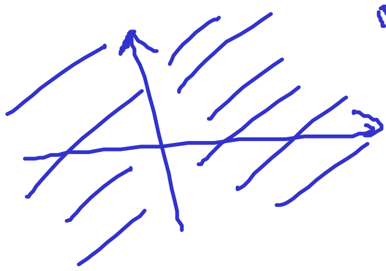
① $f(x,y)=\sqrt{x^2+y^2}\,\sin {y^2}/{x}$ , да ли може непр да се додеф на $\mathbb{R}^2$?

$\overline{D_f} = \mathbb{R}^2$  $D_f = \mathbb{R}^2 \setminus \{0\}\times\mathbb{R}$



$f$ је непр на $D_f$

$y\in\mathbb{R}, x=0$ ?   $y_0\in\mathbb{R}$ произв.

$$\lim_{(x,y)\to(0,y_0)} f(x,y) = \lim_{(x,y)\to(0,y_0)} \underbrace{\underbrace{\sqrt{x^2+y^2}}_{\to |y_0|}\,\underbrace{\sin {y^2}/{x}}_{\text{дивергира за } y_0\neq 0}}_{\text{не постоји}}$$

$y_0\neq 0$

$\Rightarrow$ $f$ се не може непр додеф у $(0,y_0), y_0\neq 0$.

$$\lim_{(x,y)\to(0,0)} \underbrace{\sqrt{x^2+y^2}}_{\to 0}\,\underbrace{\sin \frac{y^2}{x}}_{\text{ограничено}} = 0$$

$\Rightarrow$ $f$ се може непр. додеф у $(0,0)$.

② $$\lim_{\substack{x\to+\infty\\ y\to+\infty}} \frac{x+y}{x^2-xy+y^2} \quad \left(\overset{?}{=}0\right) \checkmark$$

ово очекујемо

$xy \le x^2-xy+y^2 \rightsquigarrow x^2-2xy+y^2=(x-y)^2\ge 0$

$x,y>0$

$$0 \le \frac{x+y}{x^2-xy+y^2} \le \frac{x+y}{x\cdot y} = \frac{x}{xy}+\frac{y}{xy} = \underbrace{\frac{1}{y}}_{\substack{y\to+\infty\\ \to 0}}+\underbrace{\frac{1}{x}}_{\substack{x\to+\infty\\ \to 0}}$$

$x\to+\infty$, $y\to+\infty$ $\downarrow 0$ Т о 2п.

Задаци за вежбу 1

Одредити $\lim\limits_{x\to a}\lim\limits_{y\to b} f(x,y)$, $\lim\limits_{y\to b}\lim\limits_{x\to a} f(x,y)$, $\lim\limits_{(x,y)\to(a,b)} f(x,y)$

① $f(x,y)=\dfrac{\sin(xy)}{x}$ , $a=0$, $b\in\mathbb{R}$

② $f(x,y)=(x+y)^k e^{-(x^2+y^2)}$ , $a=b=+\infty$

③ $f(x,y)=\sqrt[3]{xy-x}\ \mathrm{arctg}\, {1}/{x}$ , $a=0$, $b\in\mathbb{R}$

④ $f(x,y)=\dfrac{xy}{x^2+y^2}$ , $a=b=0$.

# Парцијални изводи. Диференцијал

$f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$ отворен скуп, $1 \le k \le n$, $a \in A$

$$\frac{\partial f}{\partial x_k}(a) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(a_1,\dots,a_{k-1}, a_k+h, a_{k+1},\dots,a_n) - f(a_1,\dots,a_n)}{h}$$

$a = (a_1,\dots,a_n)$

↓

$k$-ти парцијални извод

$\vec{h} = (h_1, h_2, \dots, h_n) \in \mathbb{R}^n \setminus \{0\}$

$$\Delta f(a,\vec{h}) = f(a+\vec{h}) - f(a) = f(a_1+h_1, a_2+h_2, \dots, a_n+h_n) - f(a_1,\dots,a_n)$$

$$\Delta f(a,\vec{h}) = L_1 \cdot h_1 + L_2 \cdot h_2 + \dots + L_n h_n + o(h)$$

$f$ диф у $a$ ако $\displaystyle\lim_{\substack{h \to 0 \\ h \in \mathbb{R}^n}} \frac{o(h)}{\|h\|} = 0$, $\|h\| = \sqrt{h_1^2 + h_2^2 + \dots + h_n^2}$

$f$ диф у $a$ $\Rightarrow$ $L_k = \dfrac{\partial f}{\partial x_k}(a)$

$$df(a)(h) = \frac{\partial f}{\partial x_1}(a) h_1 + \frac{\partial f}{\partial x_2}(a) h_2 + \dots + \frac{\partial f}{\partial x_n}(a) h_n$$

$df(a): \mathbb{R}^n \to \mathbb{R}$ линеарна

$dx_k(a)(h) = d\pi_k(a)(h) = \pi_k(h) = h_k$

$$df(a) = \frac{\partial f}{\partial x_1}(a)\, dx_1 + \frac{\partial f}{\partial x_2}(a)\, dx_2 + \dots + \frac{\partial f}{\partial x_n}(a)\, dx_n$$

↓

тотални диф. ф-је $f$ у $a$

диф $\Rightarrow$ непр, диф + диф = диф, диф $\cdot$ диф = диф, диф $\circ$ диф = диф

$\boxed{T}$ $f: A \to \mathbb{R}$, $a \in A$, $\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}, \dots, \dfrac{\partial f}{\partial x_n}$ непр у $a$

$\Rightarrow$ $f$ диф. у $a$.

1. Испитати диф ф-је $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$D_f = \mathbb{R}^2$

$f$ је непр на $\mathbb{R}^2 \setminus \{(0,0)\}$

Да ли је $f$ непр у $(0,0)$?

$$\lim_{(x,y)\to(0,0)} \underbrace{\frac{xy}{x^2+y^2}}_{f(x,y)} \overset{?}{=} 0 = f(0,0) \quad ↯$$

$(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right) \xrightarrow[n\to\infty]{} (0,0)$, $\quad f(x_n,y_n) = \dfrac{1/n^2}{1/n^2 + 1/n^2} = \dfrac{1}{2} \underset{n\to\infty}{\not\to} 0$

⇒ f има прекид у (0,0) ⇒ f није диф у (0,0).

$$\frac{\partial f}{\partial x}(x,y) = \frac{y(x^2+y^2) - 2x\cdot xy}{(x^2+y^2)^2} = \frac{y^3 - x^2y}{(x^2+y^2)^2}$$

$(x,y) \neq (0,0)$

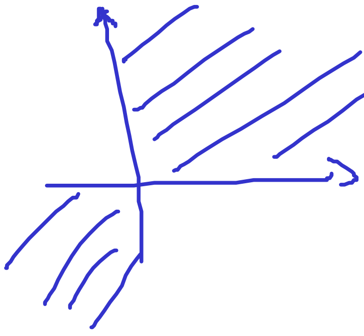
$$\frac{\partial f}{\partial y}(x,y) = \frac{x^3 - y^2x}{(x^2+y^2)^2}$$

$\Big\}$ непр на $\mathbb{R}^2 \setminus \{(0,0)\}$

$\boxed{T}$ ⇒ f диф на $\mathbb{R}^2 \setminus \{(0,0)\}$

(2) $f(x,y) = \sqrt[3]{xy}$ диф на домену фје?

Γ $\sqrt[4]{xy}$



$D_f = \mathbb{R}^2$

f је непр на $\mathbb{R}^2$

$\frac{\partial f}{\partial x}(x,y) = \frac{1}{3}\frac{1}{\sqrt[3]{(xy)^2}}\cdot y$ није деф на $\{0\}\times\mathbb{R} \cup \mathbb{R}\times\{0\}$

$\frac{\partial f}{\partial y}(x,y) = \frac{1}{3}\frac{1}{\sqrt[3]{x^2y^2}}\cdot x$

$\frac{\partial f}{\partial x}$ и $\frac{\partial f}{\partial y}$ су непр на $\mathbb{R}^2 \setminus$ (x-оса, y-оса)

⇒ f диф на $\mathbb{R}^2 \setminus (\{0\}\times\mathbb{R} \cup \mathbb{R}\times\{0\})$

$(0,y_0)$, $y_0 \in \mathbb{R}$

$$y_0 \neq 0,\ \frac{\partial f}{\partial x}(0,y_0) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(h,y_0) - f(0,y_0)}{h} = \lim_{h\to 0}\frac{\sqrt[3]{h\cdot y_0} - \overbrace{\sqrt[3]{0\cdot y_0}}^{=0}}{h}$$

$$= \sqrt[3]{y_0}\lim_{h\to 0}\frac{\sqrt[3]{h}}{h} = \text{sgn}\, y_0 \cdot \text{"}(+\infty)\text{"} \notin \mathbb{R}$$

$\frac{\partial f}{\partial x}(0,y_0)$ није добро деф ⇒ f није диф у $(0,y_0)$, $y_0 \neq 0$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{\overbrace{\sqrt[3]{h\cdot 0}}^{=0} - 0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{\sqrt[3]{0\cdot h} - 0}{h} = 0$$

$$\Delta f((0,0),(h_1,h_2)) = f(h_1,h_2) - \underset{0}{\underset{\shortparallel}{f(0,0)}} = \sqrt[3]{h_1 h_2}$$

$$\underset{(h_1,h_2)}{\underset{\shortparallel}{\theta(h)}} = \Delta f((0,0),(h_1,h_2)) - \underbrace{\frac{\partial f}{\partial x}(0,0)h_1}_{=0} - \underbrace{\frac{\partial f}{\partial y}(0,0)h_2}_{=0} = \sqrt[3]{h_1 h_2}$$

$$? \lim_{\underset{h}{\underset{\shortparallel}{(h_1,h_2)}}\to(0,0)} \frac{\theta(h)}{\|h\|} = 0 \ ? \ \not\downarrow$$

$$\frac{\theta(h)}{\|h\|} = \frac{\sqrt[3]{h_1^n \cdot h_2^n}}{\sqrt{h_1^{n2} + h_2^{n2}}} = \frac{\sqrt[3]{1/n^2}}{\sqrt{1/n^2 + 1/n^2}} = \frac{1/n^{2/3}}{\sqrt{2}/n} = \frac{n^{1/3}}{\sqrt{2}} \xrightarrow[n\to\infty]{} +\infty$$

$$h_1^n = h_2^n = 1/n$$

$$(h_1^n, h_2^n) \xrightarrow[n\to\infty]{} (0,0)$$

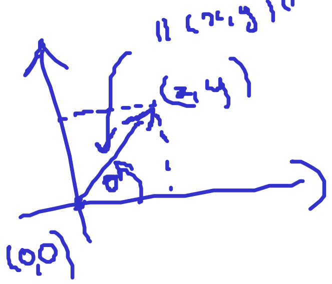
$\Rightarrow$ f није диф у (0,0).

③ $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ . испитати диф. dje.

Нема прекид у (0,0)?

$$? \lim_{(x,y)\to(0,0)} f(x,y) = 0 \ ?$$

$$\frac{xy}{\sqrt{x^2+y^2}}$$




$x = r\cos\theta$

$y = r\sin\theta$

$$(x,y) \to (0,0) \iff r = \|(x,y)\| \to 0, \quad \theta \in [0, 2\pi)$$

$$\frac{xy}{\sqrt{x^2+y^2}} = \frac{r^2 \cos\theta \sin\theta}{r} = \underbrace{r \cos\theta \sin\theta}_{\text{огран}} \xrightarrow[r\to 0]{} 0$$

$$\Rightarrow f(x,y) \xrightarrow[(x,y)\to(0,0)]{} (0,0) \Rightarrow f \text{ непр у } (0,0)$$

$$0 \le \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \le \frac{|xy|}{\sqrt{2|xy|}} = \frac{1}{\sqrt{2}}\sqrt{|xy|} \xrightarrow{(x,y)\to(0,0)} 0$$

$(x^2+y^2 \ge 2|xy|)$

$\downarrow$ То 2п

$0$

$$\frac{\partial f}{\partial x}\underset{\substack{\ne\\(0,0)}}{(x,y)} = \frac{y\sqrt{x^2+y^2} - xy\cdot\frac{2x}{2\sqrt{x^2+y^2}}}{(x^2+y^2)} = \frac{y(\cancel{x^2}+y^2) - \cancel{x^2 y}}{(x^2+y^2)^{3/2}} = \dots$$

$$\frac{\partial f}{\partial y} = \frac{x^3}{(x^2+y^2)^{3/2}}$$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ су непр на $\mathbb{R}^2\setminus\{(0,0)\}$

$\Rightarrow$ $f$ диф на $\mathbb{R}^2\setminus\{(0,0)\}$

$f$ диф у $(0,0)$ ?

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{\overset{=0}{f(h,0)}-\overset{=0}{f(0,0)}}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = 0$$

$$\Delta f((0,0),\underbrace{(h_1,h_2)}_{\vec{h}}) = f(h_1,h_2) - f(0,0) = \frac{h_1h_2}{\sqrt{h_1^2+h_2^2}}$$

$$o(\vec{h}) = \Delta f(0,\vec{h}) - \underbrace{\frac{\partial f}{\partial x}(0,0)}_{0}h_1 - \underbrace{\frac{\partial f}{\partial y}(0,0)}_{0}h_2 = \frac{h_1h_2}{\sqrt{h_1^2+h_2^2}}$$

$$\frac{o(\vec{h})}{\|\vec{h}\|} = \frac{h_1h_2}{h_1^2+h_2^2} \underset{\vec{h}\to 0}{\not\to} 0, \quad (h_1,h_2) = (1/n, 1/n) \to (0,0)$$

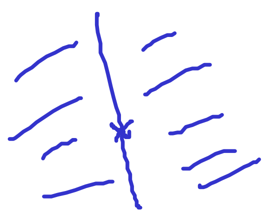
$$\frac{1/n\cdot 1/n}{1/n^2+1/n^2} = 1/2 \not\to 0$$

$\Rightarrow$ $f$ није диф у $(0,0)$

(4) $f(x,y) = \begin{cases} \sqrt{x^2+y^2}\,\sin y^2/x, & x\ne 0 \\ 0, & x = 0 \end{cases}$ диф ?

$\Rightarrow$ $f$ није непр на $\{0\}\times(\mathbb{R}\setminus\{0\})$

$\Rightarrow$ $f$ није диф на $\{0\}\times(\mathbb{R}\setminus\{0\})$



$$\frac{\partial f}{\partial x}(x,y) = \frac{x}{\sqrt{x^2+y^2}} \sin \frac{y^2}{x} + \sqrt{x^2+y^2} \cos \frac{y^2}{x} \cdot \left(-\frac{y^2}{x^2}\right)$$

$\downarrow$

непр на $\mathbb{R}^2 \setminus (\{0\}\times\mathbb{R})$

$$\frac{\partial f}{\partial y}(x,y) = \frac{y}{\sqrt{x^2+y^2}} \sin \frac{y^2}{x} + \sqrt{x^2+y^2} \cos \frac{y^2}{x} \left(\frac{2y}{x}\right)$$

$\downarrow$

непр на $\mathbb{R}^2 \setminus (\{0\}\times\mathbb{R})$

$\Rightarrow$ $f$ дифф на $\mathbb{R}^2 \setminus (\{0\}\times\mathbb{R})$

$f$ дифф у $(0,0)$?

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\overbrace{\sqrt{h^2+0^2}\cdot \sin \frac{0^2}{h}}^{=0} - 0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{\overbrace{f(0,h)}^{=0} - \overbrace{f(0,0)}^{=0}}{h} = 0$$

$$\Delta f((0,0),(h_1,h_2)) = f(h_1,h_2) - f(0,0)$$

$1^\circ$ $h_1 = 0$

$$\Delta f((0,0),(h_1,h_2)) = f(0,h_2) - f(0,0) = 0 - 0 = 0$$

$$\theta(h) = \overset{\shortparallel}{0} - \underset{\frac{\partial f}{\partial x}}{\underset{\shortparallel}{0}}\cdot h_1 - \underset{\frac{\partial f}{\partial y}}{\underset{\shortparallel}{0}}\cdot h_2 = 0$$

$$\frac{\theta(\vec{h})}{\|\vec{h}\|} = 0 \xrightarrow[\vec{h}\to 0]{} 0$$

$(0, h_2)$

$2^\circ$ $h_1 \neq 0$

$$\Delta f((0,0),(h_1,h_2)) = \sqrt{h_1^2+h_2^2} \sin \frac{h_2^2}{h_1}$$

$$\theta(\vec{h}) = \sqrt{h_1^2+h_2^2} \sin \frac{h_2^2}{h_1}$$

$\vec{h} = (h_1,h_2)$

$$\frac{\theta(h)}{\|h\|} = \frac{\cancel{\sqrt{h_1^2+h_2^2}} \sin \frac{h_2^2}{h_1}}{\cancel{\sqrt{h_1^2+h_2^2}}} = \sin \frac{h_2^2}{h_1}$$

$h_1 = 1/n, \quad h_2 = 1/\sqrt{n} \quad \xrightarrow[n\to\infty]{} 0$

$$\sin \frac{1/n}{1/n} = \sin 1 \underset{n\to\infty}{\not\to} 0$$

$\Rightarrow f$ није диф у $(0,0)$.

За вежбу:

. $f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

. $f(x,y) = \begin{cases} e^{\left(\frac{x^3}{x^2+y^2}\right)} & , (x,y) \neq (0,0) \\ 1 & , (x,y) = (0,0) \end{cases}$

. Испитати диф $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$.

$\|\cdot\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$