31. $a_0 > 0$

$$a_{n+1} = \frac{3a_n^2}{(1+a_n)^3 - 1}$$

$\lim_{n\to\infty} a_n = ?$ $\lim_{n\to\infty} n\cdot a_n = ?$

$\lim_{n\to\infty} a_n = ?$

$a_n > 0 \quad \forall n \in \mathbb{N}$

(БИ) $a_1 = \frac{3a_0^2}{\underbrace{(1+a_0)^3 - 1}_{>0}} > 0$

(ИК) $a_n > 0 \Rightarrow a_{n+1} > 0 \dots$

$a_{n+1} \boxed{\leq} a_n$

$$\frac{3a_n^2}{(1+a_n)^3 - 1} = \frac{3a_n^2}{a_n^3 + 3a_n^2 + 3a_n + \cancel{1} - \cancel{1}} = \frac{3a_n}{a_n^2 + 3a_n + 3} \;\square\; a_n \quad / : a_n > 0$$

$$\frac{3}{\underbrace{a_n^2 + 3a_n + 3}_{>0}} \boxed{\leq} 1$$

$a_n \downarrow$

$\Rightarrow a_n$ конвергира $\lim_{n\to\infty} a_n = a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{3a_n}{a_n^2 + 3a_n + 3} = \frac{3a}{a^2 + 3a + 3} \quad / (a^2+3a+3)$

$$a^3 + 3a^2 + \cancel{3a} = \cancel{3a}$$

$$a^2(a+3) = 0 \quad \boxed{a = 0} \vee a = -3$$

$\hookrightarrow$ ово не може јер $a_n > 0 \Downarrow a \geq 0$

$$\lim_{n\to\infty} n\cdot a_n = \lim_{n\to\infty} \frac{n}{\frac{1}{a_n}} \overset{*}{\underset{ШТ}{=}} \lim_{n\to\infty} \frac{n+1-n}{\frac{1}{a_{n+1}} - \frac{1}{a_n}} = \lim_{n\to\infty} \frac{1}{\frac{1}{\frac{3a_n}{a_n^2+3a_n+3}} - \frac{1}{a_n}\cdot\frac{3}{3}} =$$

$b_n = \frac{1}{a_n}$

$a_n \downarrow \Rightarrow \underline{b_n \uparrow}$

$0 < a_{n+1} \leq a_n \qquad \frac{1}{a_n} \leq \frac{1}{a_{n+1}}$

$\lim_{n\to\infty} a_n = 0,\ a_n > 0 \Rightarrow \lim_{n\to\infty} b_n = +\infty$

$$= \lim_{n\to\infty} \frac{1}{\frac{a_n^2 + 3a_n + 3 - 3}{3a_n}} = \lim_{n\to\infty} \frac{3a_n}{a_n^2 + 3a_n} =$$

$$= \lim_{n\to\infty} \frac{3}{a_n + 3} = \boxed{1}$$

($a_n \to 0$ за $n \to \infty$)

36. $\lim\limits_{n\to\infty} \frac{2n}{[n\sqrt{2}]} = \sqrt{2}$

$\ulcorner$

$[(n+1)\sqrt{2}] - [n\sqrt{2}]$

$[n\sqrt{2} + \sqrt{2}] = ?$

$\lrcorner$

$$n\sqrt{2} - 1 \leq [n\sqrt{2}] \leq n\sqrt{2} \implies \underbrace{\frac{2n}{n\sqrt{2}}}_{\downarrow\ \frac{2}{\sqrt{2}} = \sqrt{2}} \leq \underbrace{\frac{2n}{[n\sqrt{2}]}}_{\downarrow\ \text{Т.о.2П}\ \sqrt{2}} \leq \underbrace{\frac{2n}{n\sqrt{2}-1}}_{\downarrow\ \sqrt{2}}$$

$$\lim_{n\to\infty} \frac{(n+2)n^n}{(n+1)^{(n+1)}} = \lim_{n\to\infty} \frac{(n+2)n^n}{(n+1)\cdot(n+1)^n} = \lim_{n\to\infty} \underbrace{\frac{n+2}{n+1}}_{\to 1} \underbrace{\left(\frac{n}{n+1}\right)^n}_{\to \frac{1}{e}} = \frac{1}{e}$$

$$\left(\frac{n}{n+1}\right)^n = \left(1 - \frac{1}{n+1}\right)^n = \left(\underbrace{\left(1 - \frac{1}{n+1}\right)^{-(n+1)}}_{\to e}\right)^{\overbrace{-\frac{n}{n+1}}^{\to -1}} \to \frac{1}{e}$$

б) $\alpha \in \mathbb{R}\setminus\mathbb{Q},\ p_n, q_n \in \mathbb{N}$, $\lim\limits_{n\to\infty} \frac{p_n}{q_n} = \alpha$ $\overset{?}{\implies} \lim\limits_{n\to\infty} q_n = +\infty$

$\lim\limits_{n\to\infty} q_n \neq +\infty \implies \neg\,(\forall M>0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0\ \ q_n > M)$

$\exists M > 0\ \forall n_0 \in \mathbb{N}\ \exists n \geq n_0\ \ q_n \leq M$

$\implies \exists$ подниз q_{n_k} т.д. $\forall k \in \mathbb{N}\ \ q_{n_k} \leq M$

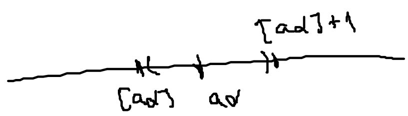
$\forall k \in \mathbb{N}:\ \ q_{n_k} \in \underbrace{\{1, 2, \dots, [M]\}}_{\text{конач}} \implies \exists$ подниз $q_{n_{k_\ell}}$ од q_{n_k} т.д. $q_{n_{k_\ell}}$ констант $q_{n_{k_\ell}} = a \in \mathbb{N}$

$\lim\limits_{\ell\to\infty} \frac{p_{n_{k_\ell}}}{q_{n_{k_\ell}}} = \alpha$

$\lim\limits_{\ell\to\infty} \frac{p_{n_{k_\ell}}}{a} = \alpha \implies \lim\limits_{\ell\to\infty} p_{n_{k_\ell}} = a\cdot\alpha \in \mathbb{R}\setminus\mathbb{Q}$

$\forall \varepsilon > 0\ \exists \ell_0\ \ \ell \geq \ell_0\ \ p_{n_{k_\ell}} \in \underbrace{(a\cdot\alpha - \varepsilon,\ a\alpha + \varepsilon)}$

за $\varepsilon = \min\{[a\cdot\alpha] + 1 - a\cdot\alpha,\ a\cdot\alpha - [a\cdot\alpha]\}$



$(a\alpha - \varepsilon, a\alpha + \varepsilon) \cap \mathbb{N} = \emptyset$

$p_{n_{k_\ell}} \notin \mathbb{N}$ $\lightning$ јер $p_{n_{k_\ell}}$ су природни

$\implies \lim\limits_{n\to\infty} q_n = +\infty$.