1. $\sup A, \inf A, \min A, \max A = ?$

a) $A_1 = \{ n + \frac{10}{n} : n \in \mathbb{N} \}$

$$n + \frac{10}{n} \underset{AG}{\geq} 2\sqrt{n \cdot \frac{10}{n}} = 2\sqrt{10}$$

$$a_n = n + \frac{10}{n}$$

$$a_{n+1} - a_n = (n+1) + \frac{10}{n+1} - n - \frac{10}{n} = 1 + \frac{10n - 10n - 10}{n(n+1)} =$$

$$= \frac{n^2 + n - 10}{n(n+1)} > 0, \quad n > x_2$$

$n \in \mathbb{N}$

$$x^2 + x - 10 = 0 \iff x_{1,2} = \frac{-1 \pm \sqrt{1+40}}{2} = \frac{-1 \pm \sqrt{41}}{2}, \quad x_1 = \frac{-1-\sqrt{41}}{2} < 0 < n$$

$$2 < x_2 = \frac{\sqrt{41}-1}{2} < 3$$

$n \geq 3 \Rightarrow a_{n+1} \geq a_n$

$$a_1 = 1 + \frac{10}{1} = 11, \quad a_2 = 2 + \frac{10}{2} = 7, \quad \boxed{a_3 = 3 + \frac{10}{3} = \frac{19}{3}}, \quad a_4 = 4 + \frac{10}{4} = 4 + \frac{5}{2} = \frac{13}{2}$$

$$\min A = \frac{19}{3}$$

$a_n \uparrow, \; n \geq 3 \qquad \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n + \frac{1}{n} = +\infty \Rightarrow \sup A = „+\infty" \; (y\ \overline{\mathbb{R}})$

не постоји у $\mathbb{R}$

$\Rightarrow$ max не постоји

б) $A_2 = \{ x + \frac{10}{x}, \; x \in \mathbb{Q}^+ \}$

$\Rightarrow A_1 \subseteq A_2 \Rightarrow \sup A_2 = \sup A_1 = +\infty \; (y\ \overline{\mathbb{R}}) \Rightarrow$ не постоји у $\mathbb{R}$

$$x + \frac{10}{x} \geq 2\sqrt{x \cdot \frac{10}{x}} = 2\sqrt{10} \Rightarrow \inf A_2 \geq 2\sqrt{10}$$

$$x, \frac{10}{x} > 0 \Rightarrow (\sqrt{x})^2 + \left(\frac{\sqrt{10}}{\sqrt{x}}\right)^2 - 2\sqrt{10} \cdot \frac{1}{\sqrt{x}} \cdot \sqrt{x} = \left(\sqrt{x} - \frac{\sqrt{10}}{\sqrt{x}}\right)^2 \geq 0$$

„=" акко $\sqrt{x} = \frac{\sqrt{10}}{\sqrt{x}}, \quad x = \frac{10}{x}$

$f(x) = x + \frac{10}{x} \to \min$ $\qquad x^2 = 10, \quad \underline{\underline{x = \sqrt{10} \notin \mathbb{Q}}}$

$\mathbb{Q}$ густ у $\mathbb{R} \Rightarrow \exists\, x_n \in \mathbb{Q}^+ \quad x_n \to \sqrt{10}, \; n \to \infty$

$$x_n + \frac{10}{x_n} \xrightarrow[n\to\infty]{} \sqrt{10} + \frac{10}{\sqrt{10}} = 2\sqrt{10}.$$

$$\Rightarrow \inf A_2 = 2\sqrt{10}.$$

2. $x>0$, $a_0>0$

$$a_n = \frac{1}{2}\left(a_{n-1}+\frac{x}{a_{n-1}}\right),\ n\in\mathbb{N} \quad \overset{\text{ПМИ}}{\Longrightarrow} \quad a_n>0,\ n\in\mathbb{N}$$

а) a_n конв?, $\lim\limits_{n\to\infty} a_n = ?$

$$a_n - a_{n-1} = \frac{1}{2}\left(a_{n-1}+\frac{x}{a_{n-1}}\right) - a_{n-1} = \frac{1}{2}\left(\frac{x}{a_{n-1}} - a_{n-1}\right) = \frac{1}{2}\,\frac{x-a_{n-1}^2}{a_{n-1}}$$

$a_n \geq a_{n-1} \iff a_{n-1}^2 \leq x$, $a_{n-1}\leq\sqrt{x}$, $\quad a_{n-1}\leq\sqrt{x} \Rightarrow a_{n-1}\leq a_n \leq \sqrt{x}$?

$\boxed{a_n \leq a_{n-1} \iff a_{n-1}^2 \geq x,\ a_{n-1}\geq\sqrt{x}}$, $\quad a_{n-1}\geq\sqrt{x} \Rightarrow a_n\leq a_{n-1}$

$(a_n \geq \sqrt{x})$?

$$\Rightarrow a_n = \frac{1}{2}\Big(\underset{>0}{a_{n-1}} + \underbrace{\frac{x}{a_{n-1}}}_{>0}\Big) \underset{АГ}{\geq} \frac{1}{2}\cdot 2\sqrt{a_{n-1}\cdot\frac{x}{a_{n-1}}} = \sqrt{x},\ n\in\mathbb{N}$$

$n\in\mathbb{N}$ $\quad \underline{\underline{a_n\geq\sqrt{x}}} \iff a_n\downarrow$ и $a_n\geq\sqrt{x} \overset{\text{т. мон. конв}}{\Longrightarrow}$ a_n конвергира

$$\lim_{n\to\infty} a_n = a = \lim_{n\to\infty}\frac{1}{2}\left(a_{n-1}+\frac{x}{a_{n-1}}\right) = \frac{1}{2}\left(a+\frac{x}{a}\right) \Rightarrow a=\sqrt{x}.$$

$$\Rightarrow \lim_{n\to\infty} a_n = \sqrt{x}.$$

б)
$$\lim_{n\to\infty}\left(e^{\frac{1}{2}\left(\frac{a_n}{n}\right)^2} - \frac{1}{\cos\frac{a_n}{n}}\right)n^4 = (0\cdot\infty) = ?$$

$$\lim_{n\to\infty}\frac{\overset{\to\sqrt{x}>0}{a_n}}{n} = 0, \quad \frac{a_n}{n} = t$$

$$e^{\frac{1}{2}t^2} = 1+\frac{1}{2}t^2+\frac{1}{2}\cdot\left(\frac{1}{2}t^2\right)^2 + o\left(\left(\frac{1}{2}t^2\right)^2\right) \underset{\frac{a_n}{n}}{=} 1+\frac{1}{2}\frac{a_n^2}{n^2}+\frac{1}{8}\frac{a_n^4}{n^4}+o\left(\frac{1}{n^4}\right)$$

$$\ulcorner\ e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+o(x^3)\ \lrcorner$$

$$\ulcorner \cos x = 1-\frac{x^2}{2}+\frac{x^4}{24}+o(x^5)\ \lrcorner$$

$$\cos\frac{a_n}{n} = 1-\frac{a_n^2}{2n^2}+\frac{a_n^4}{24n^4}+\underset{o\left(\frac{1}{n^5}\right)}{\underset{\parallel}{o\left(\frac{a_n^5}{n^5}\right)}}$$

$$\frac{1}{\cos\frac{a_n}{n}} = \left(\cos\frac{a_n}{n}\right)^{-1} = \Big(1\underbrace{-\frac{a_n^2}{2n^2}+\frac{a_n^4}{24n^4}+o\left(\frac{1}{n^5}\right)}_{x}\Big)^{-1} =$$

$$\ulcorner\ (1+x)^{-1} = 1-x+x^2-x^3+x^4+o(x^4)\ \lrcorner$$

$$= 1 - \left(-\frac{a_n^2}{2n^2} + \frac{a_n^4}{24n^4} + o\left(\frac{1}{n^5}\right)\right) + \left(-\frac{a_n^2}{2n^2} + \frac{a_n^4}{24n^4} + o\left(\frac{1}{n^5}\right)\right)^2$$

$$+ o\left(\frac{a_n^4}{n^4}\left(-\frac{1}{2} + \frac{a_n^2}{24n^2} + o\left(\frac{1}{n^3}\right)\right)^2\right)$$

$$o\left(\frac{1}{n^4}\right)$$

$$= 1 + \frac{a_n^2}{2n^2} - \frac{a_n^4}{24n^4} + \frac{1}{4}\frac{a_n^4}{n^4} + o\left(\frac{1}{n^4}\right).$$

$$\lim_{n\to\infty}\left(e^{\frac{1}{2}\frac{a_n^2}{n^2}} - \frac{1}{\cos\frac{a_n}{n}}\right)n^4 =$$

$$= \lim_{n\to\infty}\left(\left(\cancel{1} + \cancel{\frac{a_n^2}{2n^2}} + \frac{1}{8}\frac{a_n^4}{n^4} + o\left(\frac{1}{n^4}\right)\right) - \left(\cancel{1} + \cancel{\frac{a_n^2}{2n^2}} - \frac{a_n^4}{24n^4} + \frac{a_n^4}{4n^4} + o\left(\frac{1}{n^4}\right)\right)\right)n^4$$

$$\left[\ \frac{1}{8} + \frac{1}{24} - \frac{1}{4} = \frac{3+1-6}{24} = -\frac{1}{12}\ \right]$$

$$= \lim_{n\to\infty}\left(-\frac{1}{12}\cdot\frac{a_n^4}{n^4} + o\left(\frac{1}{n^4}\right)\right)n^4 = -\frac{1}{12}\cdot x^2$$

3. $$f(x) = \begin{cases} \sqrt[5]{x^5 - 2x^4}, & x \le 2 \\ 8\operatorname{arctg}\frac{1}{x} + \ln(x^2+1), & x > 2 \end{cases}$$

а) непрекидност: f непрекидна као композиција таквих на $(-\infty,2)\cup(2,+\infty)$

$x=2$ $\quad \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\sqrt[5]{x^5-2x^4} = \sqrt[5]{2^5 - 2\cdot 2^4} = 0$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 8\operatorname{arctg}\frac{1}{x} + \ln(x^2+1) = \underset{>0}{8\operatorname{arctg}\frac{1}{2}} + \underset{>0}{\ln 5} \neq 0$$

$\Rightarrow f$ има прекид у 2.

диференцијабилност:

f има прекид у 2 $\Rightarrow f$ није диференцијабилна у 2.

$x < 2$

$$f'(x) = \left((x^5 - 2x^4)^{1/5} \right)' = \frac{1}{5} (x^5 - 2x^4)^{-4/5} \cdot (5x^4 - 8x^3) =$$

$$= \frac{5x^4 - 8x^3}{5(x^5 - 2x^4)^{4/5}} , \quad x \neq 0, \quad x < 2$$

$$f'_+(0) = \lim_{\substack{h \to 0^+ \\ h < 2}} \frac{f(h) - \overset{=0}{f(0)}}{h} = \lim_{h \to 0^+} \frac{\sqrt[5]{h^5 - 2h^4}}{h} = \lim_{h \to 0^+} \frac{h \sqrt[5]{1 - \frac{2}{h}}}{h} \to +\infty = -\infty$$

$f'_-(0) = +\infty$

$\Rightarrow$ f није диф у 0.

$$x > 2 \qquad f'(x) = \left(8 \operatorname{arctg} \frac{1}{x} + \ln(x^2+1) \right)' = 8 \cdot \frac{1}{1 + \left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right) + \frac{1}{x^2+1} \cdot 2x =$$

$$= -\frac{8}{x^2+1} + \frac{2x}{x^2+1} = \frac{2x-8}{x^2+1}$$

$\Rightarrow$ f је диф на $(-\infty, 0) \cup (0,2) \cup (2, +\infty)$.

в) ток и график

1) $D_f = \mathbb{R}$

$$f(x) = \begin{cases} \sqrt[5]{x^5 - 2x^4}, & x \leq 2 \\ \underbrace{8 \operatorname{arctg} \underset{\vee\, 0}{\frac{1}{x}}}_{\vee\, 0} + \underbrace{\ln(x^2+1)}_{\vee\, 0}, & x > 2 \end{cases}$$

$f(x) > 0 \qquad x > 2$

$f(x) = 0 \iff x^5 - 2x^4 = 0 \iff x = 0, \; x = 2$

$x^4(x-2) \to f(x) < 0, \; x < 2$.

f није ни непарна, ни парна, ни периодична.

2° асимптоте:

$\to$ немамо вертикалне асимптоте из непрекидности под а)

$x \to -\infty$

$$f(x) = \sqrt[5]{x^5 - 2x^4} = x \sqrt[5]{1 - \frac{2}{x}} = x \Big(1 - \underbrace{\frac{2}{x}}_{\to 0^-} \Big)^{1/5} = x \left(1 + \frac{1}{5}\left(-\frac{2}{x}\right) + o\left(\frac{1}{x}\right)\right)$$

$$= x - \frac{2}{5} + o(1), \quad x \to -\infty$$

$\Rightarrow$ коса асимптота $y = x - \frac{2}{5}, \; x \to -\infty$

$x \to +\infty$

$$f(x) = 8\underbrace{\operatorname{arctg}\frac{1}{x}}_{0} + \ln(x^2+1) \sim \ln x^2 \sim 2\ln x$$

$$\left.\begin{array}{l} \dfrac{\ln x}{x} \to 0, \ x\to+\infty; \quad \ln x \to +\infty, \ x\to+\infty \\ \operatorname{arctg}\dfrac{1}{x} \to 0, \ x\to+\infty \end{array}\right\} \Rightarrow \text{нема косу ни хоризонталну}$$

$$\lim_{x\to+\infty} f(x) = +\infty$$

$$\lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{8\operatorname{arctg}\frac{1}{x} + \ln(x^2+1)}{x} = 0$$

3° монотоност:

$x \neq 0, \ x \neq 2$

$$f'(x) = \begin{cases} \dfrac{\overbrace{5x^4-8x^3}^{=x^3(5x-8)}}{5(x^5-2x^4)^{4/5}}, & x\neq 0, \ x<2 \\[2ex] \dfrac{2x-8}{x^2+1}, & x>2 \end{cases}$$

$f'(x)=0, \quad x = \frac{8}{5} < 2$

$f'(x)=0, \quad x = 4 > 2$

$\frac{5x^4-8x^3}{5(x^5-2x^4)^{4/5}}$ $\qquad\qquad$ $\frac{2x-8}{x^2+1}$

		0		8/5		2		4	
f'	+		−		+		−		+
f	↗		↘		↗		↘		↗

4° конвексност:

$x<2, \quad x\neq 0$

$$f''(x) = \left(\frac{5x^4-8x^3}{5(x^5-2x^4)^{4/5}}\right)' = \frac{1}{5}\,\frac{(20x^3-24x^2)\cdot(x^5-2x^4)^{4/5} - \frac{4}{5}(x^5-2x^4)^{-1/5}\cdot(5x^4-8x^3)^2}{(x^5-2x^4)^{8/5}}$$

$$= \frac{1}{5}\,\frac{(20x^3-24x^2)(x^5-2x^4) - \frac{4}{5}(25x^8-80x^7+64x^6)}{(x^5-2x^4)^{9/5}}$$

$$= \frac{1}{5}\,\frac{\cancel{20x^8} - \cancel{24x^7} - \cancel{40x^7} + 48x^6 - \cancel{20x^8} + \cancel{64x^7} - \overbrace{\frac{256}{5}}^{=\left(51+\frac{1}{5}\right)}x^6}{(x^5-2x^4)^{9/5}}$$

$$= \frac{1}{5}\,\frac{\left(-3-\frac{1}{5}\right)\cdot x^6}{(x^5-2x^4)^{9/5}}$$

$$x>2 \qquad f''(x) = \left(\frac{2x-8}{x^2+1}\right)' = \frac{2(x^2+1) - 2x(2x-8)}{(x^2+1)^2} = \frac{-2x^2+16x+2}{(x^2+1)^2} =$$

$$= -2\,\frac{x^2-8x-1}{(x^2+1)^2}, \qquad x^2-8x-1=0 \iff x_{1,2}=\frac{8\pm\sqrt{64+4}}{2}$$

$$= 4\pm\sqrt{17}$$

$x_1 = 4-\sqrt{17} < 2$

$x_2 = 4+\sqrt{17}$

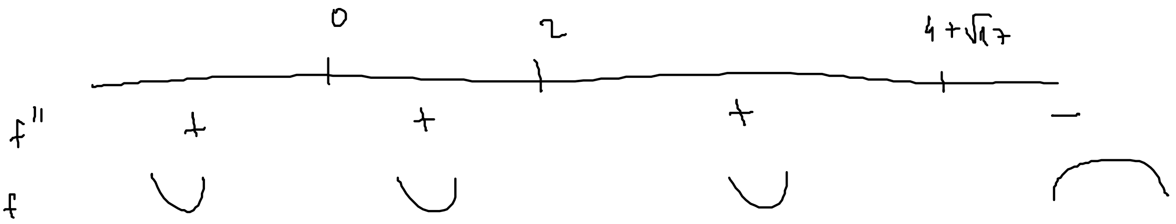
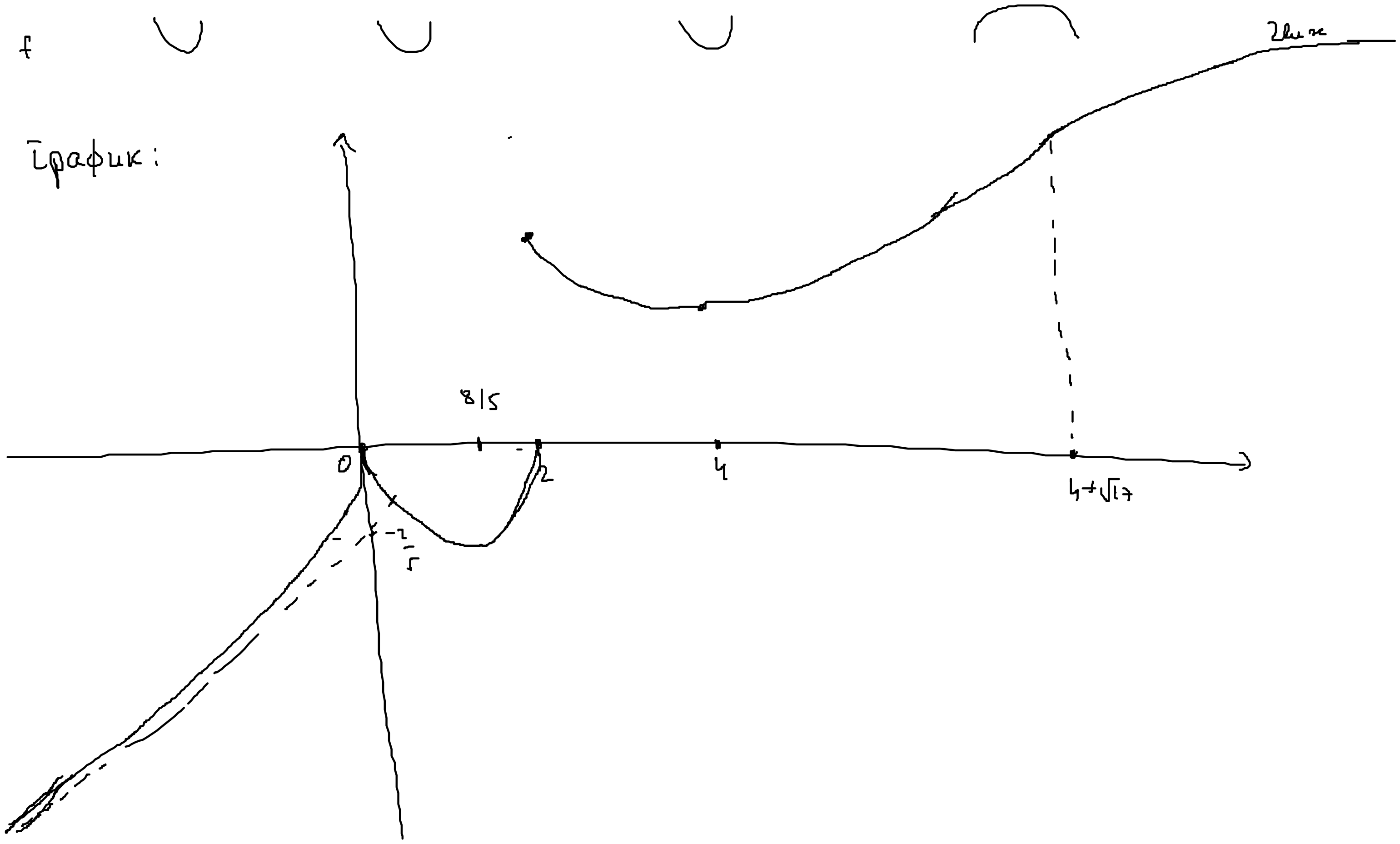


График:



④ $f : [0,1] \to [0,1]$ НА, диференцијабилна

$\overset{?}{\Rightarrow} \exists\, x_1 \neq x_2$, $x_1, x_2 \in (0,1)$: $|f'(x_1)| = |f'(x_2)| = 1$.

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f НА $\Rightarrow \exists\, a,b \in [0,1]$ $f(a)=0$, $f(b)=1$ $\Rightarrow a,b$ мин, макс. ф-је f

$\forall x \in [0,1]$ $0 \le f(x) \le 1$

$\Rightarrow a,b$ лок. мин и макс. ф-је f $\Big\}$ $\overset{\text{Ферма}}{\Rightarrow}$ $f'(a)=f'(b)=0$

f диф у a и b

$f(a)=0$, $f(b)=1$, f диф на $(0,1)$ $\overset{\text{Лагранж}}{\Rightarrow}$ $\exists\, c \in (a,b)$ $f'(c) = \dfrac{f(b)-f(a)}{b-a} = \dfrac{1}{b-a}$

$a, b \in (0,1) \Rightarrow |a-b| = |b-a| < 1$

$|f'(c)| = \frac{1}{|b-a|} > 1$

WLOG $b > a$ $\quad f'(c) > 1 \quad , \quad f'(a) = 0 \quad , \quad f'(b) = 0$

Дарбу $\Rightarrow \exists\, x_1 \in (a, c) \quad f'(x_1) = 1$

$\exists\, x_2 \in (c, b) \quad f'(x_2) = 1$

$(a,c) \cap (c,b) = \phi \Rightarrow |f'(x_1)| = |f'(x_2)| = 1 \quad , \quad x_1 \neq x_2$