Равномерна непр.

- $f: A \to \mathbb{R}^m$ равн. непр $\Leftrightarrow \forall \varepsilon>0\ \exists \delta>0\ \forall x,y\in A\ d_{\mathbb{R}^n}(x,y)<\delta \Rightarrow d_{\mathbb{R}^m}(f(x),f(y))<\varepsilon$

  $A\subseteq \mathbb{R}^n$

- f није равн непр $\Rightarrow \exists x_n, y_n \quad d_{\mathbb{R}^n}(x_n,y_n)\to 0 \wedge d_{\mathbb{R}^m}(f(x_n),f(y_n)) \geq C > 0$ ($\not\to 0$)

- Кантор:

  $f: A\to\mathbb{R}$ непр. A компактан $\Rightarrow f$ равн непр у A

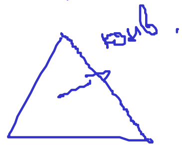
  $A\subseteq\mathbb{R}^n \quad A$ компакт $\Leftrightarrow A$ затворен + ограничен

$\boxed{T}$ $f: A\to\mathbb{R}$, A отворен и конвексан у $\mathbb{R}^n$ (конвексан $\forall x,y\in A\ \forall t\in(0,1)\ tx+(1-t)y\in A$)

диференцијабилна на A

$\exists C>0\ \forall a\in A\ \left|\frac{\partial f}{\partial x_i}(a)\right| \leq C \quad \forall 1\leq i\leq n$

$\Rightarrow f$ равн непр на A.



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(1) $\mathbb{D} = \{(x,y): x^2+y^2<1\}$

$f(x,y) = \sin\frac{1}{1-x^2-y^2}$ р.н. на $\mathbb{D}$?

$t = 1-(x^2+y^2) \longrightarrow h(t) = \sin\frac{1}{t}, \quad t\in(0,1]$

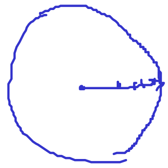
$\sin\frac{1}{t}$ није равн непр.

$t_n = \frac{1}{2n\pi} \to 0 \qquad h(t_n) = \sin 2n\pi = 0$

$u_n = \frac{1}{2n\pi+\pi/2} \to 0 \qquad h(u_n) = \sin(2n\pi+\pi/2) = 1$

$|t_n-u_n| \to 0$ и $|h(t_n)-h(u_n)| = 1 \not\to 0$

$\Rightarrow h$ није р.н.



$(a_n,b_n) \Rightarrow a_n^2+b_n^2 = 1-t_n$

$\parallel$

$(r\cos\theta, r\sin\theta) \qquad \theta=0 \quad \Rightarrow b_n=0,\ a_n = \sqrt{1-t_n} = \sqrt{1-\frac{1}{2n\pi}},$

$f(a_n,b_n) = \sin\frac{1}{1-(1-t_n)} = \sin\frac{1}{t_n} = \sin 2n\pi = 0$

$(c_n,d_n) \qquad c_n^2+d_n^2 = 1-u_n$

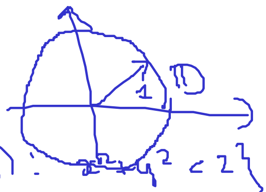
$\theta=0 \to c_n = \sqrt{1-u_n},\ d_n=0 \qquad f(c_n,d_n) = \sin\frac{1}{u_n} = 1$

$= \sqrt{1-\frac{1}{2n\pi+\pi/2}}$

$d_2((a_n,\overset{=0}{b_n}),(c_n,\overset{=0}{d_n})) = |a_n-c_n| = \Big|\underbrace{\sqrt{1-\frac{1}{2n\pi}}}_{\to 1} - \underbrace{\sqrt{1-\frac{1}{2n\pi+\pi/2}}}_{\to 1}\Big| \xrightarrow[n\to\infty]{} 0$

$|f(a_n,b_n)-f(c_n,d_n)| = 1 \underset{n\to\infty}{\not\to} 0 \quad \Rightarrow f$ није равн непр на $\mathbb{D}$

$g(x,y)=\sin\dfrac{1}{2-x^2-y^2}$, $(x,y)\in D$



$\tilde g(x,y)=\sin\dfrac{1}{2-x^2-y^2}$, $(x,y)\in A$, $A=\{(x,y): x^2+y^2<2\}=B_{d_2}(0,\sqrt{2})$

↳ непр на A

$\overline{D}=\{(x,y): x^2+y^2\le 1\}\subseteq A \Rightarrow \tilde g|_{\overline{D}}$ непр.

$\overline{D}$ компактан ?

↓

затворен и огран $\overline{D}\subseteq B(0,\sqrt{2}) \Rightarrow \overline{D}$ комп.

Кантор $\Rightarrow \tilde g$ равн непр. на $\overline{D}$ $\Rightarrow \tilde g|_{D}=g$ равн непр на D.

2. $f(x,y)=\dfrac{x^3+y^3}{x^2+y^2}$ р.н. на $\mathbb{R}^2\setminus\{(0,0)\}$?

$$\frac{\partial f}{\partial x}=\frac{3x^2(x^2+y^2)-2x(x^3+y^3)}{(x^2+y^2)^2}=\frac{x^4+3x^2y^2-2xy^3}{x^4+2x^2y^2+y^4}$$

$$=1+\frac{-y^4+x^2y^2-2xy^3}{(x^2+y^2)^2}=$$

$$=1-\frac{y^2(y^2-x^2+2xy)}{(x^2+y^2)^2}$$

$$=1-\underbrace{\left(\frac{y^2}{x^2+y^2}\right)}_{\text{огран}}\underbrace{\left(\frac{y^2+2xy-x^2}{x^2+y^2}\right)}_{\text{огран}}$$

$$C_1\ \underset{?}{\le}\ \frac{y^2+2xy-x^2}{x^2+y^2}\ \underset{?}{\le}\ C_2 \quad ✓$$

$$y^2+2xy-x^2\le C_2(x^2+y^2)$$

$$-(x^2+y^2)\le 2xy\le x^2+y^2$$

$$-2(x^2+y^2)\le -2x^2\le y^2+2xy-x^2\le 2y^2\le 2(x^2+y^2)$$

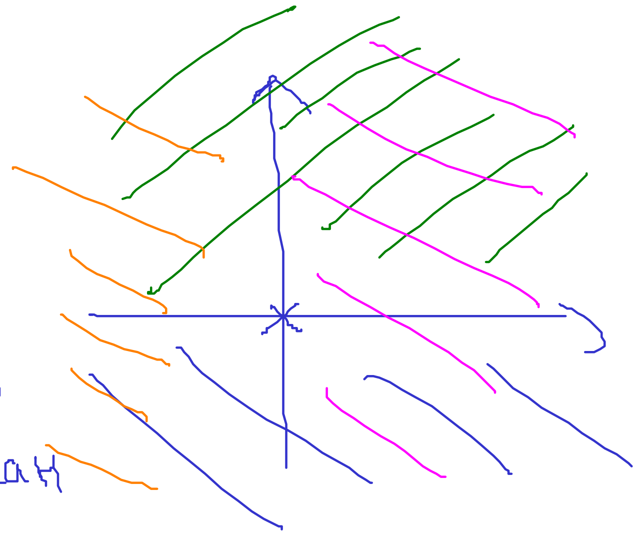
$\Rightarrow \quad \left|\frac{\partial f}{\partial x}(x,y)\right| \le C, \quad C>0, \quad (x,y)\in\mathbb{R}^2\setminus\{0\}$

$\left|\frac{\partial f}{\partial y}(x,y)\right| \le ?$

f симетрична $\quad \frac{\partial f}{\partial y}(x,y) = \frac{\partial f}{\partial x}(y,x)$

$\Rightarrow \left|\frac{\partial f}{\partial y}\right| \le C$

$\mathbb{R}^2\setminus\{0\}$ није конвексан



$A_1 = \{(x,y) : y>0\}$

$A_2 = \{ \quad y<0\}$

$A_3 = \{ \quad x>0\}$

$A_4 = \{ \quad x<0\}$

отворен конвексан

$\boxed{T} \Rightarrow f$ р.н. на A_1, A_2, A_3, A_4

$\Rightarrow f$ р.н. на $A_1\cup A_2\cup A_3\cup A_4 = \mathbb{R}^2\setminus\{(0,0)\}$

Да ли f може да се додеф непр на $\mathbb{R}^2$?

$f(x,y) = \frac{x^3+y^3}{x^2+y^2} \quad \mathbb{R}^2\setminus\{(0,0)\}$

$? \exists \lim_{(x,y)\to(0,0)} f(x,y) = ?$

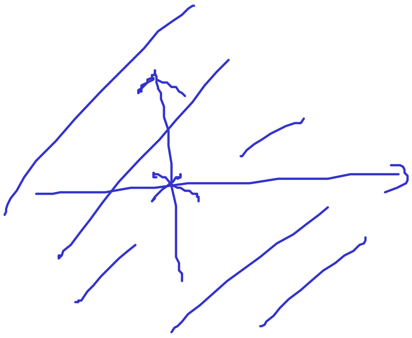
$x = r\cos\theta$

$y = r\sin\theta$

$\Rightarrow \lim_{r\to 0} \frac{r^{\cancel{3}}\cos^3\theta + r^{\cancel{3}}\sin^3\theta}{\cancel{r^2}} =$

$= \lim_{r\to 0} r\,(\underbrace{\cos\theta + \sin\theta}_{\text{огран}}) = 0$

Да ли $\tilde f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0)\end{cases}$?



$\tilde f$ непр на $\mathbb{R}^2$

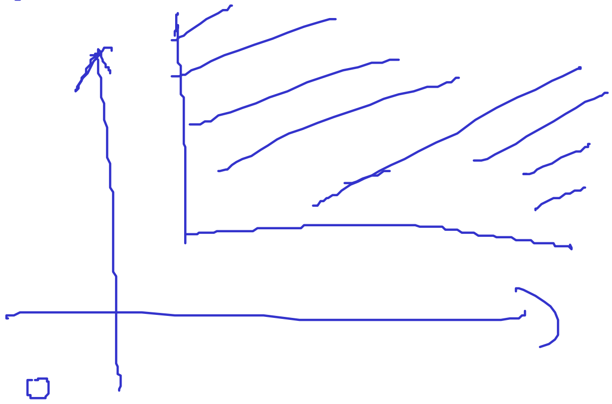
$\overline{D} = \{ (x,y) : x^2+y^2 \le 1 \}$

$\tilde f$ непр на $\overline{D}$ $\Big\}$ Кантор $\Rightarrow$ $\tilde f$ р.н. на $\overline{D}$

$\overline{D}$ комп.

$\tilde f$ р.н. на $\mathbb{R}^2\setminus\{(0,0)\}$ $\Rightarrow$ $\tilde f$ р.н. на $\mathbb{R}^2$

(3) $f(x,y) = \dfrac{2x+y}{2+xy}$, $A = \{ (x,y)\in\mathbb{R}^2 : x,y \ge 1 \}$



$$\frac{\partial f}{\partial x} = \frac{4+2xy-2xy-y^2}{(2+xy)^2} = \frac{4-y^2}{(2+xy)^2}$$

$x > a > 0$

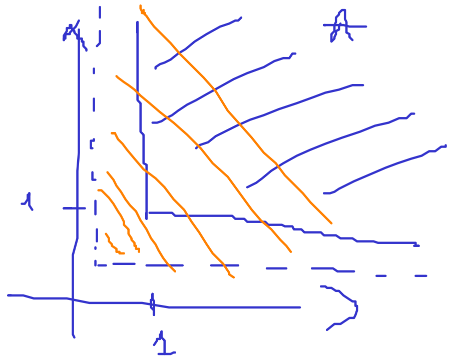
$$\frac{\partial f}{\partial x} = \frac{4-y^2}{4+4xy+x^2y^2} = \underbrace{\overbrace{\frac{4}{4+\dots}}^{0\,\le}}_{\le 1} - \underbrace{\left(\frac{y^2}{4+4xy+x^2y^2}\right)}_{\text{огран.}}$$

$$0 \le \frac{y^2}{4+4xy+y^2\cdot x^2} \le \frac{1}{x^2} \le \frac{1}{a^2}, \quad x > a$$

$$\left|\frac{\partial f}{\partial x}(x,y)\right| \le C_1, \quad C_1 > 0, \quad x > a,\ y > a$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{2+xy-2x^2-xy}{(2+xy)^2} = \frac{2-2x^2}{(2+xy)^2} = \underbrace{\overbrace{\frac{2}{(2+xy)^2}}^{0\,\le}}_{\le 1} - \underbrace{\overbrace{\frac{2x^2}{(2+xy)^2}}^{0\,\le}}_{\le\, 2/a^2}$$

$$\left|\frac{\partial f}{\partial y}\right| \le C_2 \quad , \quad x,y > a > 0$$



$a = 1/2 > 0$

$B = \{(x,y) \mid x,y > 1/2\}$

$\downarrow$

отворен

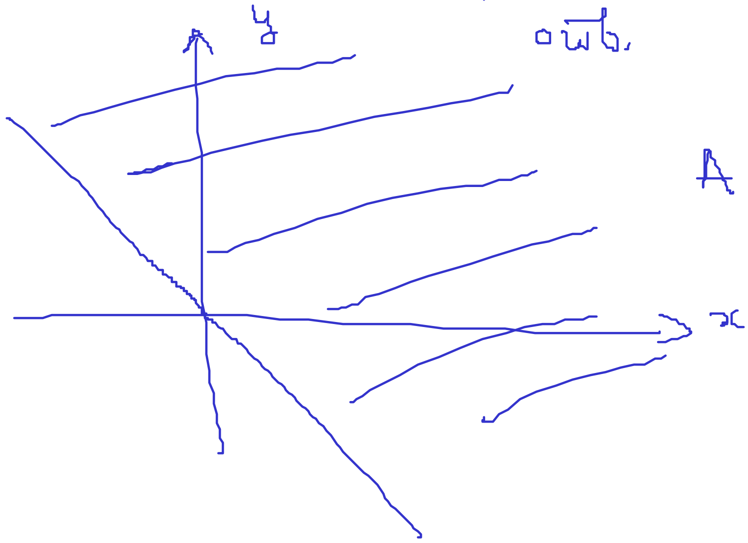
конвексан

$A \subseteq B$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ огран на B $\quad$ (T) $\Rightarrow$ f р.н. на B

$\Rightarrow$ f р.н. на A

(4) $f(x,y) = x\log(x+y)$, $\quad A = \{(x,y)\in\mathbb{R}^2 \mid x+y>0\}$ $\to$ конв. отв.



$$\frac{\partial f}{\partial x} = \log(x+y) + \frac{x}{x+y}$$

није огран

$$\frac{\partial f}{\partial y} = \frac{x}{x+y}$$

(a_n, b_n)

(x_n, y_n)

$$f(a_n,b_n) - f(x_n,y_n) = a_n\log(a_n+b_n) - x_n\log(x_n+y_n)$$

$(a_n,b_n) = (n,n)$ $\qquad n\log(2n) = n\log n + n\log 2$

$(x_n,y_n) = (n+t_n, n)$ $\qquad (n+t_n)\log(2n+t_n)$

$t_n \to 0$

$t_n = 1/n$ $\qquad (n+1/n)\log(2n+1)$

$n\log 2n$

$n\log(2n+1) + \frac{1}{n}\log(2n+1) \to 0$

$$(a_n, b_n) = (n + t_n, -n) \qquad a_n + b_n = t_n$$
$$t_n \downarrow 0$$

$$f(a_n, b_n) = (n + t_n) \log t_n = n \log t_n + t_n \log t_n$$

$$t_n = 1/n$$
$$(a_n, b_n) = (n + 1/n, -n) \Rightarrow f(a_n, b_n) = \underbrace{n \log 1/n}_{= -n \log n} + 1/n \log 1/n$$

$$t_n = 2/n$$
$$(x_n, y_n) = (n + 2/n, -n) \Rightarrow f(x_n, y_n) = \underbrace{n \log 2/n}_{= n\log 2 - n \log n} + 2/n \log 2/n$$

$$d_2((x_n, y_n), (a_n, b_n)) = 1/n \xrightarrow[n\to\infty]{} 0$$
$$|f(x_n, y_n) - f(a_n, b_n)| = |\underbrace{n \log 2}_{\to\infty} + \underbrace{2/n \log 2/n - 1/n \log 1/n}_{\to 0}| \to +\infty$$

$\Rightarrow f$ није равн. непр.

(5)
$$f(x,y) = \begin{cases} \sqrt{x^2+y^2}\, \sin \frac{1}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \quad \text{р.н. на } \mathbb{R}^2$$

$$x = r\cos\theta, \quad y = r\sin\theta$$
$$f(r,\theta) = \begin{cases} r \sin \frac{1}{r} & , r \neq 0 \\ 0 & , r = 0 \end{cases} \quad \text{непр} \quad r \in [0, +\infty)$$

$$\frac{\partial f}{\partial r}(r,\theta) = \sin 1/r - r\cos 1/r \cdot 1/r^2 \quad , \text{ огран. за } r \geq 1$$

$\xrightarrow{\text{(T)}}$ f р.н. на $(1, +\infty)$

Кантор $\Rightarrow$ f р.н. на $[0,1]$

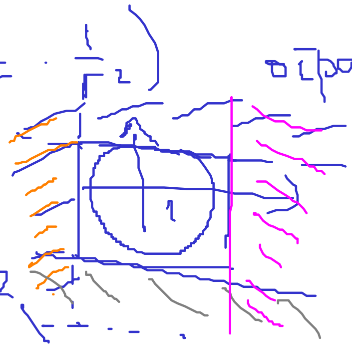
$\Big\} \Rightarrow f$ р.н. на $[0, +\infty)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{x}{\sqrt{}} \cdot \sin\frac{1}{\sqrt{}} + \sqrt{x^2+y^2} \cdot \cos\frac{1}{\sqrt{}} \cdot \frac{x}{\sqrt{}} \cdot \left(-\frac{1}{(\sqrt{})^3}\right)$$

$$= \frac{x}{\sqrt{}}\left(\sin\frac{1}{\sqrt{}} - \cos\frac{1}{\sqrt{}} \cdot \frac{1}{(x^2+y^2)}\right)$$

$$\frac{\partial f}{\partial x}(r,\theta) = \cos\theta\left(\sin\frac{1}{r} - \cos\frac{1}{r}\cdot\frac{1}{r^2}\right) \quad \text{огран } r > 1$$

$\frac{\partial f}{\partial y} \rightarrow$ огран



(T) $\Rightarrow$ f равн непр на $R^2 \setminus$ ...

Кантор $\Rightarrow$ f равн непр на $B(0,2)$

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(6) $f(x,y,z) = \sin(x^2+y^2+z^2)$, $(x,y,z) \in \mathbb{R}^3$

$x = \sqrt{r}\cos\theta\cos\varphi$

$y = \sqrt{r}\sin\theta\cos\varphi$, $z = \sqrt{r}\sin\varphi$

$$f(x,y,z) = \begin{cases} \sin r, & r > 0 \\ 0, & r = 0 \end{cases} \quad \text{р.н. на } [0,+\infty)$$

$f(r,\theta,\varphi)$

f није равн непр. $(x,y,z) \in \mathbb{R}^3$

$\sin(x^2+y^2+z^2) = 1$ $\qquad$ $\sin(x^2+y^2+z^2) = 0$

$x^2+y^2+z^2 = 2n\pi + \pi/2$ $\qquad$ $x^2+y^2+z^2 = 2n\pi$

$(a_n, b_n, c_n) = (\sqrt{2n\pi + \pi/2}, 0, 0) \rightarrow f(a_n,b_n,c_n) = 1$

$(x_n, y_n, z_n) = (\sqrt{2n\pi}, 0, 0)$ $\qquad$ $f(x_n,y_n,z_n) = 0$

? $d_2((a_n,b_n,c_n),(x_n,y_n,z_n)) = \sqrt{2n\pi+\pi/2} - \sqrt{2n\pi} = \sqrt{2n\pi}\left(\left(1+\frac{1}{4n}\right)^{\frac{1}{2}} - 1\right) \rightarrow 0$