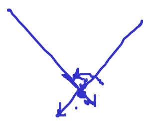
$f(x) = \sqrt[3]{x}\, \log(1+|x+x^2|)$ диф ?

$f$ непр ✓

$$f'(x) = \frac{1}{3} \cdot \underbrace{x^{-2/3}}_{\text{проблем у } 0} \cdot \operatorname{Log}(1+|x+x^2|) + \sqrt[3]{x} \cdot \frac{1}{1+|x+x^2|} \cdot \underbrace{\operatorname{sgn}(x+x^2)}_{\text{проблем } 0 \text{ и } -1} \cdot (2x+1)$$

$|x|$ није диф у $0$



$f'$ добро деф за $x \in \mathbb{R}\setminus\{0,-1\}$

$f'_+(-1) = ?$ $\qquad f'_+(0) = ?$

$f'_-(-1) = ?$ $\qquad f'_-(0) = ?$

$$f'_-(-1) = \lim_{h\to 0^+} \frac{f(-1-h) - f(-1)}{-h}$$


$$= \lim_{h\to 0^+} \frac{\sqrt[3]{-1-h}\, \ln(1+|(-1-h)(-\cancel{1}-h+\cancel{1})|) - \underbrace{\sqrt[3]{-1}\, \ln(1+|-1+1|)}_{=0}}{-h}$$

$$= \lim_{h\to 0^+} \frac{\sqrt[3]{1+h}}{h} \ln(1+\underbrace{h(1+h)}_{\sim h(1+h)}) = 1$$

$$f'_+(-1) = \lim_{h\to 0^+} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0^+} \frac{\overbrace{\sqrt[3]{-1+h}}^{\to -1}\, \ln(1+\overbrace{h(1-h)}^{\sim \cancel{h}(1-h)})}{\cancel{h}} = -1$$

$\Rightarrow f$ није диф у $-1$.

$$|-h+h^2| = h\,\overbrace{|-1+h|}^{\to 1}$$

$$f'_-(0) = \lim_{h\to 0^+} \frac{f(-h) - \overbrace{f(0)}^{=0}}{-h} = \lim_{h\to 0^+} \frac{\overbrace{\sqrt[3]{-h}}^{\to 0}\, \ln(1+|-h+h^2|)}{-\cancel{h}} = 0$$

$$|h+h^2| = \cancel{h}+h^2$$

$$f'_+(0) = \lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \frac{\sqrt[3]{h}\, \ln(1+|h+h^2|)}{\cancel{h}} = 0$$

$\Rightarrow f$ диф у $0$.

$$\sum_{n=4}^{\infty} \frac{\log(1+\pi^n)}{n^\alpha} \Big(\underbrace{\frac{x+1}{2}}_{t}\Big)^n \qquad \text{konb ?}$$

$$D = \,]-1,1[$$

$$\underbrace{\sum_{n=4}^{\infty} \frac{\log(1+\pi^n)}{n^\alpha}\, t^n}_{\underbrace{\qquad}} \longrightarrow t\in D = ?$$

$$\Downarrow$$

$$x \in D'$$

$$\frac{x+1}{2} \in \,]-1,1[$$

$$x+1 \in \,]-2,2[$$

$$\boxed{x \in \,]-3,1[}$$

$$a_n = \frac{\log(1+\pi^n)}{n^\alpha}$$

$$\lim_{n\to+\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to+\infty} \frac{\dfrac{\log(1+\pi^n)}{n^\alpha}}{\dfrac{\log(1+\pi^{n+1})}{(n+1)^\alpha}} =$$

$$= \lim_{n\to+\infty} \frac{\log(1+\pi^n)}{\log(1+\pi^{n+1})} \cdot \Big(\frac{n+1}{n}\Big)^\alpha = \boxed{1}$$

$$\frac{n+1}{n} = 1+\frac{1}{n} \xrightarrow[n\to+\infty]{} 1$$

$$\frac{\log \pi^n \left(1+\left(\frac{1}{\pi}\right)^n\right)}{\log \pi^{n+1}\left(1+\frac{1}{\pi^{n+1}}\right)} = \frac{\log\pi^n + \log\left(1+\left(\frac{1}{\pi}\right)^n\right)}{\log\pi^{n+1} + \log\left(1+\frac{1}{\pi^{n+1}}\right)}$$

$$\log\pi^n = n\log\pi, \qquad \log\left(1+\left(\frac{1}{\pi}\right)^n\right) \to 0$$

$$\log\pi^{n+1} = (n+1)\log\pi, \qquad \log\left(1+\frac{1}{\pi^{n+1}}\right) \to 0$$

$$\underbrace{\qquad\qquad} \xrightarrow[n\to+\infty]{} 1$$

$$(-1,1) \subseteq D \subseteq [-1,1]$$

$t_1 = 1$:

$$\underbrace{\sum_{n=1}^{\infty} \frac{\log(1+\pi^n)}{n^\alpha}}$$

$$\frac{\log(1+\pi^n)}{n^\alpha} \sim \frac{\log \pi^n}{n^\alpha} \sim \frac{\log \pi}{n^{\alpha-1}}, \quad n\to+\infty$$

$\log \pi^n = n \log \pi \to 1$

$\pi$ поредб.

$\alpha - 1 > 1 \iff \sum_{n=1}^{\infty} \frac{\log(1+\pi^n)}{n^\alpha}$ конв.

$\boxed{\alpha > 2} \iff \sum_{n=1}^{\infty} \frac{\log(1+\pi^n)}{n^\alpha}$ конв.

$t_2 = -1$

$$\sum_{n=1}^{\infty} \underbrace{\frac{\log(1+\pi^n)}{n^\alpha}}_{b_n} (-1)^n$$

? $b_n \to 0$ ?

$b_n$ монотоно ?

$$\frac{\log(1+\pi^n)}{n^\alpha} \sim \frac{\log \pi}{n^{\alpha-1}} \xrightarrow[n\to+\infty]{} 0, \quad \alpha - 1 > 0$$

$\Leftrightarrow \boxed{\alpha > 1}$

за $\alpha \leq 1$ $\quad b_n \not\to 0$!

Испитујемо монотоност за $(1 < \alpha \leq 2)$

$$b_n = \frac{\log(1+\pi^n)}{n^\alpha} = \frac{\log \pi^n + \log\left(1+\frac{1}{\pi^n}\right)}{n^\alpha} =$$

$$= \frac{\log \pi}{n^{\alpha-1}} \downarrow + \frac{\log\left(1+\frac{1}{\pi^n}\right)}{n^\alpha} \downarrow \qquad \frac{1}{\pi^n} \downarrow$$

$\alpha > 1$

$\alpha - 1 > 0 \qquad \frac{1}{n^{\alpha-1}} \downarrow$

$\Rightarrow b_n \downarrow$

Најдиме $\sum_{n=4}^{\infty} b_n \cdot (-1)^n$ конв.

$\alpha \leq 1 \qquad D = (-1, 1) \longrightarrow x \in D' = (-3, 1)$

$1 < \alpha \leq 2 \qquad D = [-1, 1) \qquad x \in D' = [-3, 1)$

$\alpha > 2 \qquad D = [-1, 1] \qquad x \in D' = [-3, 1]$.

$n - 4 = k$

$$\sum_{n=2}^{\infty} \underbrace{\frac{\sin n}{\ln n} \operatorname{arctg} n}_{a_n} \frac{(x+2)^n}{t^n} \quad \text{конв?}$$

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{\left|\frac{\sin n}{\ln n}\right| \operatorname{arctg} n \to \frac{\pi}{2}}{\left|\frac{\sin(n+1)}{\ln(n+1)}\right| \operatorname{arctg}(n+1) \to \frac{\pi}{2}} =$$

$$= \left|\frac{\sin n}{\sin(n+1)}\right| \cdot \underbrace{\frac{\ln(n+1)}{\ln n}}_{\to 1}$$

$\sin n \cdot \cos 1 + \cos n \cdot \sin 1$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\cos 1 + \sin 1 \frac{\cos n}{\sin n}\right| \to ?$$

$\operatorname{ctg} n$

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|} = ?$$

$$\frac{\sqrt[n]{|\sin n|}\ \sqrt[n]{\operatorname{arctg} n} \to 1 \quad \to \frac{\pi}{2} \to 1}{\sqrt[n]{\ln n}} \to 1$$

$$1 < \sqrt[n]{\ln n} < \sqrt[n]{n}$$

$\downarrow 1 \qquad \downarrow 1$

$$\overline{\lim_{n\to+\infty}} \sqrt[n]{\underbrace{|\sin n|}_{?}} \leq 1$$

$$f(x) = \underbrace{\sin x}$$

$f(\mathbb{N})$ je hustá v $[-1,1]$

$$\boxed{\sin k_n \to 1}, \quad n \to +\infty$$

$k_n \uparrow$

$k_n \in \mathbb{N}$

$$\left(2k\pi + \frac{\pi}{2}\right) \notin \mathbb{N}$$

$$\boxed{\sqrt[n]{\sin k_n} \to 1}$$

$$\overline{\lim_{n\to+\infty}} \sqrt[n]{|\sin n|} = 1$$

$$\Rightarrow R = 1$$

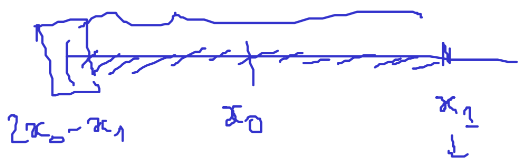
$$(x+2) \in (-1,1)$$

$x+2 = -1$ :

$x+2 = 1$ :

$x+2=\boxed{1}$

$$\sum_{n=2}^{+\infty} \frac{\sin n}{\ln n} \operatorname{arctg} n \cdot 1^n$$




конв. условно

не конв апс.

Абел:

А1: $\sum_{n=2}^{\infty} \frac{\sin n}{\ln n}$ конв,

Д1. $\sum_{n=2}^{\infty} \sin n$ огран.

Д2. $\frac{1}{\ln n} \downarrow$, $\frac{1}{\ln n} \to 0$

А2. $\operatorname{arctg} n \uparrow$, $\operatorname{arctg} n \to \frac{\pi}{2}$ огран ✓

Абел $\Rightarrow \sum_{n=2}^{\infty} \frac{\sin n}{\ln n} \operatorname{arctg} n$

Ово није потребно ако се одредило раније R!

$$\sum_{n=2}^{\infty} \frac{|\sin n| \operatorname{arctg} n}{\ln n}$$

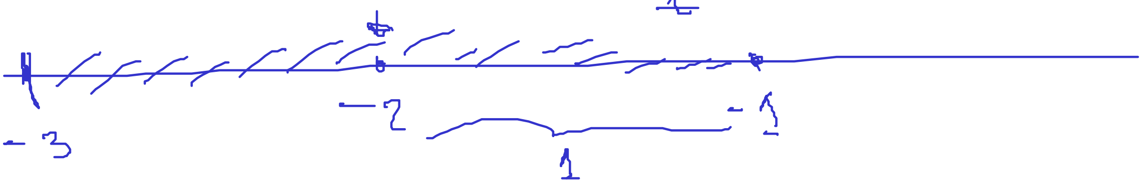
$$\frac{\operatorname{arctg} n}{\ln n} \sim \frac{\pi/2}{\ln n}$$

$$|\sin n| \geq \sin^2 n = \frac{1-\cos 2n}{2}$$

$$\underbrace{\sum \frac{|\sin n| \operatorname{arctg} n}{\ln n}}_{\text{дивергира}} \geq \underbrace{\sum \frac{\operatorname{arctg} n}{2\ln n}}_{\sim \frac{c}{\ln n} \text{ дивергира}} - \frac{1}{2}\underbrace{\sum \frac{\cos 2n \operatorname{arctg} n}{\ln n}}_{\text{Абел конв.}}$$

$\Leftarrow$

$\Rightarrow$ Ред условно конв. у $x+2=1$ ! $\boxed{R=1}$



$x \in (-3, -1) \overset{\text{лема}}{\Longrightarrow}$ ред апс. конв.

$x = -1 \Rightarrow$ конв.

$\boxed{x = -3}$ ?

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sin n \ \operatorname{arctg} n}{\ln n} \quad \text{конв ?}$$

$\alpha = 1 \quad \sin n\alpha$

$$S_n = \sum_{k=1}^{n} (-1)^k \sin k = -\sin 1 + \sin 2 - \sin 3 + \dots + (-1)^n \sin n \quad / \ 2\cos\frac{1}{2}$$

$$2\cos\tfrac{1}{2} S_n = -2\cos\tfrac{1}{2}\sin 1 + 2\sin 2\cos\tfrac{1}{2} - \dots$$

$\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$

$$= -\cancel{\sin 3/2} - \sin\tfrac{1}{2} + \cancel{\sin 5/2} + \cancel{\sin 3/2} - \cancel{\sin 5/2} - \cancel{\sin 7/2}$$

$$+ \dots + (-1)^n \cancel{\sin\tfrac{2n-1}{2}} + (-1)^n \sin\tfrac{2n+1}{2}$$

$$= (-1)^n \sin\frac{2n+1}{2} - \sin\frac{1}{2}$$

$$|S_n| = \frac{\left|(-1)^n \sin\frac{2n+1}{2} - \sin\frac{1}{2}\right|}{2\cos 1/2} \le \frac{2}{2\cos 1/2} = \frac{1}{\cos 1/2} \to \text{огран}$$

$\cos 1/2 \neq 0$

$\Rightarrow S_n$ ограничене

Абел:

А1. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \sin n}{\ln n}$

$\underline{A}$ 1. $\sum (-1)^n \sin n$ има огран парц. суме

$\underline{A}$ 2. $\frac{1}{\ln n} \downarrow, \to 0$.

А2. $\operatorname{arctg} n \uparrow, \to \pi/2$.

$\Rightarrow$ конв. у $x = -3$, $(x+2 = -1)$.

$\boxed{2}$ $\sum a_n$ конв. $\Rightarrow a_n \to 0$

$$b_n = \frac{(n+a_n)a_n}{(1+n)(1+a_n)}$$

а) $\sum a_n^2$ конв. $\Rightarrow$ ? $\sum b_n$ конв. ?

б) $\sum \frac{(-1)^n}{\sqrt{n}}$ $\Rightarrow$ ? $\sum b_n$ конв. ?

$$\sum b_n = \sum \frac{(n+a_n)a_n}{(1+n)(1+a_n)} = \sum\left(\frac{n a_n}{(n+1)(1+a_n)} + \frac{a_n^2}{(1+n)(1+a_n)}\right)$$

$$= \underbrace{\sum \frac{n a_n}{(n+1)(1+a_n)}}_{} + \underbrace{\sum \frac{a_n^2}{(n+1)(1+a_n)}}_{\text{на основу поређеног конв.}}$$

$$\sum \frac{n}{n+1} \cdot \frac{a_n}{1+a_n}$$

$\frac{a_n^2}{(n+1)(1+a_n)} \sim \frac{a_n^2}{n} \leq a_n^2$, $\frac{a_n^2}{n}$ конв., $\sum a_n^2$ конв.

$(n+1)(1+a_n) \geq n \geq 1$, конв.

$\sum a_n$ конв.

$$\boxed{\underbrace{\frac{n}{n+1}}_{} \cdot \frac{1}{1+a_n}} = \left(1 - \frac{1}{n+1}\right)\left(\frac{1}{1+a_n}\right)$$

$$= \frac{1}{1+a_n} - \frac{1}{(n+1)(1+a_n)}$$

$$\frac{a_n}{1+a_n} = 1 - \frac{1}{1+a_n}$$

$$\frac{(-1)^n}{n+(-1)^n} \cdot \frac{n-(-1)^n}{n-(-1)^n} = \frac{(-1)^n \cdot n - 1}{n^2-1} = \underbrace{\frac{(-1)^n n}{n^2-1}}_{\text{конв.}} - \underbrace{\frac{1}{n^2-1}}_{\text{конв.}}$$

не лајбниц

$\sum \underbrace{\frac{(-1)^n}{n+(-1)^n}}_{a_n}$ конв (само условно)

$\frac{n}{n^2-1} \overset{?}{>} \frac{n+1}{(n+1)^2-1}$

$n^2(n+2) \; ? \; (n-1)(n+1)^2$

$\sum a_n^2$ конв. $\cancel{n^3} + 2\cancel{n^2} \; ? \; (n-1)(n^2+2n+1) = \cancel{n^3} + 2\cancel{n^2} + n - n^2 - 2n - 1$

$|a_n|$ није монотон $\quad n^2+n+1 \; \overset{?}{>} \; 0$

$$\sum \frac{(n+a_n)\,a_n^2}{(n+1)(1+a_n)\,a_n} \text{ конв ?}$$

$\sum a_n,\ \sum a_n^2$ конв.

$$\boxed{\sum_{n=1}^{\infty} \frac{n a_n}{(n+1)(1+a_n)}} \text{ конв. ?}$$

$$\left(\frac{n}{n+1} \cdot \frac{a_n}{1+a_n}\right)$$

---

б) $a_n = \dfrac{(-1)^n}{\sqrt{n}}$

$$b_n = \frac{n \cdot \frac{(-1)^n}{\sqrt{n}}}{(n+1)\left(1+\frac{(-1)^n}{\sqrt{n}}\right)} \cdot \frac{\sqrt{n}}{\sqrt{n}} =$$

$$= \frac{(-1)^n(n+1-1)}{(n+1)(\sqrt{n}+(-1)^n)} =$$

$$= \frac{(-1)^n}{\sqrt{n}+(-1)^n} - \frac{(-1)^n}{(n+1)(\sqrt{n}+(-1)^n)}$$

$$\underset{\text{див.}}{\underset{\downarrow}{\sum b_n}} = \underbrace{\sum \frac{(-1)^n}{\sqrt{n}+(-1)^n}}_{\text{дивергира на засц.}} - \underbrace{\sum \frac{(-1)^n}{(n+1)(\sqrt{n}+(-1)^n)}}_{\text{конв. апс.}}$$

јер $\sim \dfrac{1}{n \cdot \sqrt{n}}$

$$\sum_{n=1}^{\infty} \frac{n a_n}{(n+1)(1+a_n)}$$

$$\frac{1}{1+a_n} = \frac{1+a_n-a_n}{1+a_n} = 1 - \frac{a_n}{1+a_n}$$

$$\sum_{n=1}^{\infty} \frac{n}{n+1} \frac{a_n}{(1+a_n)} = \sum_{n=1}^{\infty} \frac{n}{n+1} a_n\left(1-\frac{a_n}{1+a_n}\right) = \underbrace{\sum_{n=1}^{\infty} \frac{n a_n}{n+1}}_{\text{Абел. конв.}} - \underset{\text{због као малопре конв.}}{\underset{\downarrow}{\sum_{n=1}^{\infty} \frac{n \cdot a_n^2}{(n+1)\cdot(1+a_n)}}}$$

A1. $\sum a_n$ конв. A2. $\frac{n}{n+1}\uparrow,\ \to 1$.