(Т) о инверзној фји

$A \subseteq \mathbb{R}^m$ отв., $a \in A$, $f: A \to \mathbb{R}^m$, $f \in C^1$, $\det df(a) \neq 0$

$\Rightarrow \exists\, \underbrace{U \ni a,\ V \ni f(a)}_{\text{околине}}$, $f: U \to V$ бијекција

$f^{-1}: V \to U$ и $f^{-1} \in C^1$

$df^{-1}(\underset{f(x)}{\underset{\shortparallel}{y}}) = [df(x)]^{-1}$.

Γ $f, f^{-1} \in C^1 \Rightarrow f$ дифеоморфизам

$f: A \to B$ $\in C^1$
бијекција $f^{-1} \in C^1$

$f, f^{-1} \in C \to f$ хомеоморфизам

f је лок. дифеоморф ако $\forall a \in A$ $\exists U \ni a$ околина $f: U \to f(U)$ бијекција $\in C^1$
$f^{-1}: f(U) \to U$ $\in C^1$.

$\llcorner$

(1) Доказати да пресл. $f: \mathbb{R}^3 \to \mathbb{R}^3$ деф са $f(x,y,z) = (\underbrace{x-y-z}_{s}, \underbrace{x^2+y^2+z^2}_{t}, \underbrace{xyz}_{u})$
задаје глатку промену коорд. у околини $(1,1,0)$.
Одредити $\frac{\partial y}{\partial s}(0,2,0)$ ако су нове коорд. (s,t,u).

$f^{-1}(s,t,u) = (x,y,z)$

? $\exists f^{-1}$ у околини $(1,1,0)$?

$$df(x,y,z) = \begin{bmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} & \frac{\partial s}{\partial z} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 2x & 2y & 2z \\ yz & xz & xy \end{bmatrix}$$

$$df(1,1,0) = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \text{да ли је инвертибилно?}$$

$\det df(1,1,0) = 1 \cdot \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} = 4 \neq 0$

$\Rightarrow \exists U \ni (1,1,0)$ и $V \ni f(1,1,0) = (0,2,0)$

$f^{-1}: V \to U$ $\qquad f \circ f^{-1} = Id$, $f^{-1} \in C^1$ $\qquad f^{-1}(s,t,u) = (x,y,z)$

$\Rightarrow$ ова промена коорд. је глатка

$\frac{\partial y}{\partial s}(0,2,0) = ?$ $\qquad df^{-1}(0,2,0) = [df(1,1,0)]^{-1}$

$$B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -1 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$$

$$\left.\begin{aligned} a_{11} + 2a_{12} &= 1 \\ -a_{11} + 2a_{12} &= 0 \end{aligned}\right\} \quad a_{12} = 1/4, \quad a_{11} = 1/2$$

$$-a_{11} + a_{13} = 0 \quad \to \quad a_{13} = 1/2$$

$$a_{21} + 2a_{22} = 0$$

$$-a_{21} + 2a_{22} = 1 \quad \to \quad a_{22} = 1/4 \qquad a_{21} = -1/2$$

$$-a_{21} + a_{23} = 0 \qquad\qquad a_{23} = -1/2$$

$$a_{31} + 2a_{32} = 0$$

$$-a_{31} + 2a_{32} = 0 \quad \to \quad a_{31} = a_{32} = 0$$

$$-a_{31} + a_{33} = 1 \qquad\qquad a_{33} = 1$$

$$dF^{-1}(0,2,0) = B = \begin{bmatrix} 1/2 & 1/4 & 1/2 \\ \boxed{-1/2} & 1/4 & -1/2 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{aligned} &\to \frac{\partial y}{\partial s}(0,2,0) = -1/2 \\ &\to \frac{\partial x}{\partial u} = 1/2 \\ &\frac{\partial z}{\partial t} = 0 \end{aligned}$$

$\frac{\partial y}{\partial s}$

(2) $f \in C^2$, $f : \mathbb{R}^n \to \mathbb{R}$.
Стац. тачка $x \in \mathbb{R}^n$ је недегенерисана ако је $\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right]_{i,j=1}^{n}$ инвертиб.
Стац. тачка изолована од осталих стац. тачака.

Доказати да је свака недеген. стац. изолована од осталих стац. тачака.

$F : \mathbb{R}^n \to \mathbb{R}^n \qquad F(x) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$

$a \in \mathbb{R}^n$ стац. недеген. тачка фје $f \Rightarrow F(a) = (0, 0, \dots, 0) = 0$

$f \in C^2 \Rightarrow F \in C^1$, $[dF(a)] = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(a) \right]$ инв.

$\det dF(a) \neq 0$

(T) $\Rightarrow \exists \underbrace{U}_{\text{отворена (отв. скуп)}} \ni a$, $V \ni F(a) = 0$ $\qquad F : U \to V$ инв.

$F^{-1} : V \to U$, $F^{-1} \in C^1$

$F^{-1}(0) = a$

$b \in U \setminus \{a\} \Rightarrow F(b) \neq 0 \Rightarrow df(b) \neq 0 \Rightarrow b$ није стац.

$\boxed{3}$ (MHB) $n \in \mathbb{N}$ $\quad (M_{n\times n}(\mathbb{R}), \|\cdot\|_2) \cong (\mathbb{R}^{n^2}, \|\cdot\|_2)$

$d_2 \to$ Еуклидска метрика

а) $f(X) = X^2 + X$, $X \in M_{n\times n}$

Доказати да је $f \in C^\infty$ и одредити $df(X)$

б) Доказати да $\exists \varepsilon > 0$ $\forall A \in M_{n\times n}$ $\|A\|_2 < \varepsilon$ јна $X^2 + X = A$ има решења у $M_{n\times n}$.

$df(X) \to$ линеарно пресликавање $\quad df(X): M_{n\times n} \to M_{n\times n}$

$df(X)H \to$ линеаран део прираштаја

$$f(X+H) - f(X) = df(X)H + \underbrace{o(H)}_{\text{мало у односу на } H}$$

$$f(X+H) - f(X) = \underset{\substack{\| \\ (X+H)(X+H)}}{(X+H)^2} + (X+H) - X^2 - X$$

$$= \cancel{X^2} + HX + XH + H^2 + \cancel{X} + H - \cancel{X^2} - \cancel{X} =$$

$$= \underbrace{HX + XH + H}_{\text{линеарно по } H} + \underbrace{H^2}_{\in o(H)},\ H \to 0$$

$df(X)H = HX + XH + H \to$ непр? ✓ јер је линеарно по X

II начин:

$$X \sim (\underbrace{x_{11}, x_{12}, \dots, x_{1n}, \dots, x_{n1}, \dots, x_{nn}}_{n^2})$$

$$f(X) = X^2 + X = \left[\sum_{k=1}^{n} x_{ik}x_{kj}\right] + [x_{ij}]_{i,j=1}^{n} = \Big[\underbrace{\sum_{k=1} x_{ik}x_{kj} + x_{ij}}_{f_{ij}}\Big]_{i,j=1}^{n}$$

$$\frac{\partial f_{ij}}{\partial x_{ml}} = \begin{cases} \underset{\substack{\| \\ m}}{x_{ii}} + \underset{\substack{\| \\ l}}{x_{jj}} + 1, & i = m, l = j \\ 0, & i \neq m, l \neq j \\ x_{lj}, & i = m, l \neq j \\ x_{im}, & i \neq m, l = j \end{cases} \qquad \delta_x(y) = \begin{cases} 1, & y = x \\ 0, & y \neq x \end{cases}$$

$$x_{lj}\,\delta_i(m) + x_{im}\,\delta_j(l) + \delta_i(m)\delta_j(l)$$

$f: \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$

$$df = \underbrace{\left[x_{lj}\delta_i(m) + x_{im}\delta_j(l) + \delta_i(m)\delta_j(\dots)\right]_{(i,j)(m,l)=(1,1)}^{(n,n)}}_{n^2\times n^2}$$

$$df\,\underset{\substack{\| \\ (h_{11}, h_{12}, \dots, h_{1n}, h_{21}, \dots, h_{n1}, \dots, h_{nn})}}{H} = \left[\left(\qquad\right)\right]_{n^2\times n^2} [h_{a,b}]_{a,b=1}^{n} = \dots$$

б) ? $\exists \varepsilon>0 \quad \|A\|_2 < \varepsilon \iff A \in B(\mathbb{O}, \varepsilon)$?

$[0]^n_{ij=1}$

$X^2 + X = A$ има решење ?

$X = \mathbb{O} = [0]^n_{ij} \qquad f(\mathbb{O}) = \mathbb{O}^2 + \mathbb{O} = \mathbb{O} + \mathbb{O} = \mathbb{O} \in B(\mathbb{O}, \varepsilon)$

$df(X)H = HX + XH + H$

$df(\mathbb{O})H = H\cdot\mathbb{O} + \mathbb{O}\cdot H + H = H$

$df(\mathbb{O}) = Id \longrightarrow$ инвертибилно $\qquad df(\mathbb{O})^{-1} = Id^{-1} = Id$

$df \in C^1$

T $\Rightarrow \exists\, U \ni \mathbb{O},\ V \ni f(\mathbb{O}) = \mathbb{O}$ (околина) $\qquad f^{-1}: V \to U \qquad f^{-1}(\mathbb{O}) = \mathbb{O}$

$f^{-1} \in C^1$

$\Rightarrow V$ околина $\exists \varepsilon > 0,\ B(\mathbb{O}, \varepsilon) \subseteq V$

$\forall A \in B(\mathbb{O}, \varepsilon) \qquad X^2 + X = A$ има јединств. решење у U

$f(X) = A$

$X = f^{-1}(A) \in M_{n\times n}$

$M = \{ a : \mathbb{N} \to \{0, 1\} \}$

$d : M \times M \to \mathbb{R}$

$$d(a,b) = \begin{cases} 0 & a = b \\ \dfrac{1}{\min\{k \in \mathbb{N} : a(k) \neq b(k)\}} & a \neq b \end{cases}$$

а) Док. (M, d) м.п.

б) Испитати непр. $f : (M, d) \to (\mathbb{R}, |\cdot|)$

$$f(a) = \sum \frac{2a(k)}{3^k}$$

в) Да ли је (M, d) компактан м.п. ?

a) d метрика:

1) $d(a,b) \geq 0$ ✓

$d(a,b) = 0 \iff a = b$

јер се min достиже

2) $d(a,b) = d(b,a)$ ✓

3) неједнакост троугла:

$d(a,b) \overset{?}{\leq} d(a,c) + d(c,b)$

$a \neq b \neq c \neq a$

$$\frac{1}{\min\{k : a(k) \neq b(k)\}} = d(a,b) = \frac{1}{k_1}$$

$a, b : \mathbb{N} \to \{0,1\}$

$\min\{k : a(k) = b(k)\} = k_1$

$k < k_1 \to a(k) = b(k)$

$k = k_1 \quad a(k_1) \neq b(k_1)$

$$\frac{1}{\min\{c(k) \neq a(k)\}} = \frac{1}{k_2}$$

$k_2 = \min\{k : a(k) \neq c(k)\}$

$k_3 = \min\{k : c(k) \neq b(k)\}$

$d(c,b) = \frac{1}{k_3}$

$k_1 = k_2 = k_3 \to$ то не може да се деси

јер

$c(k_1) \in \{0,1\}$, $\{a(k_1), b(k_1)\} = \{0,1\}$

$$\underline{c(k) = b(k)}, \quad k < k_3$$

$$a(k) = b(k), \quad k < k_1$$

$$a(k) = c(k), \quad k < k_2$$

$$a(k) = b(k) \qquad k < \min\{k_2, k_3\}$$

$$k < k_1$$

$$\Rightarrow \quad k_1 \geq \min\{k_2, k_3\}$$

$$k_2 < k_3 \quad \Rightarrow \quad c(k) = b(k) \quad k < k_3, \quad k \leq k_2$$

$$c(k_3) \neq b(k_3)$$

$$c(k) = a(k), \quad k < k_2$$

$$c(k_2) \neq a(k_2)$$

$$k_1 < k_3 \qquad \Rightarrow \quad k_1 = \min\{k_2, k_3\}$$

$$k_2 = k_3 \qquad \Rightarrow \quad k_1 \geq \min\{k_2, k_3\}$$

$$\frac{1}{k_1} \leq \frac{1}{\min\{k_2, k_3\}} \leq \frac{1}{k_2} + \frac{1}{k_3}$$

$$f : (M, d) \to (\mathbb{R}, |\cdot|)$$

$$f(a) = \underbrace{\sum_{k=1}^{\infty} \frac{a(k)^{\in\{0,1\}}}{3^k}}_{\text{konv.}} \leq 2 \underbrace{\sum_{k=1}^{\infty} \frac{1}{3^k}}_{\text{konv.}}$$

f непр.:

$\forall \varepsilon > 0 \ \exists \delta > 0 \quad d(a,b) < \delta \Rightarrow d(f(a), f(b)) < \varepsilon$

$$\overset{|-|}{d(f(a),f(b))} = |f(a) - f(b)|$$

$|f(a) - f(b)| < \varepsilon$

$$2 \left| \sum_{k=1}^{\infty} \frac{a(k)}{3^k} - \sum_{k=1}^{\infty} \frac{b(k)}{3^k} \right| < \varepsilon$$

$$\underbrace{2 \left| \sum_{k=1}^{\infty} \frac{a(k) - b(k)}{3^k} \right| \leq 2 \sum_{k=1}^{\infty} \frac{|a(k) - b(k)|}{3^k}} = (*)$$

$$d(a,b) = \frac{1}{k_1} \Rightarrow \min\{k : a(k) \neq b(k)\} = k_1$$

$k < k_1 \quad a(k) = b(k)$

$a(k_1) \neq b(k_1)$

$\downarrow$

$|a(k_1) - b(k_1)| = 1$

$k < k_1, \ |a(k) - b(k)| = 0$

$$(*) = 2 \sum_{k=k_1}^{\infty} \frac{|a(k) - b(k)|}{3^k} \leq 2 \sum_{k=k_1}^{\infty} \frac{1}{3^k} =$$

$$|a(k) - b(k)| = \begin{cases} 1, & a(k) \neq b(k) \\ 0 & a(k) = b(k) \end{cases}$$

$$= \frac{2}{3^{k_1}} \sum_{k=0}^{\infty} \frac{1}{3^k} = \frac{2}{3^{k_1}} \cdot \frac{1}{1 - 1/3} = \frac{3}{3^{k_1}} = \frac{1}{3^{k_1 - 1}} < \varepsilon$$

$\downarrow$

$k_1 - 1 > -\log_3 \varepsilon$

$k_1 > -\log_3 \varepsilon + 1$

$$\Rightarrow \delta < \frac{1}{-\log_3 \varepsilon + 1}$$

✓

$\Rightarrow f$ непр.

б) (M,d) повезан ?

(M,d) није повезан јер:

$$g : (M,d) \to (\mathbb{R}, |\cdot|)$$

$$g(a) = a(1)$$

g непр:

$$d(a,b) < \delta \Rightarrow |a(1) - b(1)| < \varepsilon$$

$$|a(1) - b(1)| = \begin{cases} 0, & a(1) = b(1) \\ 1, & a(1) \neq b(1) \end{cases}$$

$$= \begin{cases} 0, & d(a,b) < \dfrac{1}{\min\{k,\ a(k) \neq b(k)\}} \le 1 \\ 1, & d(a,b) = 1 \end{cases}$$

$$\delta = \frac{1}{2}$$

$\Rightarrow g$ непр. $\quad g(M) = \{0,1\}$ и НА

$$g(0) = 0$$

$$g(1) = 1$$