$u = f \circ g \; : \mathbb{R}^2 \to \mathbb{R}$

$g : \mathbb{R}^2 \to \mathbb{R}, \quad f : \mathbb{R} \to \mathbb{R}$

$du = ?$ $\quad du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = f'(g(x,y)) \cdot \left(\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dx \right) = f'(g(x,y)) \cdot dg$

$\frac{\partial u}{\partial x}(x,y) = \frac{\partial}{\partial x}(f \circ g)(x,y) = f'(g(x,y)) \cdot \frac{\partial g}{\partial x}(x,y)$

$\frac{\partial u}{\partial y}(x,y) = f'(g(x,y)) \cdot \frac{\partial g}{\partial y}(x,y)$

- $f : A \to \mathbb{R}^m$, $A \subseteq \mathbb{R}^n$ отв., $a \in A$

$f = (f_1, f_2, \dots, f_m)$

$f_k : A \to \mathbb{R}$

$f(a+h) - f(a) = (f_1(a+h) - f_1(a), f_2(a+h) - f_2(a), \dots, f_m(a+h) - f_m(a))$

$h = (h_1, \dots, h_n)$

$= \underbrace{L(a)}_{\in M_{m \times n}} \cdot \underbrace{h}_{\in M_{n \times 1}} + o(h) \in \mathbb{R}^m$

f диф у a ако $\frac{\| o(h) \|}{\| h \|} \xrightarrow[\substack{h \to 0 \\ \text{у } \mathbb{R}^n}]{} 0$.

f диф у $a \Rightarrow L(a) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & & & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{m \times n} = Df = J_f$

(диф. где f) — J_f: Јакобијева матрица функције f

$h = (h_1, \dots, h_n) = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}_{n \times 1}$

$f : \mathbb{R}^n \to \mathbb{R}^m, \quad g : \mathbb{R}^m \to \mathbb{R}^k$

$\underbrace{D(g \circ f)}_{\in M_{k \times n}} = \underbrace{Dg(f(x,y))}_{\in M_{k \times m}} \cdot \underbrace{Df(x,y)}_{\in M_{m \times n}}$

① $\underbrace{Df(1,0)}_{\in M_{3 \times 2}} = ?$ $\quad f(x,y) = (\underbrace{x^2 - y^2}_{f_1}, \underbrace{x^2 + y^2}_{f_2}, \underbrace{(x^2 - y^2)(x^2 + y^2)}_{f_3 \,=\, x^4 - y^4})$, $\quad f : \mathbb{R}^2 \to \mathbb{R}^3$

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & -2y \\ 2x & 2y \\ 4x^3 & -4y^3 \end{bmatrix} = Df(x,y)$$

$$Df(1,0) = \begin{bmatrix} 2 & 0 \\ 2 & 0 \\ 4 & 0 \end{bmatrix}$$

(2) $h = f(u,v)$, $f(u,v) = \operatorname{arctg}(u+v)$

$u(x,y,z) = xy$

$v(x,y,z) = y/z$

$h(x,y,z) = f(u(x,y,z), v(x,y,z))$, $h: A \to \mathbb{R}$

$\underset{M}{Dh} = ?$ $\frac{\partial^2 h}{\partial x^2}$, $\frac{\partial^2 h}{\partial y \partial z} = ?$

$M_{1\times 3}$

$(u,v) = g(x,y,z) = (\overset{g_1}{xy}, \overset{g_2}{y/z}) : \mathbb{R}^3 \to \mathbb{R}^2$

$f(u,v) : \mathbb{R}^2 \to \mathbb{R}$

$$Dg(x,y,z) = \begin{bmatrix} y & x & 0 \\ 0 & \frac{1}{z} & -\frac{y}{z^2} \end{bmatrix}$$

$$Df(u,v) = \begin{bmatrix} \frac{1}{1+(u+v)^2} & \frac{1}{1+(u+v)^2} \end{bmatrix}$$

$$Dh = Df(\underset{xy}{u(x,y,z)}, \underset{y/z}{v(x,y,z)}) \cdot Dg(x,y,z) =$$

$$= \left[\frac{1}{1+(xy+y/z)^2} \quad \frac{1}{1+(xy+y/z)^2} \right] \begin{bmatrix} y & x & 0 \\ 0 & 1/z & -y/z^2 \end{bmatrix}$$

$$= \left[\frac{y}{1+(xy+y/z)^2} + \frac{0}{1+(xy+y/z)^2} \;,\; \frac{x}{1+(xy+y/z)^2} + \frac{1}{z(1+(xy+y/z)^2)} \;,\; -\frac{y}{z^2} \cdot \frac{1}{1+(xy+y/z)^2} \right]$$

$$= \frac{1}{1+(xy+y/z)^2} \left[\underset{\frac{\partial h}{\partial x}}{\underbrace{y}} \;,\; \underset{\frac{\partial h}{\partial y}}{\underbrace{x + \tfrac{1}{z}}} \;,\; \underset{\frac{\partial h}{\partial z}}{\underbrace{-\tfrac{y}{z^2}}} \right]$$

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{y}{1+(xy+y/z)^2} \right) = -\left(\frac{y}{\left(1+(xy+y/z)^2\right)^2} \right) \cdot \left(2(xy+y/z) \cdot y \right)$$

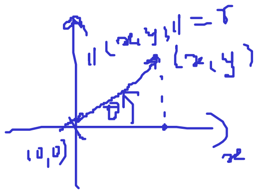
$$\frac{\partial^2 h}{\partial y \partial z} = \ldots \frac{\partial}{\partial z}\left(\frac{\partial h}{\partial y} \right) \ldots$$

(3)

$(x,y) = (r\cos\theta, r\sin\theta)$

$f : A \to \mathbb{R}^2$, $A =]0,+\infty[\times [0, 2\pi[$

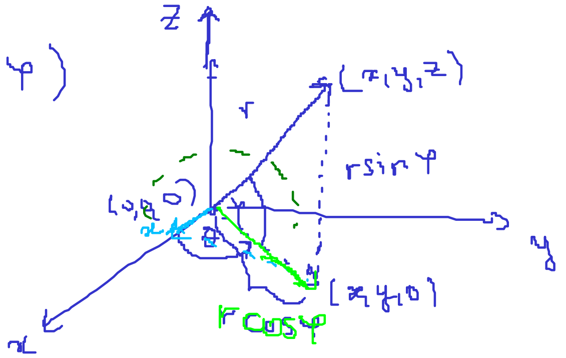
$f(r,\theta) = (r\cos\theta, r\sin\theta)$



$$J_f = Df = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\det J_f = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - (-r\sin^2\theta) = r$$

(4) $(x,y,z) = (r\cos\varphi\cos\theta,\ r\cos\varphi\sin\theta,\ r\sin\varphi)$ $\in \mathbb{R}^3 \setminus \{(0,0,0)\}$



сферне коорд.

$r \in (0,+\infty)$

$\theta \in [0,2\pi)$

$\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$f(r,\theta,\varphi) = (\underbrace{r\cos\varphi\cos\theta}_{f_1},\ r\cos\varphi\sin\theta,\ r\sin\varphi)$$

$$Df = \begin{bmatrix} \overset{+}{\cos\varphi\cos\theta} & \overset{-}{-r\sin\theta\cos\varphi} & \overset{+}{-r\cos\theta\sin\varphi} \\ \overset{-}{\cos\varphi\sin\theta} & \overset{+}{r\cos\theta\cos\varphi} & \overset{-}{-r\sin\theta\sin\varphi} \\ \underset{+}{\sin\varphi} & \underset{-}{0} & \underset{+}{r\cos\varphi} \end{bmatrix}$$

$$\det Df = \sin\varphi \begin{vmatrix} -r\sin\theta\cos\varphi & -r\cos\theta\sin\varphi \\ r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \end{vmatrix} + r\cos\varphi \begin{vmatrix} \cos\varphi\cos\theta & -r\sin\theta\cos\varphi \\ \cos\varphi\sin\theta & r\cos\theta\cos\varphi \end{vmatrix}$$

$$= \sin\varphi \cdot (r^2\sin^2\theta\cos\varphi\sin\varphi + r^2\cos^2\theta\cos\varphi\sin\varphi)$$
$$+ r\cos\varphi(r\cos^2\theta\cos^2\varphi + r\sin^2\theta\cos^2\varphi)$$

$$= \sin\varphi \cdot r^2\cos\varphi\sin\varphi + r^2\cos^3\varphi$$

$$= r^2\cos\varphi(\underbrace{\sin^2\varphi + \cos^2\varphi}_{=1}) = r^2\cos\varphi$$

(5) $f : \mathbb{R}^n \to \mathbb{R}^2$

$$f(x_1, x_2, \dots, x_n) = (\underbrace{x_1x_3 + \cos(x_2x_n)}_{f_1},\ \underbrace{e^{x_1x_2} + x_3x_n}_{f_2})$$

$Df = ?$

$$Df = \begin{bmatrix} x_3 & -\sin(x_2x_n)\cdot x_n & x_1 & 0 & \cdots & 0 & -\sin(x_2x_n)\cdot x_2 \\ x_2e^{x_1x_2} & x_1e^{x_1x_2} & x_n & 0 & \cdots & 0 & x_3 \end{bmatrix}$$

n×1

⑥ Трансформисати ј-ну $x^2\frac{\partial^2 f}{\partial x^2} - y^2\frac{\partial^2 f}{\partial y^2} - 2y\frac{\partial f}{\partial y} = 0$ узимајући за нове независне променљиве $u = xy$, $v = y/x$.

$$\frac{\partial f}{\partial x} = ?\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right)$$

$f(x,y) = g(u(x,y), v(x,y))$ [суштински „$f = g$"]

$$Df(x,y) = Dg(u,v) \cdot (D(u,v)(x,y)) = Df(u,v) \cdot (D(u,v)(x,y))$$

$$= \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \underbrace{\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}}_{f'_x} & \underbrace{\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}}_{f'_y} \end{bmatrix}$$

скраћене ознаке $\frac{\partial f}{\partial x} = f'_x$, $\frac{\partial f}{\partial y} = f'_y$

$\frac{\partial^2 f}{\partial x^2} = f''_{xx}$, $\frac{\partial^2 f}{\partial x \partial y} = f''_{xy}$, $\frac{\partial^2 f}{\partial y \partial x} = f''_{yx}$.

$$\Rightarrow f'_x = f'_u \cdot u'_x + f'_v \cdot v'_x$$

$$f''_{xx} = \frac{\partial}{\partial x}\left(f'_u \cdot u'_x + f'_v \cdot v'_x\right) = \frac{\partial}{\partial x}(f'_u(u,v)) \cdot u'_x + f'_u \cdot u''_{xx} + \frac{\partial}{\partial x}(f'_v) \cdot v'_x + f'_v \cdot v''_{xx}$$

$$= (f''_{uu} \cdot u'_x + f''_{uv} \cdot v'_x)u'_x + f'_u \cdot u''_{xx} + (f''_{vu} u'_x + f''_{vv} v'_x)v'_x + f'_v v''_{xx}$$

$$= f''_{uu}(u'_x)^2 + (f''_{uv} + f''_{vu})u'_x v'_x + f''_{vv} \cdot (v'_x)^2 + f'_u u''_{xx} + f'_v v''_{xx}$$

$$f''_{yy} = f''_{uu}(u'_y)^2 + (f''_{uv} + f''_{vu})u'_y v'_y + f''_{vv}(v'_y)^2 + f'_u u''_{yy} + f'_v v''_{yy}$$

$u'_x = y$, $u''_{xx} = 0$, $u'_y = x$, $u''_{yy} = 0$, $v'_x = -y/x^2$, $v''_{xx} = 2y/x^3$

$v'_y = 1/x$, $v''_{yy} = 0$.

$$\Rightarrow f'_y = f'_u \cdot x + f'_v \cdot \frac{1}{x}$$

$$f''_{xx} = f''_{uu} \cdot y^2 + (f''_{uv} + f''_{vu}) \cdot (-y^2/x^2) + f''_{vv}\, y^2/x^4 + f'_v \cdot 2y/x^3$$

$$f''_{yy} = f''_{uu} \cdot x^2 + (f''_{uv} + f''_{vu}) \cdot 1 + f''_{vv} \cdot 1/x^2$$

нова ј-на:

$$x^2\left(f''_{uu}\, y^2 + (f''_{uv} + f''_{vu}) \cdot (-y^2/x^2) + f''_{vv}\, y^2/x^4 + f'_v\, 2y/x^3\right)$$

$$-y^2\left(f''_{uu} \cdot x^2 + (f''_{uv} + f''_{vu}) + f''_{vv} \cdot 1/x^2\right) - 2y \cdot (f'_u x + f'_v\, 1/x) = 0$$

$\Rightarrow \quad -2y^2\left(f''_{uv} + f''_{vu}\right) - 2yx f'_u = 0$ $\longrightarrow$ ово није трансф. ј-на
још увек зависи од x и y

$y^2 = u \cdot v$
$xy = u$

$\Rightarrow \quad \boxed{-2uv\left(f''_{uv} + f''_{vu}\right) - 2u f'_u = 0}$ $\longrightarrow$ трансф. ј-на

* За вежбу трансф. ј-ну $\Delta u = 0$ $\left[\Delta u = u''_{xx} + u''_{yy}\right]$, ако $u \in C^2(\mathbb{R}^2 \setminus \{(0,0)\})$
увођењем нових коорд. r, θ, где $x = r\cos\theta$, $y = r\sin\theta$.