$\cdot\ f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$ отворен, $a \in A$, $\vec{\ell} \in \mathbb{R}^n \setminus \{0\}$

$$\frac{\partial f}{\partial \vec{\ell}}(a) = \lim_{t \to 0} \frac{f(a+t\vec{\ell}) - f(a)}{t}$$

① $f(x,y,z) = e^{z/x}$, $\vec{\ell} = (2/3, -1/3, 2/3)$, $a = (3, 0, -1)$

$$\frac{\partial f}{\partial \vec{\ell}}(a) = \lim_{t \to 0} \frac{f(a+t\vec{\ell}) - f(a)}{t} = (*)$$

$$a + t\vec{\ell} = \left(3 + t \cdot 2/3,\ 0 + t\cdot(-1/3),\ -1 + \frac{2t}{3}\right)$$

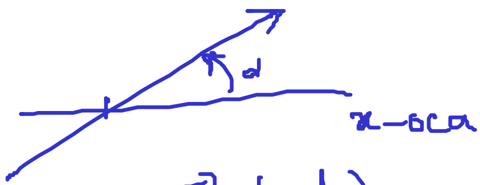
$$f(a+t\vec{\ell}) = e^{\frac{\left(-1+\frac{2t}{3}\right)}{3+2t/3}}$$

$$(*) = \lim_{t\to 0} \frac{e^{\left(\frac{-1+2t/3}{3+2t/3}\right)} - e^{-1/3}}{t} \underset{\left(\frac{0}{0}\right)}{\overset{ЛП}{=}} \lim_{t \to 0} \frac{\boxed{e^{\frac{-1+\frac{2t}{3}}{3+2t/3}}}^{\to e^{-1/3}} \cdot \dfrac{\frac{2}{3}\left(3+\frac{2t}{3}\right) - \frac{2}{3}\left(-1+\frac{2t}{3}\right)}{\left(3+\frac{2t}{3}\right)^2}}{1}$$

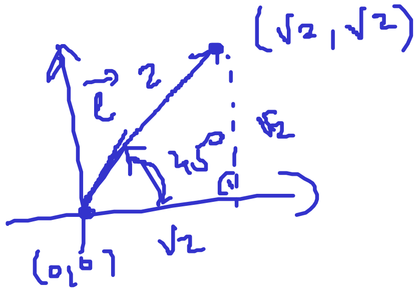
$$= e^{-1/3} \lim_{t \to 0} \frac{\overbrace{2 + \frac{4t}{9} + \frac{2}{3}}^{=8/3} - \frac{4t}{9}}{\underbrace{\left(3+\frac{2t}{3}\right)^2}_{\to 9}} = \frac{8}{27} e^{-1/3}$$

② $\vec{\ell}$ са позитивним делом x-осе гради угао 45°

$\|\vec{\ell}\| = 1$



$\vec{x} = (a,b)$



$\frac{\partial f}{\partial \vec{\ell}}(1,1) = ?$, $f(x,y) = x^2 - y^2$

$$\frac{\partial f}{\partial \vec{\ell}}(1,1) = \lim_{t\to 0} \frac{f(1+t\sqrt{2}, 1+t\sqrt{2}) - f(1,1)}{t}$$

$$= \lim_{t \to 0} \frac{(1+t\sqrt{2})^2 - (1+t\sqrt{2})^2 - (1^2 - 1^2)}{t} = 0$$

③ $$f(x,y) = \begin{cases} x^2 + y^2 - 2x^2 y - \dfrac{4x^6y^2}{(x^4+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Наћи извод у правцу сваког вектора у $(0,0)$. / Испитати диф. ф-је f.

$\vec{v} = (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$ (ta, tb)

$$\frac{\partial f}{\partial \vec{v}}(0,0) = \lim_{t\to 0} \frac{f((0,0)+t\vec{v}) - f(0,0)}{t} = \lim_{t\to 0} \frac{t^2a^2 + t^2b^2 - 2t^3a^2b - \frac{4t^8a^6b^2}{t^4(t^2a^4+b^2)^2}}{t} = 0$$

$f = g - h$, $g(x,y) = x^2+y^2-2x^2y \to$ диференцијабилно $\mathbb{R}^2$

$$h(x,y) = \begin{cases} \frac{4x^6y^2}{(x^4+y^2)^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0)\end{cases}$$

За вежбу диференцијабилност h на $\mathbb{R}^2\setminus\{(0,0)\}$

за $(0,0)$:

$$\frac{\partial h}{\partial x}(0,0) = \lim_{t\to 0}\frac{h(t,0)-h(0,0)}{t} = \lim_{t\to 0}\frac{\frac{4t^6\cdot 0}{\cdots} - 0}{t} = 0$$

$$\frac{\partial h}{\partial y}(0,0) = \lim_{t\to 0}\frac{h(0,t)-h(0,0)}{t} = 0$$

$$o(h_1,h_2) = \Delta h((0,0),(h_1,h_2)) - \underbrace{\frac{\partial h}{\partial x}}_{0}h_1 - \underbrace{\frac{\partial h}{\partial y}}_{0}h_2$$

$$= h(h_1,h_2) - h(0,0) = \frac{4h_1^6h_2^2}{(h_1^4+h_2^2)^2}$$

$$? \quad \frac{o(h_1,h_2)}{\|(h_1,h_2)\|} \xrightarrow[(h_1,h_2)\to(0,0)]{} 0 \quad ?$$

$$\| \quad \frac{4h_1^6h_2^2}{(h_1^4+h_2^2)^2\sqrt{h_1^2+h_2^2}}$$

БУО

$1 > h_1, h_2 > 0$

$$0 \le \frac{4h_1^6h_2^2}{\underbrace{(h_1^4+h_2^2)^2}_{\ge \max\{h_1^4,h_2^2\}}\cdot\underbrace{\sqrt{h_1^2+h_2^2}}_{\ge \max\{h_1^2,h_2^2\}}} \le \underbrace{\frac{4h_1^6h_2^2}{\max\{h_1^2,h_2\}^4\max\{h_1,h_2\}}}_{A(h_1,h_2)} \to 0$$

$\max\{h_1^4,h_2^2\}$

$\max\{h_1^2,h_2\}^2$

$h_2 \le h_1^2 \le h_1$

$h_1^2 \le h_2 \le h_1$

$h_1^2 \le h_1 \le h_2$

$(h_1,h_2) \downarrow [0,0]$ $\downarrow 0$ $\ell \cap$

1° $h_2 \le h_1^2 \le h_1$

$$A(h_1,h_2) = \frac{4h_1^6 h_2^2}{h_1^8 \cdot h_1} = \frac{4h_1^6 h_2^2}{h_1^3} = 4\frac{h_2^2}{h_1^3} = 4\underbrace{\left(\frac{h_2}{h_1^2}\right)^2}_{\le 1} \cdot \underset{\substack{\downarrow \\ 0}}{h_1}$$

$$\downarrow \; h_1, h_2 \to 0$$
$$0$$

2° $h_1^2 \le h_2 \le h_1$

$$A(h_1,h_2) = \frac{4h_1^6 h_2^2}{h_2^4 \cdot h_1} = 4\frac{h_1^5}{h_2^2} = 4\underbrace{\left(\frac{h_1^2}{h_2}\right)^2}_{\le 1} \cdot \underset{\substack{\downarrow \\ 0}}{h_1}$$

$$\downarrow$$
$$0$$

3° $h_1^2 \le h_1 \le h_2$

$$A(h_1,h_2) = \frac{4h_1^6 h_2^2}{h_2^4 \cdot h_2} = 4\frac{h_1^6 h_2^2}{h_2^5} = 4\frac{h_1^6}{h_2^3} =$$

$$= 4\underbrace{\left(\frac{h_1}{h_2}\right)^3}_{\le 1} \cdot \underset{\substack{\downarrow \\ 0}}{h_1^3}$$

$$\downarrow$$
$$0$$

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4) $f(x,y) = \begin{cases} \dfrac{x^2 y}{\sqrt{x^4+y^4}}, & (x,y) \ne (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Наћи извод у сваком правцу у $(0,0)$.
Испитати диф у $(0,0)$.

$\vec{v} = (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$\frac{\partial f}{\partial \vec{v}}(0,0) = \lim_{t\to 0} \frac{f(ta,tb) - \overbrace{f(0,0)}^{=0}}{t} = \lim_{t\to 0} \frac{\frac{t^3 a^2 b}{\sqrt{t^4a^4+t^4b^4}}}{t} =$$

$$= \frac{a^2 b}{\sqrt{a^4+b^4}}$$

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial \vec{e_1}}(0,0) = \frac{1^2 \cdot 0}{\sqrt{1^4+0^4}} = 0$$

$$\vec{e_1} = (1,0)$$

$$\frac{\partial f}{\partial y}(0,0) = \frac{\partial f}{\partial \vec{e_2}}(0,0) = \frac{0^2 \cdot 1}{\sqrt{1^4+0^4}} = 0$$

$$\vec{e_2} = (0,1)$$

$$\Delta f((0,0),(h_1,h_2)) = f(h_1,h_2) - \underbrace{f(0,0)}_{=0} = \frac{h_1^2 h_2}{\sqrt{h_1^4+h_2^4}}$$

$$\frac{o(h_1,h_2)}{\|h\|} \xrightarrow{h\to 0} 0 \ ?$$

$$\| $$

$$\frac{h_1^2 h_2}{\sqrt{h_1^4+h_2^4}\cdot\sqrt{h_1^2+h_2^2}} \xrightarrow{h_1,h_2\to 0} 0 \ ?$$

$$, \quad (h_1,h_2) = \left(\frac{1}{n}, \frac{1}{n}\right) \xrightarrow[n\to\infty]{} (0,0)$$

$$\frac{\frac{1}{n^2}\cdot\frac{1}{n}}{\sqrt{\frac{1}{n^4}+\frac{1}{n^4}}\cdot\sqrt{\frac{1}{n^2}+\frac{1}{n^2}}} = \frac{1/n^3}{\frac{\sqrt{2}}{n^2}\cdot\frac{\sqrt{2}}{n}} = \frac{1}{2} \underset{n\to\infty}{\not\to} 0$$

$\Rightarrow$ није диф у $(0,0)$.

за вежбу

$$f(x,y) = \begin{cases} \sqrt[3]{xy-x}\ \operatorname{arctg}^{1/x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Испитати диф f на $\mathbb{R}^2$.

$$\frac{\partial f}{\partial \vec{e}}(1,1) = ?, \quad \vec{e} = (2,2).$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$\longrightarrow$ мешовити парц. извод

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

① $f(x,y) = x^{y^2} = e^{y^2 \ln x}$, $D_f = (0,+\infty) \times \mathbb{R}$

$$\frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial f}{\partial x} = e^{y^2 \ln x} \cdot y^2 \cdot \frac{1}{x}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(e^{y^2 \ln x} \cdot y^2 \frac{1}{x}\right) =$$

$$= e^{y^2 \ln x} \cdot 2y \ln x \cdot \frac{y^2}{x} + 2y \cdot e^{y^2 \ln x} \cdot 1/x$$

$$\parallel$$

$$\frac{\partial f}{\partial y} = e^{y^2 \ln x} \cdot 2y \ln x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(e^{y^2 \ln x} \cdot 2y \ln x\right) = e^{y^2 \ln x} \cdot \frac{1}{x} y^2 \cdot 2y \ln x$$

$$+ e^{y^2 \ln x} \cdot 2y/x$$

$\boxed{T}$ $\exists \ \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ непр. $\Longrightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

② $f(x,y) = \begin{cases} xy \dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$$\frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial f}{\partial x}(x,y) \underset{(x,y)\neq(0,0)}{=} y\cdot\frac{x^2-y^2}{x^2+y^2} + xy\,\frac{2x(\cancel{x^2}+y^2)-2x(\cancel{x^2}-y^2)}{(x^2+y^2)^2}$$

$$= \frac{y(x^4-y^4)+4x^2y^3}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0}\frac{\overbrace{f(t,0)}^{=0}-\overbrace{f(0,0)}^{=0}}{t} = 0$$

$$\frac{\partial^2 f}{\partial x\,\partial y}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) =$$

$$= \lim_{t\to 0}\frac{\frac{\partial f}{\partial x}(0,t)-\frac{\partial f}{\partial x}(0,0)}{t} = \lim_{t\to 0}\frac{-\frac{t^5}{t^4}-0}{t} = -1$$

$$\frac{\partial^2 f}{\partial y\,\partial x}$$

$$g(y_0,x_0) := f(x_0,y_0) = \begin{cases} x_0y_0\,\dfrac{x_0^2-y_0^2}{x_0^2+y_0^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$$= \begin{cases} x_0y_0\cdot\dfrac{y_0^2-x_0^2}{x_0^2+y_0^2}, & \neq(0,0)\\ 0 \end{cases} = f(y_0,x_0)$$

$$\frac{\partial g}{\partial y}(y_0,x_0) = -\frac{\partial f}{\partial y}(x_0,y_0)$$

$$\frac{\partial g}{\partial x}(y_0,x_0) = \frac{\partial f}{\partial x}(y_0,x_0)$$

$$\frac{\partial f}{\partial x}(y_0,x_0) = -\frac{\partial f}{\partial x}(x_0,y_0), \quad (x,y)\in\mathbb{R}^2$$

$$\frac{\partial f}{\partial y}(x,y) = -\frac{y(x^4-y^4)+4x^2y^3}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = x\,\frac{x^2-y^2}{x^2+y^2} + xy\,\frac{-2y(x^2+y^2)-2y(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{x(x^4-y^4)-2y^2x^3}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial y}f(x,y) = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}f\right)(0,0) = -\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}f\right)(0,0)$$

$$= -(-1) = 1$$

$\Rightarrow$ не важи услов $\boxed{T}$

$\Rightarrow$ $\dfrac{\partial^2 f}{\partial x\,\partial y}$ има прекид у $(0,0)$