$f : \mathbb{R}^n \to \mathbb{R}$ диф у a

$df(a) : \mathbb{R}^n \to \mathbb{R}$ линеарно

$df(a)(\underset{h}{h}) = L_1 h_1 + L_2 h_2 + \dots + L_n h_n$, $\exists \frac{\partial f}{\partial x_k}(a) \Rightarrow L_k = \frac{\partial f}{\partial x_k}(a) \in \mathbb{R}$

$(h_1, \dots, h_n)$

$dx_k(h) = d(\pi_k)(h) = h_k$

$df(a) = \frac{\partial f}{\partial x_1}(a) dx_1 + \frac{\partial f}{\partial x_2}(a) dx_2 + \dots + \frac{\partial f}{\partial x_n}(a) dx_n \to$ тотални диф.

$\frac{\Delta f(a,h) - df(a)h}{\|h\|} \to 0 \Rightarrow f$ диф у a

$\boxed{T}$ $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$ отв., $a \in A$, $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$ непр у a

$\Rightarrow f$ диференцијабилно у a.

f, g диф $\Rightarrow f+g, f-g$ диф.

„f, g диф $\Rightarrow f \circ g$ диф" $\to$ пазити о домену и кодомену фја

1. $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$ диф на $\mathbb{R}^2$?

$(1/n, 1/n) \to (0,0)$, $f(1/n, 1/n) = \frac{1/n \cdot 1/n}{1/n^2 + 1/n^2} = 1/2 \not\to 0 \Rightarrow f$ има прекид у $(0,0)$

$\Rightarrow f$ није диф у $(0,0)$.

$\frac{\partial f}{\partial x}(x,y) = \left(\frac{xy}{x^2+y^2} \right)'_x = \frac{y(x^2+y^2) - 2x \cdot (xy)}{(x^2+y^2)^2} = \frac{y^3 - x^2 y}{(x^2+y^2)^2}$

$(x,y) \neq (0,0)$

$y \neq 0$ непр на $\mathbb{R}^2 \setminus \{(0,0)\}$

$\frac{\partial f}{\partial y}(x,y) = \frac{x^3 - y^2 x}{(x^2+y^2)^2}$ непр. на $\mathbb{R}^2 \setminus \{(0,0)\}$

$(x,y) \neq (0,0)$

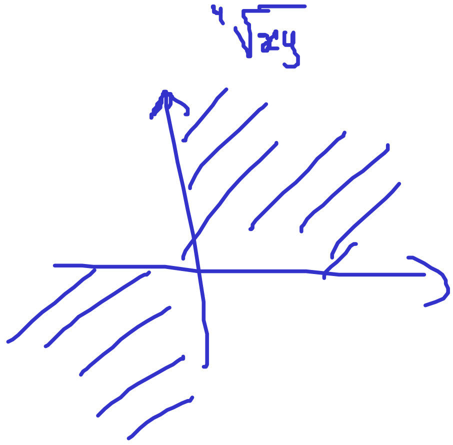
$\Rightarrow f$ диф на $\mathbb{R}^2 \setminus \{(0,0)\}$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h+0,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h\cdot 0}{h^2} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$x_n = 1/n, \; y_n = 2/n$

$$\frac{\partial f}{\partial x}(x_n, y_n) = \frac{8/n^3 - 2/n^3}{\left(1/n^2 + 4/n^2\right)^2} = \frac{6/n^3}{\frac{25}{n^4}} \longrightarrow +\infty$$

$\frac{\partial f}{\partial x}$ има прекид у 0.



(2) $f(x,y) = \sqrt[3]{xy}$, непр на домену? диф?

$D_f = \mathbb{R}^2$ домен

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{3}(xy)^{-2/3}\cdot y = \frac{1}{3}\,\frac{y}{\sqrt[3]{(xy)^2}},$$

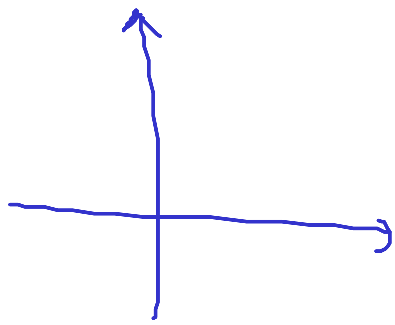
$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{3}\,\frac{x}{\sqrt[3]{(xy)^2}}$$

$x \neq 0$ и $y \neq 0 \Rightarrow \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ непр у (x,y)

(T) $\Rightarrow f$ диф у (x,y)

? шта се дешава за $x = 0, \; y \in \mathbb{R}$

и $y = 0, \; x \in \mathbb{R}$?



1° $x = 0, \; y = y_0 \in \mathbb{R}$

$$\frac{\partial f}{\partial x}(0, y_0) = \lim_{h\to 0} \frac{f(0+h, y_0) - f(0,y_0)}{h} =$$

$$= \lim_{h\to 0} \frac{\sqrt[3]{h\cdot y_0} - 0}{h} = \begin{cases} +\infty\cdot \operatorname{sgn} y_0, & y_0 \neq 0 \\ 0, & y_0 = 0 \end{cases}$$

$\Rightarrow \frac{\partial f}{\partial x}(0, y_0)$ не постоји за $y_0 \neq 0 \Rightarrow$ ф-ја није диф у $(0, \overset{\neq 0}{y_0})$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{\sqrt[3]{0h}-0}{h}=0$$

$$\Delta f((0,0),(h_1,h_2))=f(h_1,h_2)-\underset{0}{f(0,0)}=\sqrt[3]{h_1h_2}$$

$$?\quad \frac{\Delta f((0,0),(h_1,h_2))-\frac{\partial f}{\partial x}(0,0)h_1-\frac{\partial f}{\partial y}(0,0)h_2}{\|h\|}\underset{?}{\xrightarrow{h\to 0}}0\ ?\ \text{не важи!}$$

$$\frac{\sqrt[3]{h_1h_2}}{\sqrt{h_1^2+h_2^2}}\xrightarrow{h\to 0}0\ ?$$

$h_1=h_2=1/n$ $\quad \frac{\sqrt[3]{1/n^2}}{\sqrt{2/n^2}}=\frac{n}{\sqrt{2}\cdot n^{2/3}}=\frac{1}{\sqrt{2}}n^{1/3}\xrightarrow[n\to+\infty]{}+\infty$

$\left(\frac{1}{n},\frac{1}{n}\right)\to(0,0)$

$\Rightarrow f$ није диф у $(0,0)$!

(3) $f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$ диф? непр парц. изводи?

$$\frac{\partial f}{\partial x}(x_0,y_0)=2x_0\sin\frac{1}{x_0^2+y_0^2}+(x_0^2+y_0^2)\cos\frac{1}{x_0^2+y_0^2}\cdot\left(-\frac{2x_0}{(x_0^2+y_0^2)^2}\right)$$

$(x_0,y_0)\neq(0,0)$ $\rightarrow$ ово је непр као комп. таквих на $\mathbb{R}^2\setminus\{(0,0)\}$

$$\frac{\partial f}{\partial y}(x_0,y_0)=2y_0\sin\frac{1}{x_0^2+y_0^2}-\frac{2y_0}{(x_0^2+y_0^2)}\cos\frac{1}{x_0^2+y_0^2}$$

$\rightarrow$ ово је непр као малопре на $\mathbb{R}^2\setminus\{(0,0)\}$

$\Rightarrow f$ диф на $\mathbb{R}^2\setminus\{(0,0)\}$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{h^2\sin 1/h^2}{h}=\lim_{h\to 0}h\sin 1/h^2=0$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=0$$

$$\frac{\partial f}{\partial y}(x,y)=\begin{cases}2y\sin\frac{1}{x^2+y^2}-\frac{2y}{(x^2+y^2)}\cos\frac{1}{x^2+y^2}\\ 0, & (x,y)=(0,0)\end{cases}$$

$x_n=y_n=1/n$ $\quad (x_n,y_n)\xrightarrow[n\to\infty]{}(0,0)$

$$\frac{\partial f}{\partial y}(x_n, y_n) = \underbrace{\frac{2}{n}\sin(2n^2)}_{\downarrow\ 0} - \frac{2/n}{2/n^2}\cos(2n^2) \not\to 0 = \frac{\partial f}{\partial y}(0,0)$$

$\frac{2/n}{2/n^2} = n \to +\infty$, дивергира

$\frac{\partial f}{\partial y}(x,y)$ има прекид у $(0,0)$

f диф у $(0,0)$?

$$\Delta f((0,0),\underbrace{(h_1,h_2)}_{=h}) = f(h_1,h_2) - f(0,0) = (h_1^2+h_2^2)\sin\frac{1}{h_1^2+h_2^2} - 0$$

$$\frac{\Delta f(0,h) - \underbrace{\frac{\partial f}{\partial x}(0)}_{=0} h_1 - \underbrace{\frac{\partial f}{\partial y}(0)}_{=0} h_2}{\|h\| = \sqrt{h_1^2+h_2^2}} = \frac{(h_1^2+h_2^2)\sin\frac{1}{h_1^2+h_2^2}}{\sqrt{h_1^2+h_2^2}}$$

$$= \sqrt{h_1^2+h_2^2}\ \sin\frac{1}{h_1^2+h_2^2} \xrightarrow[h\to 0]{} 0$$

$\Rightarrow f$ диф у $(0,0)$!

(5) $$f(x,y) = \begin{cases} \sqrt{x^2+y^2}\,\sin y^2/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$ диф ?

f није непр на $\{0\}\times(0,+\infty)$, и $\{0\}\times(-\infty,0)$

$\Rightarrow f$ није диф на $\{0\}\times(\mathbb{R}\setminus\{0\})$

$x \neq 0$

$$\frac{\partial f}{\partial x}(x,y) = \frac{2x}{2\sqrt{x^2+y^2}}\sin y^2/x + \sqrt{x^2+y^2}\left(\cos y^2/x\right)\left(-\frac{y^2}{x^2}\right)$$

$\Rightarrow$ непр на $\mathbb{R}^2\setminus(\{0\}\times\mathbb{R})$

$$\frac{\partial f}{\partial y}(x,y) = \frac{2y}{2\sqrt{x^2+y^2}}\sin y^2/x + \sqrt{x^2+y^2}\cos y^2/x \cdot 2y/x$$

$\Rightarrow$ непр на $\mathbb{R}^2\setminus(\{0\}\times\mathbb{R})$

$\Rightarrow f$ диф на $\mathbb{R}^2\setminus(\{0\}\times\mathbb{R})$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\sqrt{h^2}\,\overbrace{\sin 0^2/h}^{=0} - 0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{\overset{=0}{f(0,h)} - \overset{=0}{f(0,0)}}{h} = 0$$

$$\frac{\Delta f((0,0),h) - \overset{=0}{\frac{\partial f}{\partial x}(0,0)}h_1 - \overset{=0}{\frac{\partial f}{\partial y}(0,0)}h_2}{\|h\|} = \frac{f(h_1,h_2)}{\sqrt{h_1^2+h_2^2}}$$

1° $h_1 = 0$ $\quad f(h_1,h_2) = f(0,h_2) = 0$

$$\frac{f(h_1,h_2)}{\|h\|} = \frac{0}{\|h\|} = 0 \xrightarrow[h_2\to 0]{} 0$$

2° $h_1 \neq 0$

$$\frac{f(h_1,h_2)}{\|h\|} = \frac{\cancel{\sqrt{h_1^2+h_2^2}}\sin\frac{h_2^2}{h_1}}{\cancel{\sqrt{h_1^2+h_2^2}}} = \sin\frac{h_2^2}{h_1} \xrightarrow[?]{h\to 0} 0 \quad \text{не важи}$$

$$(h_1,h_2)_n = \left(\frac{1}{n}, \frac{1}{\sqrt{n}}\right) \xrightarrow[n\to\infty]{} (0,0) \qquad \sin\frac{1/n}{1/n} = \sin 1 \not\xrightarrow[n\to\infty]{} 0$$

$\Rightarrow$ f није диф у $(0,0)$.

(6) $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ је непр. Испитати диф фје?

$\|x\| = \sqrt{x_1^2+x_2^2+\dots+x_n^2}$ непр као комп. таквих

$\|\cdot\|$ диф?, $f(x) = \|x\|$

$$\frac{\partial f}{\partial x_k}(x) = \frac{2x_k}{2\sqrt{x_1^2+\dots+x_n^2}} = \frac{x_k}{\|x\|}, \quad 1 \le k \le n$$

$\|0\| = 0 \Rightarrow \frac{\partial f}{\partial x_k}(0)$ није добро деф изразом

$\frac{\partial f}{\partial x_k}$ непр за $x \in \mathbb{R}^n \setminus \{0\}$, $1 \le k \le n$

(Т) $\Rightarrow$ f диф на $\mathbb{R}^n \setminus \{0\}$

$$\frac{\partial f}{\partial x_k}(0) = \lim_{h\to 0} \frac{f(0,\dots,\overset{k}{h},0,\dots) - f(0)}{h} = \lim_{h\to 0} \frac{\overset{\sqrt{0^2+\dots+h^2+0^2}}{\|(0,\dots,h,0,\dots,0)\|}}{h} = \lim_{h\to 0}\frac{|h|}{h}$$

$\Rightarrow \frac{\partial f}{\partial x_k}(0)$ не постоји $\Rightarrow$ f није диф у $\underline{\underline{0}}$.

За вежбу

Испитати непр. и диф. ф-је на њеном домену.

$$f(x,y) = \begin{cases} e^{\left(-\frac{x^3}{x^2+y^2}\right)}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$$