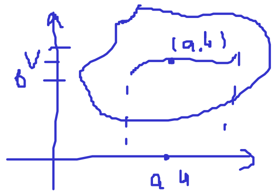
(T) $A \subseteq \mathbb{R}^2$ отв., $(a,b)\in A$, $F: A\to \mathbb{R}$ непр. $F(a,b)=0$



$\frac{\partial F}{\partial y}(a,b) \neq 0$, $\frac{\partial F}{\partial y} \in C(A)$

$\Rightarrow \exists\, U \ni a,\ V \ni b$, $f: U \to V$ непр. $f(a)=b$, $F(x, f(x)) = 0$

ако још $\frac{\partial F}{\partial x} \in C(A) \Rightarrow f'(x) = -\dfrac{\frac{\partial F}{\partial x}(x,f(x))}{\frac{\partial F}{\partial y}(x,f(x))}$, $x \in U$.

(T) $A \subseteq \mathbb{R}^{m+1}$ отв., $(\underset{\substack{\cap\\ \mathbb{R}^m}}{\overset{(a_1,\dots,a_m)}{a}}, \overset{\in\mathbb{R}}{b}) \in A$, $F: A \to \mathbb{R}$ непр, $F(a,b)=0$, $F(\underset{(x_1,\dots,x_m)}{x}, y)$

$\frac{\partial F}{\partial y}(a,b) \neq 0$, $\frac{\partial F}{\partial y} \in C(A)$

$\Rightarrow \exists\, \underset{\substack{\cap\\ \mathbb{R}^m}}{U} \ni a$, $\underset{\substack{\cap\\ \mathbb{R}}}{V} \ni b$, $f: U \to V$, $f(a)=b$ и $F(\underset{(x_1,\dots,x_m)}{x}, f(x)) = 0$, $x \in U$

$\frac{\partial F}{\partial x_k} \in C(A) \Rightarrow \frac{\partial f}{\partial x_k}(x) = -\dfrac{\frac{\partial F}{\partial x_k}(x,f(x))}{\frac{\partial F}{\partial y}(x,f(x))}$, $x \in U$

(1) Доказати да ј-на $\cos(xyz) - z^2 + 2z = \cos(xy) + e^y - e$ задаје имплиц. ф-ју $z = z(x,y)$ у околини $(0,1,2)$. Одр. $\frac{\partial z}{\partial x}(0,1)$, $\frac{\partial z}{\partial y}(0,1)$, $\frac{\partial^2 z}{\partial x^2}(0,1)$.

$F(x,y,z) = \cos(xyz) - z^2 + 2z - \cos(xy) - e^y + e$

$F(0,1,2) = \cos 0 - 2^2 + 4 - \cos 0 - e^1 + e = 0$

$F: \mathbb{R}^3 \to \mathbb{R}$

$\frac{\partial F}{\partial z}(x,y,z) = -\sin(xyz)\cdot(xy) - 2z + 2 \in C(\mathbb{R}^3)$

$\frac{\partial F}{\partial z}(0,1,2) = 0 - 4 + 2 = -2 \neq 0$

(T) $\Rightarrow \exists\, U \ni (0,1)$, $V \ni 2$, $f: U \to V$, $f(0,1) = 2$ и $F(x,y,\overset{z(x,y)}{f(x,y)}) = 0$
↳ непр.

$\frac{\partial F}{\partial x}(x,y,z) = -yz\sin(xyz) + y\sin(xy) \in C(\mathbb{R}^3)$

$\frac{\partial f}{\partial x}^{=z}(0,1) = -\dfrac{\frac{\partial F}{\partial x}(\overset{0}{x}, \overset{1}{y}, \overset{2}{f(x,y)})}{\frac{\partial F}{\partial z}(0,1,2)} = \frac{1}{2}\cdot 0 = 0$

$\frac{\partial f}{\partial y}^{=z}(0,1) = -\dfrac{\partial F/\partial y\,(0,1,2)}{\frac{\partial F}{\partial z}(0,1,2)} = \dots = -\dfrac{e}{-2} = \frac{1}{2}e$

$\frac{\partial F}{\partial y}(x,y,z) = \dots$

$\frac{\partial^2 z}{\partial x^2} = \left(\frac{\partial z}{\partial x}\right)'_x = \left(-\dfrac{-yz\sin(xy\overset{z(x,y)}{z}) + y\sin(xy)}{-xy\sin(xy\underset{z(x,y)}{z}) - 2\underset{z(x,y)}{z} + 2}\right)_x = \dots \to$ за вежбу

② Одредити тотални диф ф-је $x(y,z)$ задате ј-ном

$$e^x y + xz = y^2\cos z.$$

$dx(h) = \frac{\partial x}{\partial y} h_1 + \frac{\partial x}{\partial z} h_2$, $h = (h_1,h_2)$ → диференцијал у x од h

$dx = \frac{\partial x}{\partial y} dy + \frac{\partial x}{\partial z} dz$, $x = x(y,z)$ → тотални диференцијал

$F(x,y,z) = e^x y + xz - y^2\cos z \in C(\mathbb{R}^3)$, $F:\mathbb{R}^3 \to \mathbb{R}$

$\mathbb{R}^3 \ni (x_0,y_0,z_0)$: $F(x_0,y_0,z_0)=0$

$\frac{\partial F}{\partial x}(x_0,y_0,z_0) = e^x y + z \underset{?}{\neq} 0$ $\quad z \neq -e^x y$

? $z = -e^x y$ и $F(x,y,-e^x y) = 0$?

$e^x y - e^x xy - y^2\cos(ye^x) = e^x(y - xy) - y^2\cos(-ye^x) = 0$ → за неке x и y

$\Rightarrow$ знамо да постоји импл. ф-ја $x(y,z)$ у околинама (x_0,y_0,z_0), $z_0 \neq -e^{x_0} y_0$

$\frac{\partial F}{\partial y} = e^x - 2y\cos z \in C(\mathbb{R}^3)$, $\frac{\partial F}{\partial z} = x + y^2\sin z \in C(\mathbb{R}^3)$

(T) $\Rightarrow \exists$ импл. $x: U \to V$ $\quad x = x(y,z)$ и $F(x,y,z) = 0$, $z \neq e^{-x} y$

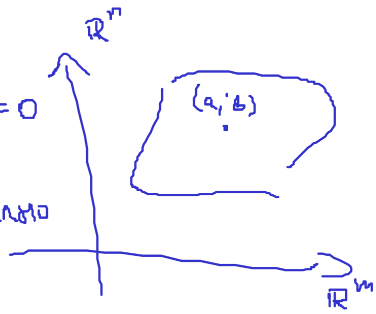
$$\frac{\partial x}{\partial y} = -\frac{\frac{\partial F}{\partial y}(x(y,z),y,z)}{\frac{\partial F}{\partial x}(x(y,z),y,z)} = \frac{2y\cos z - e^x}{e^x y + z} \quad \text{и} \quad \frac{\partial x}{\partial z} = -\frac{x + y^2\sin z}{e^x y + z}$$

$$\Rightarrow \quad dx = -\frac{\frac{\partial F}{\partial y}(x(y,z),y,z)\cdot dy + \frac{\partial F}{\partial z}(x(y,z),y,z)\,dz}{\frac{\partial F}{\partial x}(x(y,z),y,z)}$$

$\boxed{T}$ $A \subseteq \mathbb{R}^{m+n}$ отв. $\underset{\in \mathbb{R}^n}{(x,y)}$, (a,b), $a = (a_1,\dots,a_m)$, $b = (b_1,\dots,b_n)$

$F: A \to \mathbb{R}^n$, $F(a,b) = 0$, $d_yF = \left[\frac{\partial F_i}{\partial y_j}\right]_{i,j=1}^n$ $\in C(A)$ $\leftrightarrow$ $\det d_yF \neq 0$ инвертибилно

$F = F(x,y) = (F_1(x,y),\dots,F_n(x,y))$, $y = (y_1,\dots,y_n)$, $F_k: A \to \mathbb{R}$



$\Rightarrow \exists\, U \ni a$, $V \ni b$ отв., $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$, $f: U \to V$ $\quad f(a) = b$ и $F(x,f(x)) = 0$

и $\frac{\partial F}{\partial x_k} \in C(A)$, $1 \le k \le m \Rightarrow$ $\quad df(x) = -[d_yF]^{-1}_{n\times n}\cdot[d_xF]_{n\times m}$.

$$d_xF(x,y) = \left[\frac{\partial F_k}{\partial x_i}\right]_{n\times m}$$

1. Показати да систем

$$\begin{cases} x^2+y^2 = \frac{1}{2}z^2 \\ x+y+z=2 \end{cases}$$

задаје имплиц. фју $f(z) = (x(z), y(z))$ у околини $(1,-1,2)$

Одредити $x'(z), y'(z), x''(z), y''(z)$.

$$F(x,y,z) = (\underbrace{x^2+y^2-\tfrac{1}{2}z^2}_{F_1}, \underbrace{x+y+z-2}_{F_2}) \in C(\mathbb{R}^3)$$

$$F(1,-1,2) = (1+1 - \tfrac{1}{2}\cdot 4, 1-1+2-2) = (0,0)$$

$$d_{(x,y)}F = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 1 & 1 \end{bmatrix}$$

$$d_{(x,y)}F(1,-1,2) = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \longrightarrow \det(d_{(x,y)}F(1,-1,2)) = 4 \neq 0$$

$\Rightarrow d_{(x,y)}F(1,-1,2)$ инвертибилно

$\Rightarrow \exists\, U \ni 2$ и $V \ni (1,-1)$, $f: U \to V$ непр. и $f(2) = (1,-1) = (x(2), y(2))$

$f(z) = (x(z), y(z))$ и $F(f(z), z) = 0$

$x'(z), y'(z) = ?$

$$df(z) = \begin{bmatrix} x'(z) \\ y'(z) \end{bmatrix}$$

$$d_{(x,y)}F(x,y,z) = \begin{bmatrix} 2x & 2y \\ 1 & 1 \end{bmatrix} \longrightarrow \det d_{(x,y)}F(x,y,z) = 2x-2y = 2(x-y)$$

$x \neq y$

$\Vert F$

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$(d_{(x,y)}F(x,y,z))^{-1} = \frac{1}{2(x-y)}\begin{bmatrix} 1 & -2y \\ -1 & 2x \end{bmatrix}$$

$$d_z F(x,y,z) = \begin{bmatrix} \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial z} \end{bmatrix} = \begin{bmatrix} -z \\ 1 \end{bmatrix}$$

$$df = -(d_{(x,y)}F(x,y,z))^{-1} \cdot d_z F(x,y,z) = -\frac{1}{2(x-y)}\begin{bmatrix} 1 & -2y \\ -1 & 2x \end{bmatrix}\begin{bmatrix} -z \\ 1 \end{bmatrix}$$

$$= -\frac{1}{2(x-y)}\begin{bmatrix} -z-2y \\ z+2x \end{bmatrix}$$

$$\Rightarrow x'(z) = \frac{z+2y(z)}{2(x(z)-y(z))}, \quad y'(z) = -\frac{z+2x(z)}{2(x(z)-y(z))}$$

$x'(2) = \dots$ $x(2) = 1$, $y(2) = -1$, $y'(2) = \dots$

$$y''(2) = -\left(\frac{z+2x(z)}{2(x(z)-y(z))}\right)' = -\frac{(1+2x'(z))\,2(x(z)-y(z)) - 2(x'(z)-y'(z))(z+2x(z))}{4(x(z)-y(z))^2}$$

$$x'(2) = \frac{2-2}{--} = 0, \quad y'(2) = -\frac{4}{4} = -1$$

$$\Rightarrow y''(2) = -\frac{(1+0)\cdot 2 - (-1)\cdot(4)}{2\cdot 4} = -\frac{2+4}{8} = -\frac{6}{8}$$

---

(2) $z = z(x,y)$ задата са $F(x+z, y+z) = 0$, $F \in C^\infty$. Одредити dz.

$$dz = \left[\frac{\partial z}{\partial x} \quad \frac{\partial z}{\partial y}\right] = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} = ?$$

$$\begin{tikzcd}
(x+z,y+z) \in \mathbb{R}^2 \arrow[r, "F"] & \mathbb{R} \\
\mathbb{R}^3 \ni (x,y,z) \arrow[u, dashed, "G"] \arrow[ur, "H"'] &
\end{tikzcd}$$

$$F = F(u,v), \quad u = x+z, \quad v = y+z$$

$$G = (u,v)$$

$$G: \mathbb{R}^3 \to \mathbb{R}^2$$

$$G(x,y,z) = (x+z, y+z) \in C^\infty$$

$$H = F\circ G : \mathbb{R}^3 \to \mathbb{R}$$

$H(x,y,z) = 0 \to$ задаје имплиц. ф-ју $z = z(x,y)$

$$\quad = F(x+z, y+z)$$

$$H \in C^\infty \Rightarrow \frac{\partial z}{\partial x} = -\frac{\frac{\partial H}{\partial x}}{\frac{\partial H}{\partial z}} = -\frac{\frac{\partial(F\circ G)}{\partial x}}{\frac{\partial (F\circ G)}{\partial z}} = -\frac{\frac{\partial F}{\partial u}\cdot\overbrace{\frac{\partial u}{\partial x}}^{=1} + \frac{\partial F}{\partial v}\cdot\overbrace{\frac{\partial v}{\partial x}}^{=0}}{\frac{\partial F}{\partial u}\cdot\underbrace{\frac{\partial u}{\partial z}}_{=1} + \frac{\partial F}{\partial v}\cdot\underbrace{\frac{\partial v}{\partial z}}_{=1}}$$

$$= -\frac{\frac{\partial F}{\partial u}(x+z(x,y), y+z(x,y))}{\frac{\partial F}{\partial u}(---) + \frac{\partial F}{\partial v}(---)}$$

$$dF \cdot dG \qquad u = x+z, \quad v = y+z$$

$$dF(u,v) = \left[\frac{\partial F}{\partial u} \quad \frac{\partial F}{\partial v}\right]$$

$$\underset{(u,v)}{dG(x,y,z)} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{bmatrix}$$

$$\left[\frac{\partial F}{\partial u} \quad \frac{\partial F}{\partial v}\right]\begin{bmatrix} \frac{\partial u}{\partial x} & -- \\ \frac{\partial v}{\partial x} & -- \end{bmatrix} = \left[\underbrace{\frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x}}_{\frac{\partial(F\circ G)}{\partial x}} \; : \; \frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial y} \; : \; ---\right]$$

$$\frac{\partial z}{\partial y} = \cdots$$

(3) Одредити лок. екстр. имплиц. ф-је $z = z(x,y)$ задате $x^2+y^2+e^z-2x+2y-4z = 10$

$$f(x,y,z) = x^2+y^2+e^z-2x+2y-4z-10 = 0$$

$F \in C(\mathbb{R}^3)$ $\qquad\qquad\qquad\qquad (x_0, y_0, z_0)$

$$\frac{\partial F}{\partial z}(x,y,z) = e^z - 4 \neq 0 \longrightarrow \underline{\underline{z_0 \neq \ln 4}}$$

$$\Rightarrow \exists \underset{(x_0,y_0)}{\underset{\cup\!\!\!\!\,\shortmid}{U}}, V \ni z_0 \qquad \exists\, z(x,y) : U \to V$$

$$z(x_0, y_0) = z_0$$

$$\frac{\partial F}{\partial x} = 2x - 2 \;,\quad \frac{\partial F}{\partial y} = 2y + 2 \in C(\mathbb{R}^3)$$

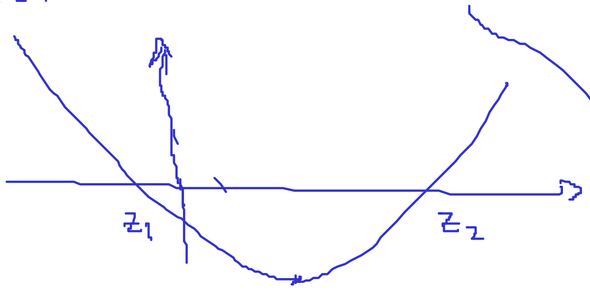
$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{2x-2}{e^{z(x,y)}-4} \;,\quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{2y+2}{e^{z(x,y)}-4}$$

$(x_1, y_1) \to$ лок. екст. за z

$$\Rightarrow \underline{2x_1 - 2 = 0} \qquad \frac{2y_1+2}{e^{z(x,y)}-4} = 0 \qquad \Rightarrow x_1 = 1 \,,\, y_1 = -1$$

$$F(1,-1,z) = 0 = \cancel{1}+\cancel{1}+e^z \cancel{-2} - 2 - 4z - 10 = 0$$

$$g(z) = e^z - 4z - 12 = 0 \longrightarrow g'(z) = e^z - 4$$



$\underline{g(\ln 4) = -12} < 0$

$z \to -\infty \Rightarrow g(z) \to +\infty$

$z \to +\infty \Rightarrow g(z) \to +\infty$

$\Downarrow$

има два решења $z_1, z_2 \neq \ln 4$

(x_1, y_1) био локални екст. локалне имп. ф-је $z(x,y)$

онда z мора бити имп. у околине (x_1, y_1, z_1)

или (x_1, y_1, z_2)

... испитати за вежбу друге изводе ...