21. $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$

$x_1 = b$, $a \in \mathbb{R}$, $b > 0$

x_n конвергира?

$\lim\limits_{n\to\infty} x_n = ?$

$x_{n+1} \square x_n$

$\frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \square x_n$

$\frac{a}{2x_n} \square \frac{1}{2} x_n \quad / \cdot 2 > 0$

$\frac{a}{x_n} \square x_n$

$\frac{a}{x_n} - x_n \square 0$

$\frac{a - x_n^2}{x_n} \square 0$

1° $a < 0$

$x_n < 0 \Rightarrow x_{n+1} > x_n$

$x_n > 0 \Rightarrow x_{n+1} < x_n$

1.1° $x_n < 0$

$x_{n+1} = \frac{1}{2} \frac{x_n^2 + a}{x_n}$

$x_n^2 \square -a$?

$|x_n| \square \sqrt{|a|}$

$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ → не може АГ јер x_n и $\frac{a}{x_n}$ различитог знака

2° $a > 0$, $b > 0$

$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$

$x_1 > 0$

$\forall n \in \mathbb{N}$ $x_n > 0$ → ПМИ

$x_{n+1} \square x_n$

$\frac{a - x_n^2}{x_n} \square 0$

$x_n^2 \le a \Rightarrow \square$ „=“ $\ge$

$x_n^2 \ge a \Rightarrow \square$ „=“ $\le$

x_n и $\frac{a}{x_n}$ истог знака $\quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \ge \sqrt{x_n \cdot \frac{a}{x_n}} = \sqrt{a}$, $n \ge 1$

$x_n^2 \ge a$, $n \ge 2$

$x_{n+1} \le x_n$

$x_n \downarrow$

$x_n \downarrow$ и $x_n > 0 \Rightarrow x_n$ конвергира и $\lim\limits_{n\to\infty} x_n = x = \lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} \frac{x_n^2 + a}{2x_n} = \frac{x^2 + a}{2x}$

$x = \frac{x+a}{2x} \rightsquigarrow 2x^2 = x^2 + a$

$x^2 = a \Rightarrow x = \sqrt{a}, \quad x = -\sqrt{a}$

$x = -\sqrt{a}\ ?, \quad x_n > 0 \Rightarrow x \geq 0 \Rightarrow x \neq -\sqrt{a}$

$\Rightarrow x = \sqrt{a}, \quad a > 0$

2° $a = 0$

$x_{n+1} = \frac{x_n}{2}$

$x_1 = b \in \mathbb{R} \Rightarrow x_{n+1} = \frac{b}{2^n}$ ПМИ ..... $\Rightarrow \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{b}{2^n} = 0$

3° $a < 0$

$x_{n+1} \ \square\ x_n$

$0 \ \square\ x_n$

$x_1 = b > 0$

$x_2 < x_1$

$x_2 < b$

$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$

Ако би $\lim\limits_{n\to\infty} x_n = x \neq 0$

$x = \lim\limits_{n\to\infty} x_{n+1} = \frac{1}{2}\frac{x^2+a}{x} \rightsquigarrow 2x^2 = x^2 + a$

$x^2 = a, \quad a < 0$ ↯

Уколико низ конв. његов лимес је једино могуће да буде 0. Да ли је то могуће?

$|x+y| \geq \big||x| - |y|\big|$

$x, y \geq 0$ ✓

$x > 0, y < 0$

$|x+y| = |x - |y|| = \big||x| - |y|\big| \geq |x| - |y|$

$\lim\limits_{n\to\infty} x_n = 0$

$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0\ |x_n| < \varepsilon$

нека: $|x_n| < \varepsilon \Rightarrow |x_{n+1}| = \frac{1}{2}\left|x_n + \frac{a}{x_n}\right| \geq \frac{1}{2}\left|\frac{a}{x_n}\right| - |x_n| = \frac{1}{2}\frac{|a|}{|x_n|} - |x_n| > \frac{1}{2}\frac{|a|}{\varepsilon} - |x_n|$

$> \frac{1}{2}\frac{|a|}{\varepsilon} - \varepsilon$

$\Rightarrow$ постоји $\varepsilon > 0$ $\quad |x_n| < \varepsilon$ и $|x_{n+1}| > \varepsilon$

$\varepsilon \leq 1 \quad |x_{n+1}| > \frac{1}{2}\frac{|a|}{\varepsilon} - \varepsilon \geq \frac{1}{2}|a| - \varepsilon, \quad \varepsilon = \frac{1}{4}|a|$

$\Rightarrow |x_n| < \frac{1}{4}|a|$ и $|x_{n+1}| > \frac{1}{4}|a|$ ↯ $\Rightarrow \lim\limits_{n\to\infty} x_n \neq 0$

$\Rightarrow x_n$ дивергира.