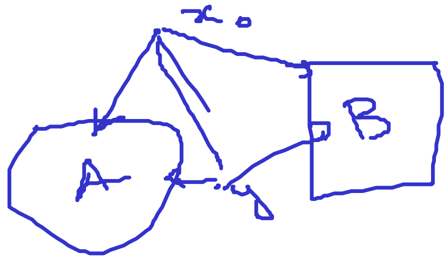
(1) (M,d) м.п. $A, B \neq \emptyset$, $A \cap B = \emptyset$

$f : M \to \mathbb{R}$ $\quad f(x) = |d(x,A) - d(x,B)|$



а) (M,d) комп $\Rightarrow \exists \min\limits_M f \leq d(A,B)$

б) (M,d) повезан $\Rightarrow \exists \min\limits_M f = 0$.

$g(x) = d(x,A) - d(x,B)$ непр ?

$x_0 \in M$, $\varepsilon > 0$ $\quad \delta(\varepsilon) = ?$ $\quad d(x_0,y) < \delta \Rightarrow |g(x_0) - g(y)| < \varepsilon$

$$|g(x_0) - g(y)| = |\underline{d(x_0,A)} - d(x_0,B) - (\underline{d(y,A)} - d(y,B))|$$

$$\leq \underbrace{|d(x_0,A) - d(y,A)|}_{\leq d(x_0,y)} + \underbrace{|d(y,B) - d(x_0,B)|}_{\leq d(x_0,y)} \leq 2\underbrace{d(x_0,y)}_{<\delta} < \varepsilon$$

$\delta = \varepsilon/2$

$d(x_0,a) \leq d(y,a) + d(x_0,y) \quad / \inf\limits_{a \in A}$

$d(x_0,A) \leq d(y,A) + d(x_0,y)$

$\Rightarrow g$ рав н непр $\Rightarrow$ $f(x) = |g(x)|$ непр на M

а) M комп $\Rightarrow \exists \min f, \max f$

$\Rightarrow \exists x_0 \in M$ $\quad \min f = f(x_0) \overset{?}{\leq} d(A,B)$

$$f(x_0) = |d(x_0,A) - d(x_0,B)|$$

$$d(A,B) = \inf_{\substack{a \in A \\ b \in B}} d(a,b) = \inf_{a \in A} d(a,B)$$

$f(x_0) \leq f(a)$, $a \in A$

$f(a) = |d(a,A) - d(a,B)| = d(a,B) \quad / \inf\limits_{a \in A}$, $\quad d(a,A) = 0$ ✓

$\Rightarrow f(x_0) \leq d(A,B)$

б) (M,d) повезан

? $\exists x_0 \in M$ $f(x_0) = 0$? $\quad$ знамо $f(x) = |g(x)| \geq 0$

$A \neq \emptyset \Rightarrow \exists a \in A$ $\quad g(a) = \underset{=0}{d(a,A)} - \underset{\geq 0}{d(a,B)} \leq 0$

$B \neq \emptyset \Rightarrow \exists b \in B$ $\quad g(b) = \underset{\geq 0}{d(b,A)} - \underset{=0}{d(b,B)} \geq 0$

M пов., g непр $\Rightarrow$ $\underset{\subseteq \mathbb{R}}{g(M)}$ пов $\Rightarrow g(M)$ интервал, $g(a), g(b) \in g(M)$

$\Rightarrow 0 \in g(M) \Rightarrow \exists x_0 \in M \quad g(x_0) = 0 \quad \Rightarrow 0 = |g(x_0)| = f(x_0)$

$\Rightarrow \exists \min f(x) = 0$.

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⑤ $f : M \to M$, (M, d) компактан

f изометрија ($\underline{d(f(x), f(y)) = d(x,y)}$, $x, y \in M$)

$\Rightarrow f$ бијекција , непр, f^{-1} непр.

$(\mathbb{N}, d)$, $d = |\cdot|$, $f(x) = x+1 \to$ изомет. $\to$ није НА

---

f непр ✓ (равн непр.)

f 1-1 :

ппс. f није 1-1 $\Rightarrow f(x) = f(y)$, $x \neq y$

f изомет $\Rightarrow \underbrace{d(f(x), f(y))}_{=0} = \underbrace{d(x,y)}_{>0}$ ↯

f НА :

ппс. $\exists x_0 \in M$, $x_0 \notin f(M)$, M компк $\Rightarrow f(M)$ комп.

$y_0 \in M$, $f(y_0) \in f(M)$

$y_0 \neq x_0$

$0 < d(y_0, x_0) = d(f(x_0), f(y_0)) = d(f(f(x_0)), f^2(y_0))$

$= \dots = d(\underbrace{f^n}_{\underbrace{f \circ \dots \circ f}_{n}}(x_0), f^n(y_0))$

$f^n(y_0) \to$ да ли конвергира ?

$f^n(y_0)$ низ у M , M компакт

$f^n(y_0)$ има конв подниз

$f^{n_k}(y_0) \to c^{\in M}$

$f^{n_k}(x_0)$ низ у M, M комп.

$\exists\ f^{n_{k_l}}(x_0)$ конв у M

$f^{n_{k_l}}(x_0) \to d$ $\qquad \boxed{d = c\ ?}$

$d(f^{n_{k_l}}(x_0), f^{n_{k_l}}(y_0)) \to d(d, c)$

$d(\underset{\{x_0\}}{\underset{\shortparallel}{x_0}}, f(M)) > 0$ → растојање два комп. скупа

$d(x_0, f(M)) = \underbrace{d(d, c)}_{=0}$

$x_0, f(x_0), \dots, f^n(x_0)$

$d(x_0, f^n(x_0)) > d(x_0, f(M))$

$d = d(x_0, f(M)) > 0$

$f^n(x_0) \to$ има конв подниз $f^{n_k}(x_0)$

$\forall m \in \mathbb{N}$

$d(x_0, f^m(x_0)) \geq d$

$\Downarrow$

$f^{n_k}(x_0)$ Кошијев низ

$\forall \varepsilon > 0\ \exists k_0\ \ n_k \geq n_{k_0}\ \ d(f^{n_k}(x_0), f^{n_{k_0}}(x_0)) < \varepsilon$

$\varepsilon < d$ фикс.

k фиксирано тд $n_k \geq n_{k_0}$

$m = n_k - n_{k_0}$

$d < d(x_0, f^m(x_0)) = d(f^{n_{k_0}}(x_0), f^{n_{k_0}}(f^m(x_0))) = d(f^{n_{k_0}}(x_0), f^{n_k}(x_0)) < \varepsilon$

④ $(C[0,1], d_\infty)$

$F: C[0,1] \to C[0,1]$

$F(f) = \frac{1}{5}\int_0^x \sin(x-t) f(t)\,dt$ → диференцијаб. ⇒ непр.

F контракција ? , $F(f) = f$ има јединствено реш.

$\left[\ \exists\, q \in [0,1)\quad d(F(f), F(g)) \le q\, d(f,g)\ \right]$

$$d_\infty(F(f), F(g)) = \max_{x\in[0,1]} |F(f)(x) - F(g)(x)|$$

$$|F(f)(x) - F(g)(x)| = \left|\frac{1}{5}\int_0^x \sin(x-t) f(t)\,dt - \frac{1}{5}\int_0^x \sin(x-t) g(t)\,dt\right|$$

$$= \frac{1}{5}\left|\int_0^x \sin(x-t)\,(f(t)-g(t))\,dt\right| \le \frac{1}{5}\int_0^x \underbrace{|\sin(x-t)|}_{\le 1}\,|f(t)-g(t)|\,dt$$

$$\le \frac{1}{5}\int_0^x \underbrace{|f(t)-g(t)|}_{\le d_\infty(f,g)}\,dt \le \frac{1}{5} d_\infty(f,g) \underbrace{\int_0^x dt}_{x} \qquad / \max_{x\in[0,1]}$$

$$d_\infty(f,g) = \max_{x\in[0,1]} |f(x)-g(x)|$$

$$\Rightarrow\ d_\infty(F(f), F(g)) \le \underbrace{\boxed{\frac{1}{5}}}_{q\in[0,1)} d_\infty(f,g) \ \Rightarrow\ F \text{ је контракција}$$

$f \equiv 0$

$$F(f)(x) = \frac{1}{5}\int_0^x 0\cdot\sin(x-t)\,dt = \frac{1}{5}\int_0^x 0\,dt = 0 = f(x)$$

$\Rightarrow f \equiv 0$ фиксна тачка

ппс. $\exists\, f \not\equiv 0$ тд $F(f) = f$ $\Rightarrow$ фиксна т. је јединствена

$$\underbrace{d_\infty(\underbrace{F(f)}_{f}, \underbrace{F(0)}_{0})}_{d_\infty(f,0)} \le \frac{1}{5}\underbrace{d_\infty(f,0)}_{>0} \quad ↯$$

Т о фиксној тачки:

(M,d) комплетан

$f: M \to M$ контракција $\left(d(f(x),f(y)) \le q\, d(x,y),\ q \in (0,1) \right)$

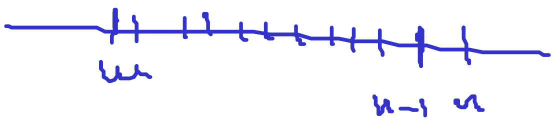
$\Rightarrow f$ има јединствену фиксну тачку

$? \ \exists x_0 \in M \quad f(x_0) = x_0$

$y \in M \quad f^n(y)$ низ

$$(*) \quad d(\underbrace{f^n(y)}_{=f^{n-1}(f(y))}, \underbrace{f^{n+1}(y)}_{=f^n(f(y))}) \le q\, d(f^{n-1}(y), f^n(y)) \le \dots \le q^n d(y, f(y))$$

$f^n(y)$ је Кошијев:



$\varepsilon > 0$

$n, m \ge n_0 \quad d(f^n(y), f^m(y)) < \varepsilon$

$$n > m \quad d(f^n(y), f^m(y)) \le d(f^n(y), f^{n-1}(y)) + d(f^{n-1}(y), f^{n-2}(y)) + \dots + d(f^{m+1}(y), f^m(y))$$

$$\overset{(*)}{\le} q^{n-1} d(y,f(y)) + q^{n-2} d(y,f(y)) + \dots + q^m d(y,f(y))$$

$$= \boxed{\sum_{k=m}^{n-1} q^k} \cdot d(y,f(y)) \le \underbrace{\sum_{k=m}^{\infty} q^k d(y,f(y))}_{\substack{\text{остатак конв. реда} \\ \to 0 \\ m \to \infty}}$$

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q} < \infty, \quad q \in (0,1)$$

за $m \ge n_0 \Rightarrow \sum_{k=m}^{\infty} q^k d(y,f(y)) < \varepsilon$

$\Rightarrow f^n(y)$ је Кошијев $\underset{M \text{ комплетан}}{\Rightarrow}$ $f^n(y)$ конв ка x_0

$\lim\limits_{n\to\infty} f^n(y) = x_0 \ /\ f$ непр $\Rightarrow \lim\limits_{n\to\infty} f^{n+1}(y) = f(x_0)$, $\lim\limits_{n\to\infty} f^{n+1}(y) = x_0$

$\Rightarrow x_0 = f(x_0)$. $\to$ јединственост за вежбу.

Фје са више променљивих

$f:(\mathbb{R}^n, d_p) \to (\mathbb{R}^m, d_q)$

$\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

$(x_1, \dots, x_n)$

f непр у x_0 $\Rightarrow$ $\forall \varepsilon > 0 \ \exists \delta > 0$ $\quad d_p(x_0, x) < \delta \Rightarrow d_q(f(x_0), f(x)) < \varepsilon$

f непр $\Leftrightarrow$ $\forall U \in \mathbb{R}^m$ отв. (затв.) $\Rightarrow f^{-1}(U)$ отв (затв.) у $\mathbb{R}^n$

$f(x) = (f_1(x), f_2(x), \dots, f_m(x))$

f непр. $\Leftrightarrow$ $f_1, f_2, \dots, f_m$ непр. $\quad f_k : (\mathbb{R}^n, d_p) \to (\mathbb{R}, |\cdot|)$, $1 \le k \le m$

f непр у x_0 $\overset{\text{Хајне}}{\Leftrightarrow}$ $\forall$ низ $x_n \xrightarrow[n\to\infty]{} x_0 \Rightarrow f(x_n) \xrightarrow[n\to\infty]{} f(x_0)$

f непр у x_0 $\Rightarrow$ $\lim\limits_{x\to x_0} f(x) = f(x_0)$.

$f: \mathbb{R}^2 \to \mathbb{R}$

$$\lim_{x\to a}\lim_{y\to b} f(x,y) \ne \lim_{y\to b}\lim_{x\to a} f(x,y) \ne \lim_{(x,y)\to(a,b)} f(x,y)$$

1) $f(x,y) = \dfrac{x-y}{x+y}$, $\quad \lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y) = ?$

$\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y) = ?$

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = ?$

I $\quad \lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} \dfrac{x-y}{x+y}\right) = \lim\limits_{x\to 0} 1 = 1 \quad \longrightarrow (0, 1/n)$

$\lim\limits_{y\to 0}\lim\limits_{x\to 0} \dfrac{x-y}{x+y} = \lim\limits_{y\to 0} -1 = -1 \quad \longrightarrow (1/n, 0)$

$$f\left(0,\tfrac{1}{n}\right) = \frac{0-\frac{1}{n}}{0+\frac{1}{n}} = -1 \to -1, \quad \left(0,\tfrac{1}{n}\right) \to (0,0)$$

$$f\left(\tfrac{1}{n},0\right) = \frac{\frac{1}{n}-0}{\frac{1}{n}+0} = 1 \to 1, \quad \left(\tfrac{1}{n},0\right) \to (0,0)$$

$\Rightarrow$ не постоји $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

(2) $f(x) = x + y\sin\frac{1}{x}, \quad x \neq 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \underset{\to 0}{x} + \overset{\to 0}{y}\,\underbrace{\sin\tfrac{1}{x}}_{\text{огран}} = 0$$

$\to$ формално исписати за вежбу

$$\lim_{x\to 0}\lim_{y\to 0}\left(x + y\underbrace{\sin\tfrac{1}{x}}_{\text{огран}}\right) = \lim_{x\to 0} x = 0$$

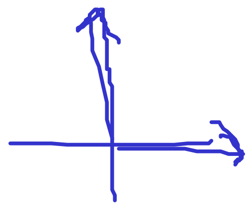
$$\lim_{y\to 0}\underbrace{\lim_{x\to 0}\left(\underset{\to 0}{x} + y\underbrace{\sin\tfrac{1}{x}}_{\text{дивергира}}\right)}_{\text{не постоји}}$$

$\to$ не постоји!

(3) $f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

непр дуж сваке праве кроз $(0,0)$ али неће бити непр у тачки

$\left.\begin{array}{l} y = kx,\ k\in\mathbb{R} \\ x = 0 \end{array}\right\}$ праве кроз $(0,0)$



$f(x,kx) = \begin{cases} \dfrac{x^2\,kx}{x^4+k^2x^2}, & x\neq 0 \\ 0, & x=0 \end{cases}$ $\to$ ово је непр у 0

$f(0,y) = \begin{cases} 0, & y\neq 0 \\ 0, & y=0 \end{cases}$ $\to$ непр.

f није непр у $(0,0)$: $\quad \left(\tfrac{1}{n},\tfrac{1}{n^2}\right) \xrightarrow{n\to\infty} (0,0)$

$f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$, $\quad x = \frac{1}{n},\ y = \frac{1}{n^2}$

$$f\left(\tfrac{1}{n},\tfrac{1}{n^2}\right) = \frac{\frac{1}{n^2}\cdot\frac{1}{n^2}}{\frac{1}{n^4}+\frac{1}{n^4}} = \frac{1}{2} \neq 0$$

$\Rightarrow$ f има прекид у $(0,0)$