5. $\lim\limits_{n\to+\infty} \frac{1}{\ln n} \int\limits_1^n \underbrace{(e^{1/x}-1)}_{f(x)}\,dx = ?$ $\quad e^{1/x} \sim 1+\frac{1}{x}, \; x\to+\infty$

$\ulcorner \int e^{1/x}\,dx = \ulcorner u = e^{1/x}$

$du = e^{1/x}\cdot\left(-\frac{1}{x^2}\right)dx$

$\frac{1}{x} = \ln u$

$dx = -\frac{du}{u\ln^2 u} \lrcorner$

$= \int -\frac{\cancel{u}}{\cancel{u}\ln^2 u}\,du = -\int\frac{du}{\ln^2 u}$

= тешко израчунати... $\lrcorner$

$\int\limits_1^{+\infty} e^{1/x}-1\,dx \sim \int\limits_1^{+\infty}\frac{dx}{x} \to$ дивергира

$\lim\limits_{n\to+\infty}\frac{\int\limits_1^n (e^{1/x}-1)\,dx}{\ln n} = \ulcorner$ ШТ. $\ln n \uparrow +\infty \lrcorner =$

$= \lim\limits_{n\to+\infty}\frac{\int\limits_1^{n+1}(e^{1/x}-1)\,dx - \int\limits_1^n (e^{1/x}-1)\,dx}{\ln(n+1)-\ln n} = \lim\limits_{n\to+\infty}\frac{\int\limits_n^{n+1}(e^{1/x}-1)\,dx}{\ln\left(\frac{n+1}{n}\right)} =$

$\ulcorner e^{1/x}-1 \sim \frac{1}{x}, \; x\to+\infty \lrcorner$

Т о ср. вр. $= \lim\limits_{n\to+\infty}\frac{(e^{1/\xi_n}-1)\cdot\int\limits_n^{n+1}dx}{\ln\left(\frac{n+1}{n}\right)}$, $\int\limits_n^{n+1}dx = 1$, $\xi_n\in(n,n+1)$

$= \lim\limits_{n\to+\infty}\frac{e^{1/\xi_n}-1}{\ln(1+\frac{1}{n})}$, $1/\xi_n \to 0$, $e^{1/\xi_n}-1 \sim 1/\xi_n$, $\ln(1+\frac{1}{n}) \sim 1/n$

$= \lim\limits_{n\to+\infty}\underbrace{\frac{e^{1/\xi_n}-1}{1/\xi_n}}_{\downarrow\,1}\cdot\underbrace{\frac{1/\xi_n}{1/n}}_{\frac{n}{\xi_n}}\cdot\underbrace{\frac{1/n}{\ln(1+1/n)}}_{\downarrow\,1}$

$\xi_n\in(n,n+1)$

$= \underbrace{\lim\limits_{n\to+\infty}\frac{n}{\xi_n}}= 1.$

$\frac{n}{n+1} \le \frac{n}{\xi_n} \le \frac{n}{n}$

$\frac{n}{n+1}\to 1$, $\frac{n}{n}\to 1$, $\frac{n}{\xi_n} \xrightarrow{\text{Т о 2п}} 1$

3. [2019.]

a_n

$a_1 = 1$

$a_{n+1} = \ln(1+\operatorname{arctg}(a_n))$

а) a_n конв. ? $\lim\limits_{n\to+\infty} a_n = ?$ $\to$ за вежбу

$\Rightarrow \lim\limits_{n\to+\infty} a_n = 0$, $a_n \downarrow$

б) $c, \alpha = ?$ $a_n \sim c n^{\alpha}$, $n\to+\infty$ (очекујемо да је $\alpha<0$)

$$\lim_{n\to+\infty} \frac{a_n}{cn^{\alpha}} = 1, \quad c,\alpha = ?$$

$$(1+\tfrac{1}{n})^{-\alpha} = 1-\frac{\alpha}{n}+o(\tfrac{1}{n})$$

$$\lim_{n\to+\infty} \frac{n^{-\alpha}}{\frac{c}{a_n}} \overset{\text{Шт.}}{=} \lim_{n\to+\infty} \frac{(n+1)^{-\alpha} - n^{-\alpha}}{\frac{c}{a_{n+1}} - \frac{c}{a_n}} = \frac{1}{c}\lim_{n\to+\infty} \frac{n^{-\alpha}\left((1+\frac{1}{n})^{-\alpha}-1\right)}{\frac{1}{\ln(1+\operatorname{arctg}a_n)} - \frac{1}{a_n}}$$

$\underbrace{\frac{c}{a_n}\uparrow, \ \to+\infty, n\to+\infty}$

$$= \frac{1}{c}\lim_{n\to+\infty} \frac{n^{-\alpha}\cdot\left(-\frac{\alpha}{n}+o(\frac{1}{n})\right)}{\frac{a_n - \ln(1+\operatorname{arctg}a_n)}{a_n\cdot\ln(1+\operatorname{arctg}a_n)}} =$$

$$= \frac{1}{c}\lim_{n\to+\infty} \frac{\left(-\frac{\alpha}{n^{\alpha+1}} + o(\frac{1}{n^{\alpha+1}})\right)\cdot \overbrace{a_n\cdot\ln(1+\operatorname{arctg}a_n)}^{\sim a_n^2+o(a_n)^2}}{\frac{a_n^2}{2}+o(a_n^2)}$$

$$\ln(1+\operatorname{arctg}a_n) \underset{a_n\to 0}{=} \ln(1+\underbrace{a_n+o(a_n)}_{\to 0}) = a_n+o(a_n)+\underbrace{o(a_n+o(a_n))}_{o(a_n)}$$

$$= a_n+o(a_n)$$

$$a_n - \underbrace{\ln(1+\operatorname{arctg}a_n)} = a_n - a_n - o(a_n) = \underline{\underline{+o(a_n)}} \ ?$$

$\operatorname{arctg} x = x - \frac{x^3}{3} + o(x^3)$

$f(x) = \operatorname{arctg} x$

$$f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = -\frac{2x}{(1+x^2)^2}$$

$$f'''(x) = -2x\cdot\left((1+x^2)^{-2}\right)' - \frac{2}{(1+x^2)^2}$$

$$f'(0)=1$$

$$f''(0)=0$$

$$f'''(0)=-2$$

$$\ln(1+x) = \underbrace{x-\frac{x^2}{2}+}\underbrace{\frac{x^3}{3}+o(x^3)}$$

$$\ln(1+\operatorname{arctg} a_n) = \ln\Big(1+\underbrace{a_n - \frac{a_n^3}{3}+o(a_n^3)}\Big) \qquad = a_n^2+o(a_n^2)$$

$$= a_n - \underbrace{\frac{a_n^3}{3}+o(a_n^3)} - \frac{\left(a_n-\frac{a_n^3}{3}+o(a_n^3)\right)^2}{2} + o\left(\left(a_n-\frac{a_n^3}{3}+o(a_n^3)\right)^2\right)$$

$$\qquad = o(a_n^2)$$

$$= \underbrace{a_n - \frac{a_n^2}{2}+o(a_n^2)}$$

$$a_n - \ln(1+\operatorname{arctg} a_n) = \frac{a_n^2}{2}+o(a_n^2)$$

$$\overset{*}{=} \frac{1}{e}\lim_{n\to+\infty}\frac{\left(-\frac{\alpha}{n^{\alpha+1}}+o\left(\frac{1}{n^{\alpha+1}}\right)\right)\cdot\left(a_n^2+o(a_n^2)\right)}{\frac{a_n^2}{2}+o(a_n^2)} \qquad (=1)$$

ово хотемо

$$= \frac{1+o(1)}{\frac{1}{2}+o(1)} \xrightarrow{n\to+\infty} \frac{1}{\frac{1}{2}} = 2$$

$$= \frac{1}{e}\underbrace{\lim_{n\to+\infty}\left(-\frac{\alpha}{n^{\alpha+1}}+o\left(\frac{1}{n^{\alpha+1}}\right)\right)}\ \underline{=1}$$

$$\downarrow$$

хотемо да ово буде коначно $\Rightarrow \alpha+1=0$

и не нула $\qquad \boxed{\alpha=-1}$

$-(-1)=1$

$\lim\to 0$ за $\alpha>-1$

$\lim\to+\infty$ за $\alpha<-1$

$$\boxed{c=2}$$

$a_n \sim \frac{2}{n}$, $n \to +\infty$

в) $\sum\limits_{n=1}^{+\infty} (-1)^n \underbrace{\arcsin \frac{1}{\sqrt{n}} \cos a_n}_{>0}$ условна + абс. конв ?

абсолютна конв.

$\arcsin \frac{1}{\sqrt{n}} \cos a_n \sim \underbrace{\frac{1}{\sqrt{n}}}\cdot 1$, $n \to +\infty$

$\arcsin \frac{1}{\sqrt{n}} \downarrow \frac{1}{\sqrt{n}}$, $\cos a_n \downarrow 1$

$\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ див. $\overset{\text{II ч.к.}}{\Longrightarrow}$ $\sum\limits_{n=1}^{\infty} \arcsin \frac{1}{\sqrt{n}} \cos a_n$ див.

условна конв.

A1. Лайбниц $\sum\limits_{n=1}^{\infty} (-1)^n \arcsin \frac{1}{\sqrt{n}}$ конв. ✓

$\arcsin \frac{1}{\sqrt{n}} \to 0$, $\arcsin \frac{1}{\sqrt{n}} \downarrow$

A2. $\cos a_n \uparrow$, $\cos a_n \to 1$, $n \to +\infty$ ✓

$\Uparrow$ $\qquad \downarrow 0$

$a_n \to 0$, $\downarrow$

$a_n > 0$

Абел $\Longrightarrow$ $\sum\limits_{n=1}^{\infty} (-1)^n \operatorname{arctg} \frac{1}{\sqrt{n}} \cos a_n$

г)

$\sum\limits_{n=1}^{\infty} a_n x^n$ → област конв ?

$x = ?$

$\sum\limits_{n=1}^{\infty} \frac{x^n}{n}$, → не конв у 1

ами конв. у -1

$D = [-1, 1)$

$\frac{a_n}{a_{n+1}} \sim \frac{2/n}{2/(n+1)} \sim 1$, $n \to +\infty$

$\Rightarrow R = 1$

$x = 1$: $\sum_{n=1}^{+\infty} a_n \cdot 1^n$

$a_n \sim \frac{2}{n}$, $\sum \frac{2}{n}$ див. $\xRightarrow{\text{II п.к.}}$ $\sum a_n$ див.

$x = -1$: $\sum_{n=1}^{\infty} a_n \cdot (-1)^n$

$a_n \downarrow$, $\lim_{n \to +\infty} a_n = 0$

Лајбниц $\Rightarrow$ $\sum_{n=1}^{\infty} a_n (-1)^n$ конв.

$D = [-1, 1)$ → област конв. реда

AI ЈУН 2017

$\boxed{3}$ $I_n = \int_0^{\pi/2} \frac{\sin 2x \cdot \sin^{n-2} x}{1 + \sin x} dx$, $n \geq 2$

а) Одредити везу I_n и I_{n+1} ?

$\sin^{n-1} x = \sin^{n-2} x (\underbrace{\sin x + 1} - 1)$

$$I_{n+1} = \int_0^{\pi/2} \frac{\sin 2x \cdot \sin^{n-1} x}{1 + \sin x} dx =$$

$$= \int_0^{\pi/2} \frac{\sin 2x \cdot \sin^{n-2} x \cdot (\cancel{\sin x + 1})}{\cancel{1 + \sin x}} dx - \underbrace{\int_0^{\pi/2} \frac{\sin 2x \cdot \sin^{n-2} x}{1 + \sin x} dx}_{I_n}$$

$$\int_0^{\pi/2} \sin 2x \cdot \sin^{n-2} x \, dx = 2 \int_0^{\pi/2} \sin^{n-1} x \cdot \cos x \, dx = \left\lceil \begin{matrix} t = \sin x \\ dt = \cos x \, dx \end{matrix} \right\rfloor =$$

$$= 2 \int_0^1 t^{n-1} dt = 2 \left. \frac{t^n}{n} \right|_0^1 = \frac{2}{n}$$

$$I_{n+1} = \frac{2}{n} - I_n$$

$$I_2 = \int_0^{\pi/2} \frac{\sin 2x \cdot \sin^{2-2=1} x}{1+\sin x}\,dx = \int_0^{\pi/2} \frac{2\sin x\cos x}{1+\sin x}\,dx = \begin{bmatrix} t=\sin x \\ dt=\cos x\,dx \end{bmatrix}$$

$$= \int_0^1 \frac{2t}{1+t}\,dt = 2\int_0^1 1 - \frac{1}{1+t}\,dt = 2\big(t - \ln(1+t)\big)\Big|_0^1$$

$$= 2 - 2\ln 2.$$

$$\lim_{n\to+\infty} I_n = ?$$

Ако би $\lim\limits_{n\to+\infty} I_n$ постојао, онда $\lim\limits_{n\to+\infty} I_n = l = \lim\limits_{n\to+\infty} I_{n+1} = \lim\limits_{n\to+\infty}\left(\frac{2}{n} - I_n\right)$

$= 0 - l$

$\Rightarrow$ Ако l постоји онда l мора бити 0.

$$I_n = \int_0^{\pi/2} \underbrace{\frac{\sin 2x \cdot \sin^{n-2} x}{1+\sin x}}_{>0 \text{ за } (0,\frac{\pi}{2})}\,dx > 0 \quad \forall n\in\mathbb{N}$$

$I_{n+1} > 0 \quad \Rightarrow \quad I_{n+1} = \frac{2}{n} - I_n > 0 \quad \Rightarrow \quad 0 < I_n < \frac{2}{n} \xrightarrow{n\to+\infty} 0$

Т о 2П $\Rightarrow \lim\limits_{n\to+\infty} I_n = 0.$

e) $\sum\limits_{n=2}^{\infty} I_n \cdot \frac{\ln n \cos n\pi}{n}$

$I_n \cdot \frac{\ln n}{n}$ $\downarrow$, $\to 0$ ✓

Да ли је $I_n \downarrow$?

$\sum\limits_{n=2}^{+\infty} \left|I_n \frac{\ln n}{n} \cos n\pi\right|$ конв?

$\left|I_n \frac{\ln n}{n} \underset{(-1)^n}{\underset{\parallel}{\cos n\pi}}\right| \le I_n \frac{\ln n}{n} < \frac{2\ln n}{n^2}$, $\sum\limits_{n=1}^{\infty} \frac{2\ln n}{n^2}$ конв.

Т.п.к. $\Rightarrow \sum\limits_{n=2}^{+\infty} I_n \frac{\ln n}{n} \cos n\pi$ конв. апсолутно.

$\boxed{2}$ $\alpha \in \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$ $g(x) = \begin{cases} \operatorname{arctg} x, & |x| \geq 1 \\ \frac{\pi}{4} \underline{\underline{\operatorname{sgn} x}} + \frac{(1-x)^\alpha}{2}, & |x| < 1 \end{cases}$

а) непрекидност:

$|x| > 1$ или $0 < |x| < 1$ $\to$ g непр и диф. у x као комп. елем. ф-ја у околини те тачке.

$x_1 = 1$, $x_2 = -1$, $x_3 = 0$

$x_1 = 1$: $\lim_{x \to 1^+} g(x) = ?$ $\lim_{x \to 1^-} g(x) = ?$

$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (\operatorname{arctg} x) = \operatorname{arctg} 1 = \frac{\pi}{4}$$

само у овом случ. имамо непр у 1

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} \Big(\frac{\pi}{4} \underset{\substack{\shortparallel \\ 1}}{\operatorname{sgn} x} + \frac{(1-x)^\alpha}{2}\Big) = \begin{cases} \boxed{\frac{\pi}{4}, \quad \alpha > 0} \\ \frac{\pi}{4} + \frac{1}{2} & \alpha = 0 \\ +\infty & \alpha < 0 \end{cases}$$

$x_2 = -1$: $\lim_{x \to -1^-} g(x) = \lim_{x \to -1^-} \operatorname{arctg} x = -\frac{\pi}{4}$

$$\lim_{x \to -1^+} g(x) = \lim_{x \to -1^+} \Big(\frac{\pi}{4} \underset{\substack{\shortparallel \\ -1}}{\operatorname{sgn} x} + \frac{(1-x)^\alpha}{2}\Big) = -\frac{\pi}{4} + \underset{\substack{\shortparallel \\ 2^{\alpha-1} \\ \not= 0}}{\frac{2^\alpha}{2}}, \quad \alpha \in \mathbb{R}$$

у -1 увек имамо прекид ф-је

$x_3 = 0$: $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \frac{\pi}{4} \operatorname{sgn} x + \frac{(1-x)^\alpha}{2} = \frac{\pi}{4} + \frac{1}{2}$

$\not\shortparallel$

$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-}$ —— $\shortparallel$ —— $= -\frac{\pi}{4} + 1/2$

у 0 увек имамо прекид

б) диференцијабилност:

након кратке дискусије, треба проверити шта се дешава у $x_1 = 1$ у случају $\alpha > 0$.

$$f'_+(1) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{\operatorname{arctg}(1+h) - \pi/4}{h} = \lim_{h\to 0^+} \frac{\frac{1}{2}h + o(h)}{h} = \frac{1}{2}$$

~~осим~~ $\alpha = 1$

~~осим~~ $\alpha = 1$

$$f'_-(1) = \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{-h} = \lim_{h\to 0^-} \frac{\pi/4 + \left(\frac{-h}{2}\right)^{\alpha} - \pi/4}{-h} = \lim_{h\to 0^-} \frac{(h)^{\alpha-1}}{2}$$

$$\operatorname{arctg}(1+h) = \operatorname{arctg} 1 + \frac{1}{2}h + o(h)$$

$$(\operatorname{arctg} x)' = \frac{1}{1+x^2}$$

$\Rightarrow$ f није диф у $\underbrace{1,\ \text{нити}}_{\text{осим за } \alpha = 1}$ у 0 и -1.

б) испитати р. н. за ~~верту~~ ...