrow x=0, y=0 \Rightarrow C_\pm(0,0,\pm 100)$

$f(A_\pm)=100^2+0^2+0^2=100^2$

$f(B_\pm)=0^2+2\cdot 100^2+0^2=2\cdot 100^2$

$f(C_\pm)=0^2+0^2+3\cdot 100^2=\underline{3\cdot 100^2}$ → макс. међу стац.

$\Rightarrow f(D)=[0, 3\cdot 100^2]$

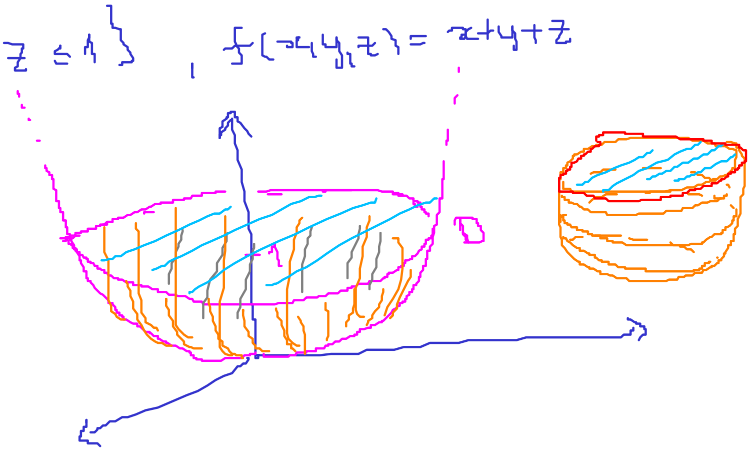
(4) $D=\{(x,y,z): x^2+y^2 \le z \le 1\}$, $f(x,y,z)=x+y+z$

$f(D)=?$

D – компакт + повезан

$\Downarrow$

$f(D)=[a,b]$



1° $\underline{\mathrm{int}D}$

$$\left.\begin{array}{l}\frac{\partial f}{\partial x}=1\\ \frac{\partial f}{\partial y}=1\\ \frac{\partial f}{\partial x}=1\end{array}\right\} \Rightarrow f \text{ нема лок екст на } \mathbb{R}^3$$

2° $\partial D=\{(x,y,z): x^2+y^2=z, 0\le z\le 1 \lor x^2+y^2\le 1, z=1\}$

$C=\{(x,y,z): x^2+y^2=1, z=1\}$

$=P_1\cup P_2\cup C$, $P_1=\{(x,y,z): x^2+y^2=z, 0<z<1\}$

$P_2=\{(x,y,z): x^2+y^2<1, z=1\}$

2.1° C :

$$f(x,y,z) = x+y+z$$

$$\varphi_1(x,y,z) = x^2+y^2-1=0, \quad \varphi_2(x,y,z) = z-1=0$$

$$\hookrightarrow z=1 \qquad f(x,y) = x+y+1$$

$$\varphi_1(x,y) = x^2+y^2-1=0$$

$$F(x,y,\lambda) = x+y+1 + \lambda(x^2+y^2-1)$$

$$\frac{\partial F}{\partial x} = 1+2\lambda x = 0 \rightarrow x,\lambda,y \neq 0 \qquad x = -\frac{1}{2\lambda}$$

$$y = -\frac{1}{2\lambda}$$

$$\frac{\partial F}{\partial y} = 1+2\lambda y = 0$$

$$\frac{\partial F}{\partial \lambda} = x^2+y^2-1 = 0 \qquad \frac{2}{4\lambda^2} = 1$$

$$\lambda = \pm\frac{1}{\sqrt{2}}$$

$$x = \mp\frac{1}{\sqrt{2}}$$

$$y = \mp\frac{1}{\sqrt{2}} \qquad f(x,y,z) = x+y+z$$

$$A_\pm\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 1\right) \longrightarrow f(A_-) = -\frac{2}{\sqrt{2}}+1 = -\sqrt{2}+1$$

$$\longrightarrow f(A_+) = \frac{2}{\sqrt{2}}+1 = \sqrt{2}+1$$

2.2° P_1 : $\varphi(x,y,z) = x^2+y^2-z=0$

$$0 < z < 1$$

$$f(x,y,z) = x+y+z$$

$$F(x,y,z,\lambda) = x+y+z + \lambda(x^2+y^2-z)$$

$$\frac{\partial F}{\partial x} = 1+2\lambda x = 0 \longrightarrow x = -1/2$$

$$y = -1/2$$

$$\frac{\partial F}{\partial y} = 1+2\lambda y = 0$$

$$\frac{\partial F}{\partial z} = 1-\lambda = 0 \longrightarrow \lambda = 1$$

$$\frac{\partial F}{\partial \lambda} = x^2+y^2-z = 0 \Longrightarrow z = x^2+y^2 = 1/2 < 1$$

$$f(-1/2,-1/2,1/2)_{\in P_1} = -1/2 - 1/2 + 1/2 = -1/2 < f(A_-) = -\sqrt{2}+1$$

2.3° P_2: $z = 1,\ x^2+y^2<1$

$f(x,y,z) = x+y+z$

$g(x,y) = f(x,y,1) = x+y+1$

$$\left.\begin{aligned} \frac{\partial g}{\partial x} &= 1 \\ \frac{\partial g}{\partial y} &= 1 \end{aligned}\right\} \text{ нема лок. екстр.}$$

$$\Rightarrow\ f(D) = \left[-\tfrac{1}{2},\ \sqrt{2}+1\right]$$