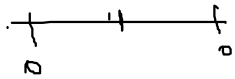
$f : [0,1] \to \mathbb{R}$, $f \in C^2_{[0,1]}$, $f(0)=0$, $f(1)=1$ $\to \exists \xi$ $f'(\xi) = \frac{f(1)-f(0)}{1-0} = \underline{\underline{1}}$

$f'(0) = f'(1) = 0$

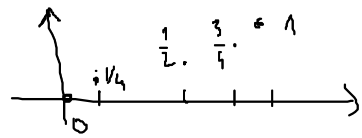


$f'(\xi) = 1$

? $f' \geq 2$ $\quad \frac{1}{4} \quad \frac{1}{2} \quad \frac{3}{4}$

? $\exists x_0 \in [0,1]$ $\quad \underbrace{|f''(x_0)| \geq 4}$?

$\exists \xi \quad |f'(\xi)| > 2$



nnc.

$\forall x \quad |f''(t)| < 4$

$-4 < f''(t) < 4 \quad / \int_0^x dt \quad / \int_x^1 dt$

$-4(1-x) < \underset{0}{f'(1)} - f'(x) < 4(1-x) \quad / \cdot (-1)$

$\underbrace{\int_0^x -4\,dt} < \int_0^x f''(t)\,dt < 4\int_0^x dt$

$-4(1-x) < f'(x) < 4(1-x) \quad / \int_u^1 dx$

$1 - f(u) < 4x\Big|_u^1 - 2x^2\Big|_u^1$

$-4x < f'(x) - \underset{0}{f'(0)} < 4x$

$|f'(x)| < \min\{\underbrace{4(1-x), 4x}_{\leq 2}\}$

$-4x < f'(x) < 4x \quad / \int_0^u dx \quad / \int_u^1 dx$

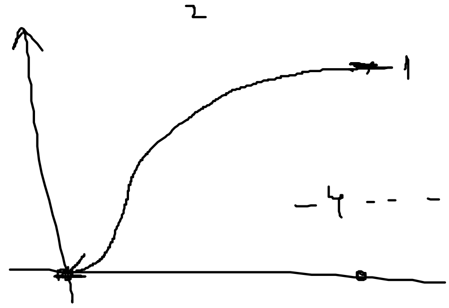
$\boxed{|f'(x)| < 2}$

$-4\underset{\frac{u^2}{2}}{\int_0^u x\,dx} < f(u) - \underset{0}{f(0)} < 4\underset{\frac{u^2}{2}}{\int_0^u x\,dx}$

$\longrightarrow \quad -2u^2 < f(u) < 2u^2$, $u \in [0,1]$

$u = 1$

$-2 < \underset{1}{f(1)} < 2$



$-4 \cdots < \underset{1}{f(1)} - f(u) < 4\underbrace{\int_u^1 x\,dx}_{\frac{x^2}{2}\Big|_u^1} = 2 - 2u^2$, $u \neq 1$

$-2 + 2u^2 < 1 - f(u) < 2 - 2u^2$, $u \in (0,1)$

$2u^2 - 1 < f(u) < 3 - 2u^2$

$-2u^2 < f(x) < 2u^2$

$-2x^2 + 4x - 2 < 1 - f(x) < 4(1-x) - 4 \cdot \frac{1-x^2}{2} = 2 - 4x + 2x^2$

$-2x^2 + 4x - 3 < -f(x) < 1 - 4x + 2x^2$

$f(x) < \min\{2x, \underbrace{2x^2 - 4x + 3}_{2x-4}\}$

$-2x^2 + 4x - 1 < f(x) < \underbrace{2x^2 - 4x + 3}_{g(x) - g(0)} \quad / \int_u^1 d4x$

$f(0) = 0$ $\quad g(0) = 3$

$< g'(x)$

$f(x)$

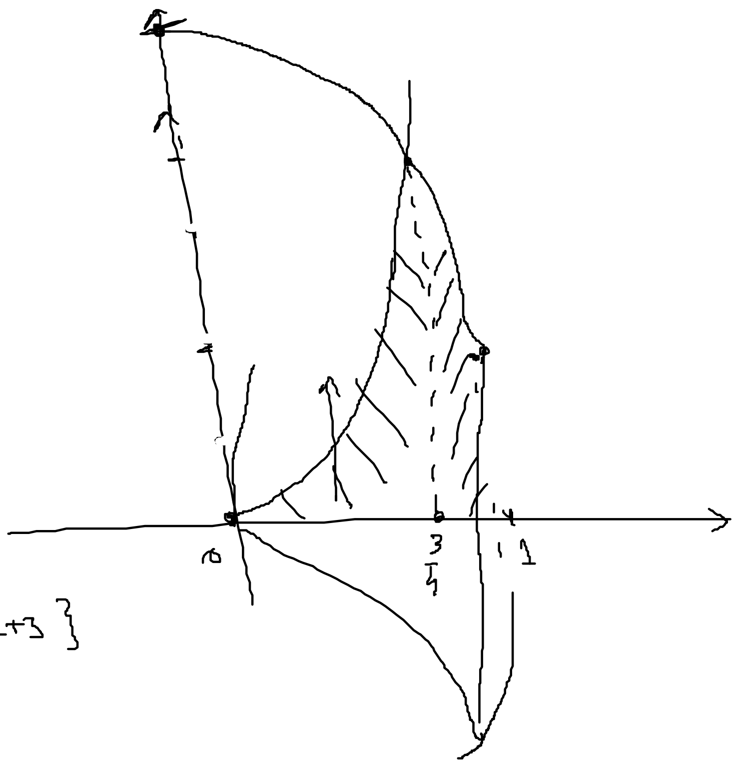
$|f'(x)| < 4x$

$|f'(x)| < 1$

$f(0)=0$, $f(1)=1$ $f'(0)=f'(1)=0$

$-2x^2+4x-1 < f(x) < \underbrace{2x^2-4x+3}$

$2(x-1)^2+1$



$-2x^2 < f(x) < 2x^2$

$f(x) < \min\{2x^2, 2x^2-4x+3\}$

$\cancel{2x^2} = \cancel{2x^2}-4x+3$

$\boxed{x=\frac{3}{4}}$

$\int_0^x \underbrace{f'(t)}\,dt = f(x)-\underset{0}{f(0)} < \int_0^x \underbrace{4t}\,dt = 2x^2 - \underline{\quad}$

$f'(t) < 4t$ go $x=3/4$

$x=\frac{3}{4}$ $f'\left(\frac{3}{4}\right) < 3$ ✓

$f(0)=0=f'(0)=f'(1)$

$f(1)=1$

$? \exists x_0 \; |f''(x_0)| \geq 4$?

ПНС. $-4 < f''(x) < 4$ $/ \int_0^u dx$ $/ \int_u^1 dx$

$-4u < f'(u) < 4u$

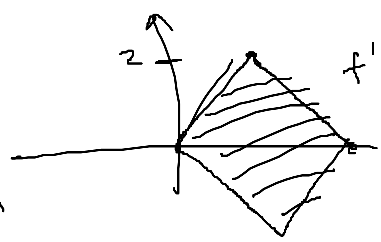
$-4(1-u) < -f'(u) < 4(1-u)$

$f'(x) < 4(1-x)$ $x \in (0,1)$

$-\min\{4x, 4(1-x)\} < f'(x) < \min\{4x, 4(1-x)\}$ $/ \int_0^1 du$

$< \underbrace{f(1)-f(0)}_{1} < \int_0^{1/2} 4x\,dx + \int_{1/2}^1 4(1-x)\,dx$ $\int_0^1 \min\{4x, 4(1-x)\}\,du$

$4 \cdot \frac{x^2}{2}\Big|_0^{1/2} + 4\cdot\frac{1}{2} - 4\cdot\frac{x^2}{2}\Big|_{1/2}^1 = \int_0$



$$4 \cdot \frac{\frac{1}{4}}{2} + x - 2 + 4 \cdot \frac{\frac{1}{4}}{2} = 1 \quad ✓$$

- $f \in C^1[a,b]$, $f(a)=0$ $\Rightarrow$ $\frac{2}{(b-a)^2} \int_a^b |f(x)|\,dx \le \max_{x\in[a,b]} |f'(x)|$

$$\int_a^b |f(x)|\,dx \overset{?}{\le} \underbrace{\frac{(b-a)^2}{2} \max |f'(x)|}$$

$$|f(x) - \underset{0}{\underset{\shortparallel}{f(a)}}| = \left|\int_a^x f'(t)\,dt\right| \le \max_{[a,x]} |f'(t)| \cdot \left|\underset{x-a}{\underset{\shortparallel}{\int_a^x dt}}\right| \quad \Big/ \int_a^b dx$$

$$\int_a^b |f(x)|\,dx \le \max|f'(t)| \cdot \int_a^b (x-a)\,dx = \frac{(b-a)^2}{2}$$

- $f, g : [a,b] \to \mathbb{R}$, $D(a,b) \cap C[a,b]$, $f', g' \ge 0$, $\uparrow$, $C(a,b)$ $\overset{?}{\Rightarrow} \exists c : \underset{f'(\xi_1)}{\frac{f(b)-f(a)}{b-a}} \cdot \underset{g'(\xi_2)}{\frac{g(b)-g(a)}{b-a}} = f'(c)g'(c)$

$f'(\xi_1) \cdot g'(\xi_2) = A$

Лагранж

Коши је Т. $\quad \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$

$f', g' \uparrow \ge 0$ $\quad \overset{\ge 0}{f'(\xi_1)}\, g'(\xi_1) \le f'(\xi_1)\boxed{g'(\xi_2)} \le f'(\xi_2)\boxed{g'(\xi_2)}$

WLOG $\xi_1 < \xi_2$

$F(x) = f'(x)g'(x) \in C(a,b)$

$F(\xi_1) \le A \le F(\xi_2) \Rightarrow \exists c : F(c) = A$

Вајерштрас на F $\qquad F(c) = f'(c) \cdot g'(c)$

$$\int_0^\infty \frac{\sin x}{x} \ln^p(2+x)\, dx$$

$p > -1 \qquad -1 < p \leq 0$

$\frac{|\sin x|}{x} \ln^p(2+x)$

$|\sin x| \geq \sin^2 x = \frac{1 - \cos 2x}{2} \rightarrow$

$\int_0^\infty \frac{\cos 2x}{2x} \ln^p(2+x)dx \rightarrow$ конв. условно (Дирихле)

$\ln(2+x) \sim \ln x, \; x \to +\infty$

$\int_2^\infty \frac{\ln^p(2+x)}{2x} dx \qquad \frac{\ln^p(2+x)}{2x} \sim \frac{1}{x \cdot \ln^{-p} x}, \quad x \to +\infty$

конв акко $-p > 1$

$p < -1$

$$\int_1^n \frac{dx}{x^3\sqrt{1+x^4}} = \int_1^n \frac{x\, dx}{x^4 \sqrt{1+x^4}} = \left[t = x^2 \right] = \int \ldots$$

$\frac{1}{2}dt$, $x^4 = t^2$, $\sqrt{1+t^2}$

$t = x^4$

$dt = 4x^3 dx$ $\qquad \int \frac{4(x^3 dx)}{4x^8 \sqrt{1+x^4}}$

dt

$4x^8 = t \cdot t^{1/2} \sqrt{1+t}$

$t^{3/2}\sqrt{1+t}$

---

a) $\int_0^1 \frac{\ln(1-x^2)}{(\alpha - x)^2} dx$ + парцијална (конта гора са $\lim_{h \to 1^-}$)

$\alpha > 1$

$\downarrow$

1 је једини сингл.

$\frac{\ln(1-x^2)}{(\alpha - x)^2} \sim \ln(1-x^2) = \ln(1-x) + \ln(1+x), \quad x \to +\infty$

$\ln 2$

$\int_0^1 \ln(1-x)dx$ конвергира

b) $0 < \alpha \le 1$

- $\alpha = 1$; 1 jedina sing.

$$\int_0^1 \frac{\ln(1-x^2)}{(1-x)^2}\,dx = \left[u = 1-x\right] = \int_0^1 \frac{\ln u + \ln(2-u)}{u^2}\,du$$

$\ln(1-x^2) = \ln(1-x) + \ln(1+x)$

$$\frac{\ln u + \ln(2-u)}{u^2} \sim \frac{\ln u}{u^2} > \frac{1}{u}, \quad u \to 0$$

$$\Rightarrow \int_0^1 \frac{1}{u}\,du \quad \text{div.} \Rightarrow \text{div.} \int_0^1 \frac{\ln(\ )}{(1-x)^2}\,dx$$

- $0 < \alpha < 1$

sing. α u 1

$$\underbrace{\int_0^1 \ldots}_{\text{div.}} := \int_0^\alpha + \int_\alpha^1 = \int_0^\alpha + \int_\alpha^c + \int_c^1$$

$$\int_0^\alpha \frac{\ln(1-x^2)}{(\alpha-x)^2}\,dx = \left[\begin{matrix}\alpha - x = t\\ x = \alpha - t\end{matrix}\right] = \int_0^\alpha \frac{\ln(\ldots)}{t^2}\,dt \sim \frac{1}{t^2}, \; t \to 0$$

$1-(\alpha-t)^2 \to 1$

$$\int_0^\alpha \frac{\ln(\ldots)}{(\alpha-x)^2}\,dx \text{ div.}$$

$$\sum_{n=1}^{\infty}\left(\sqrt[n]{a} - \frac{\sqrt[n]{b} + \sqrt[n]{c}}{2}\right), \quad a, b, c > 0$$

$a = \sqrt{bc}$ konv.

$a \ne \sqrt{bc}$ div.

$\ln \frac{a}{\sqrt{bc}} \ne 0$

$$a^{1/n} - \frac{b^{1/n} + c^{1/n}}{2}$$

$$a^{1/n} = e^{\frac{\ln a}{n}} \sim 1 + \frac{\ln a}{n} + \frac{(\ln a)^2}{2n^2} + \ldots$$

$$b^{1/n} \approx 1 + \frac{\ln b}{n} + \frac{(\ln b)^2}{2n^2}$$

$$c^{1/n} \approx 1 + \frac{\ln c}{n} + \frac{(\ln c)^2}{2n^2}$$

$\ln\sqrt{bc}$

$$a^{1/n} - \frac{\ldots}{2} \sim \left(\frac{\ln a - \frac{\ln b + \ln c}{2}}{n} + \frac{(\ln a)^2 - \frac{(\ln b)^2 + (\ln c)^2}{2}}{n^2}\right)$$

$a_1 = 1, \quad a_{n+1} = \ln(1+\mathrm{arctg}(a_n))$, $\quad a_1 > 0$, ПП. $a_n > 0$

$a_{n+1} = \ln(1+\mathrm{arctg}\, a_n) > 0$

$\Rightarrow a_n > 0$

конв ? $\quad \mathrm{arctg}\, a_n$?

lim ?

$\mathrm{arctg}\, x$? $\quad x$

$f(x) = x - \mathrm{arctg}\, x$

$f'(x) = 1 - \frac{1}{1+x^2} > 0, \quad f(0) = 0 \Rightarrow f(x) \geq 0$

$x \geq \mathrm{arctg}\, x$

$a_{n+1} \leq \ln(1+\mathrm{arctg}\, a_n) \leq \ln(1+a_n) \leq a_n$

$g(x) = x - \ln(1+x), \quad g'(x) = 1 - \frac{1}{1+x} \underbrace{> 0}_{x>0} \quad g(x) > \underset{g(0)}{0}, \; x > 0$

$x > \ln(1+x)$

$\underline{a_n \downarrow}, \; \underline{>0} \Rightarrow$ конв.

$a = \lim_{n\to\infty} a_n$

$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \ln(1+\mathrm{arctg}\, a_n)$

$= \ln(1+\mathrm{arctg}\, a)$

$a = \ln(1+\mathrm{arctg}\, a)$

$f(x) = \ln(1+\mathrm{arctg}\, x)$

$\ln(1+\mathrm{arctg}\, x) \leq x$

$F(x) = \underset{a_n}{x} - \underset{a_{n+1}}{\ln(1+\mathrm{arctg}\, x)}, \quad x \geq 0$

$F'(x) = 1 - \frac{1}{1+\mathrm{arctg}\, x} \cdot \frac{1}{1+x^2} > 0$, $\; x > 0$

$F(x) > 0, \quad x > 0$

$F(0) = 0 \Rightarrow \boxed{a = 0}$

$a_n \sim c n^{\alpha}$

$\frac{a_n}{a_{n+1}} = 1 + \frac{\mu}{n \ln n}$

$n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{\mu}{\ln n} \; \ldots$

$\frac{a_n}{a_{n+1}} = 1 + \underbrace{\frac{\mu}{n} + \frac{\theta_n}{n^{\alpha}}}$

$n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq r > 1$

≤ 1