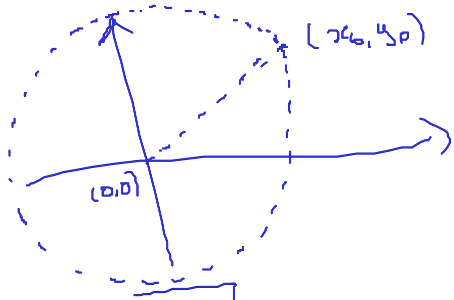
① $f(x,y) = (x^2+y^2)e^{-x^2-y^2}$

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \quad \begin{array}{l} \mathbb{R}^2\setminus\{(0,0)\} \\ r>0 \end{array}$$



$g(r) = f(r,\theta) = r^2 e^{-r^2}$

$g'(r) = 2r e^{-r^2} - 2r^3 e^{-r^2} = 0$

$= 2r(1-r^2)e^{-r^2} = 0 \iff r=0 \vee \boxed{r=1} \vee r=-1$

$$\begin{array}{c|ccccc} g' & 0 & + & 1 & - & \infty \\ g & & \nearrow & & \searrow & \end{array}$$

$\Rightarrow r=1 \Rightarrow$ локални максимум за g

$\Rightarrow S(0,1)$ - скуп лок. максимума за f

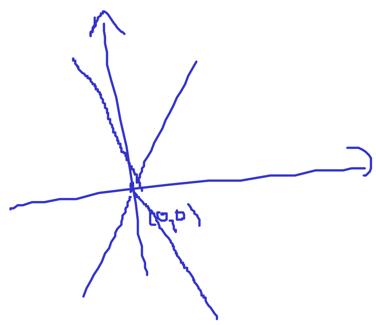
$f(0,0) = (0^2+0^2)e^{-0} = 0$ $\Rightarrow (0,0)$ строг лок. минимум

$\underset{(x,y)\neq(0,0)}{f(x,y)} = (x^2+y^2)e^{-x^2-y^2} > 0 = f(0,0)$

② $f(x,y) = (x-y^2)(2x-y^2)$

$(0,0)$ дуж сваке праве лок екст?

$(0,0)$ на $\mathbb{R}^2$?



$y = kx$, $x=0$

$f(x,kx) = (x-k^2x^2)(2x-k^2x^2)$

$= x^2(1-k^2x)(2-k^2x)$

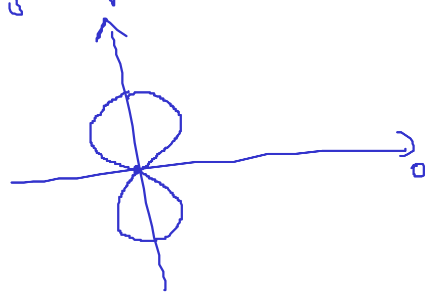
$f(0,0) = 0$, $|x| < \frac{1}{k^2}$ $\Rightarrow f(x,kx) = x^2(1-k^2x)(2-k^2x) > 0 = f(0,0)$

$\Rightarrow (0,0)$ лок. мин $f(x,kx)$ на $y=kx$

$x=0$ $f(0,y) = -y^2\cdot(-y^2) = y^4 > 0$

$\Rightarrow (0,0)$ лок. мин.

$(0,0)$ лок. мин за f ? $\Rightarrow$ на основу претходног решење мислимо ....



$f(x,y) = (x-2y^2)(x-y^2)$

$(0,\varepsilon) \Rightarrow f(0,\varepsilon) = \varepsilon^4 > 0$

$y^2 < x < 2y^2 \rightarrow \varepsilon^2 < x < 2\varepsilon^2$

$f(\frac{3}{2}\varepsilon^2, \varepsilon) = -\frac{\varepsilon^2}{2}\cdot\frac{\varepsilon^2}{2} = -\frac{\varepsilon^4}{4} < 0$, $x = \frac{3}{2}\varepsilon^2$

г) $(0,0)$ није лок. екст за $\mathbb{R}^2$

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За вежбу одр. слику домена и лок. екст. ф-ја

$f(x,y) =$

а) $xy\sqrt{4-x^2-y^2}$

б) $x^4+y^4-x^2-2xy-y^2$

в) $2x^4+y^4-x^2-2y^2$

г) $x^2y^3(6-x-y)$.

Тејлоров развој

$f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$ (отворен), $a \in A$, $f \in C^m(A)$

$$df(a)h = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(a)\, h_k = \left(\sum_{k=1}^{n} h_k \frac{\partial}{\partial x_k}\right)(f)(a)$$

$$d^2 f(a) h = \sum_{i,j=1}^{n} \frac{\partial^2 f(a)}{\partial x_i \partial x_j} h_i h_j = \left(\sum_{k=1}^{n} h_k \frac{\partial}{\partial x_k}\right)^2 f(a)$$

$$d^l f(a) h = \left(\sum_{k=1}^{n} h_k \frac{\partial}{\partial x_k}\right)^l f(a) = \sum_{i_1,\dots,i_l=1}^{n} \frac{\partial^l f(a)}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_l}} h_{i_1} h_{i_2}\cdots h_{i_l}$$

$$\underset{f \in C^l}{=} \boxed{\sum_{\substack{j_1,\dots,j_n=0 \\ j_1+\dots+j_n=l}} \frac{l!}{j_1!\cdots j_n!} \frac{\partial^l f(a)}{\partial x_1^{j_1}\partial x_2^{j_2}\cdots\partial x_n^{j_n}} h_1^{j_1}\cdots h_n^{j_n}}$$

$h \to$ довољно мало

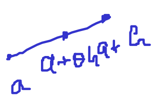
$$f(a+h)-f(a) = \sum_{k=1}^{m} \frac{1}{k!} d^k f(a)(h) + R_m(a,h) \to \text{Тејлоров развој } f \text{ у } a$$

$R_m(a,h) \sim o(\|h\|^m) \to$ Пеанов остатак

ф-ја од h

$$\frac{o(\|h\|^m)}{\|h\|^m} \underset{h\to 0}{\Longrightarrow} 0$$

Лагранжев остатак: $R_m(a,h) = \dfrac{1}{(m+1)!} d^{(m+1)} f(a+\theta h)$, $\theta \in (0,1)$



① $f(x,y) = e^x \sin y$ у околини $(0,0)$.

$$\frac{\partial f}{\partial x}(x,y) = e^x \sin y = f(x,y) \to \frac{\partial^k f}{\partial x^k} = f$$

$$\frac{\partial^k f}{\partial x^l \partial y^{k-l}}(x,y) = \frac{\partial^{k-l} f}{\partial y^{k-l}}(x,y)$$

$$\frac{\partial f}{\partial y}(x,y) = e^x \cos y, \quad \frac{\partial^2 f}{\partial y^2} = -e^x \sin y, \quad \frac{\partial^3 f}{\partial y^3} = -e^x \cos y, \quad \frac{\partial^4 f}{\partial y^4} = e^x \sin y$$

$$\frac{\partial^k f}{\partial y^k}(0,0) = \begin{cases} 0, & k=2n \\ (-1)^n, & k=2n+1 \end{cases}, \quad \frac{\partial f(0,0)}{\partial x} = f(0,0) = 0 = \frac{\partial^k f}{\partial x^k}(0,0)$$

$$\frac{\partial^k f}{\partial x^{k-l}\partial y^l} = \begin{cases} 0, & l=2n \\ (-1)^l, & l=2n+1 \end{cases}$$

$$\Rightarrow \quad f(a+h)-f(a) = \sum_{k=1}^{m} \frac{1}{k!} d^k f(0,0)h, \quad m\in\mathbb{N}$$

$$d^k f(0,0)h = \sum_{i=0}^{k} \binom{k}{i} \frac{\partial^k f(0,0)}{\partial x^{k-i}\partial y^i} h_1^{k-i} h_2^i = \sum_{i=0}^{k} \binom{k}{i} \frac{\partial^i f(0,0)}{\partial y^i} h_1^{k-i} h_2^i$$

$$\underset{i=2n+1}{=} \sum_{n=0}^{\left[\frac{k-1}{2}\right]} \binom{k}{2n+1} (-1)^n h_1^{k-2n-1} h_2^{2n+1}$$

$i=2n \Rightarrow \frac{\partial^i f(0,0)}{\partial y^i} = 0$

$a=(0,0)$, $f(0,0)=0$

$$f(h) = \sum_{k=1}^{m} \frac{1}{k!} \sum_{n=0}^{\left[\frac{k-1}{2}\right]} \binom{k}{2n+1} (-1)^n h_1^{k-2n-1} h_2^{2n+1} + R_m(h)$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \sin y = \sum_{l=0}^{\infty} \frac{(-1)^n y^{2l+1}}{(2l+1)!}$$

$$f(x,y) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{l=0}^{\infty} \frac{(-1)^h y^{2l+1}}{(2l+1)!} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^l x^k y^{2l+1}}{k!\cdot(2l+1)!}$$

② $f(x,y) = x^2 \underbrace{\sin y}_{\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!}} + \underbrace{\cos(x+y)}_{\sum_{n=0}^{\infty} \frac{(-1)^n (x+y)^{2n}}{(2n)!}}, \quad (0,0)$

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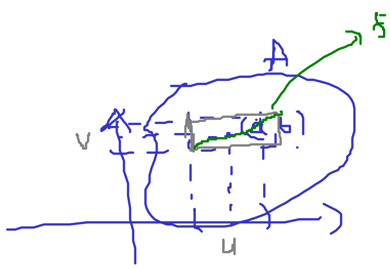
<u>Имплицитне ф-је</u>



$A \subseteq \mathbb{R}^2$ отв. $(a,b)\in A$

$F: A\to\mathbb{R}$ непрекидна $F(a,b)=0$

? $\exists\, U\times V \subseteq A$, $f: U\to V$ непр.

$U = (a-\delta, a+\delta)$

$V = (b-h, b+h)$

$F(x, f(x)) = 0$, $x\in U$

$y = f(x)$, $F(x,y)=0$, $x\in U$, $y\in V$?

T $A \subseteq \mathbb{R}^2$ отв. $(a,b)\in A$ $\quad F: A\to\mathbb{R}$ непр. $\quad F(a,b)=0$

$\frac{\partial F}{\partial y}\in C(A)$, $\frac{\partial F}{\partial y}(a,b)\neq 0$

$\Rightarrow \exists\, U\ni a,\ V\ni b$, $f:U\to V$ непр. $\quad f(a)=b$ и $F(x,f(x))=0$, $x\in U$

ако још $\frac{\partial F}{\partial x}\in C(A) \Rightarrow f'(x) = -\dfrac{\frac{\partial F}{\partial x}(x,f(x))}{\frac{\partial F}{\partial y}(x,f(x))}$.

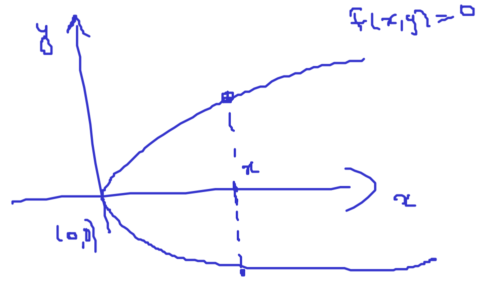
(1) $F(x,y)=x-y^4$

$F(x,y)=0$

$x(y)=x=y^4$ на $\mathbb{R}$

$y\geq 0 \Rightarrow y=\sqrt[4]{x}$

$y\leq 0 \Rightarrow y=-\sqrt[4]{x}$



(x_0,y_0) $\quad F(x_0,y_0)=0$

$y_0\neq 0 \Rightarrow y_0>0 \longrightarrow [0,+\infty)=U$, $V=[0,+\infty) \Rightarrow$ $f_1: U\to V$ $\quad f_1(x)=\sqrt[4]{x}=y$

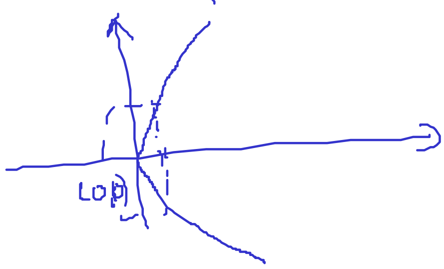
$\qquad\qquad y_0<0 \longrightarrow [0,+\infty)=U$, $V=(-\infty,0) \Rightarrow$ $f_2: U\to V$ $\quad f_2(x)=-\sqrt[4]{x}=y$

$F(x,f_1(x))=0$

$F(x,f_2(x))=0$

$y_0=0 \rightsquigarrow x_0=0$

у $(0,0)$ немамо имп. фју $y(x)$.



(2) $F(x,y)=y+\frac{1}{2}\sin y - x=0$ $\;\to$ да ли имамо имплицитну фју $y(x)$ глобално? локално? и где?

$y'(\pi),\ y''(\pi)=?$

$x(y)=y+\frac{1}{2}\sin y$ бијекција?

$x'(y)=1+\frac{1}{2}\cos y\geq\frac{1}{2}>0 \Rightarrow x\uparrow$, x непр. $\Rightarrow x$ бијективно $\mathbb{R}\to\mathbb{R}$

$y\to-\infty\Rightarrow x\to-\infty$

$y\to+\infty\Rightarrow x\to+\infty$

$\Rightarrow \exists\, y(x)=x(y)^{-1}$ $\to$ глобална имп. фја

$\frac{\partial F}{\partial x}=-1\in C(\mathbb{R}^2)$, $\frac{\partial F}{\partial y}=1+\frac{1}{2}\cos y\neq 0$

$\boxed{\text{I}} \Rightarrow \quad y'(x) = -\dfrac{\frac{\partial F}{\partial x}(x, y(x))}{\frac{\partial F}{\partial y}(x, y(x))} = \dfrac{1}{1 + 1/2 \cos y(x)}$

$$y''(x) = (y'(x))' = -\frac{-1/2 \sin y(x) \cdot y'(x)}{(1 + 1/2 \cos y(x))^2} = \frac{1/2 \sin y(x)}{(1 + 1/2 \cos y(x))^3}$$

$$x = \pi \Rightarrow y''(\pi) = \frac{1/2 \sin y(\pi)}{(1 + 1/2 \cos y(\pi))^3}$$

$$F(x, y(x)) = 0 \Rightarrow F(\pi, y(\pi)) = \underset{\pi}{\underset{\parallel}{y(\pi)}} + 1/2 \underset{0}{\underset{\parallel}{\sin y(\pi)}} - \pi = 0$$

$$\underline{\underline{y(\pi) = \pi}}$$

$$y''(\pi) = 0$$