17. $\sup C = ?$

$A = \{ (n+1)^{2/n^2} \mid n \in \mathbb{N} \}$

$B = \{ x(\alpha - x) \mid x \in (0,1) \}$, $\alpha \in \mathbb{Q}$

$B \not\subseteq [1, +\infty)$

$x(\alpha - x) \geq 0 \Rightarrow x^2 - \alpha x \leq 0$

$x \in (0, \alpha)$

$\alpha \geq 1 \Rightarrow x \cdot (\alpha - x) \geq 0$

$\alpha \geq 1 \quad B \subseteq (0, +\infty) \Rightarrow \inf B = 0$

$\exists x_n \in (0,1) : x_n(\alpha - x_n) \to 0 \qquad x_n \cdot (\alpha - x_n),\ x_n = \frac{1}{n} \Rightarrow \frac{1}{n} \cdot (\alpha - \frac{1}{n}) < \varepsilon$

за n довољно велико

$m = 1 \qquad (m+1)^{2/m^2} \in A$

$2^2 = 4 \in A \Rightarrow \frac{4}{x_n(\alpha - x_n)} \qquad \frac{1}{n} \cdot \underbrace{(\alpha - \frac{1}{n})}_{<\alpha} < \frac{\alpha}{n} < \varepsilon$

$n \geq [\frac{\alpha}{\varepsilon}] + 1$

$$\frac{4}{\frac{1}{n} \cdot (\alpha - \frac{1}{n})} = \frac{4n}{\alpha - \frac{1}{n}} \xrightarrow[n \to \infty]{} +\infty$$

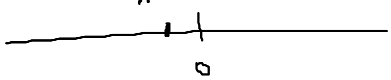
$C = \{ \frac{a}{b} \mid a \in A, b \in B \} \Rightarrow \sup C = „+\infty"$.

2° $\alpha \in (0, 1)$

$x = \alpha \Rightarrow 0 \in B$

$\Rightarrow \frac{a}{0}$ није добро деф $\Rightarrow C$ није добро деф скуп.

3° $\alpha \leq 0$



$x(\alpha - x) < 0 \Rightarrow B \subset (-\infty, 0)$

$x = \frac{1}{n} \qquad \underbrace{\frac{1}{n}(\alpha - \frac{1}{n})}_{<0} \to 0 \qquad \frac{a}{\frac{1}{n}(\alpha - \frac{1}{n})} \longrightarrow -\infty \Rightarrow \inf C = -\infty$

$\sup C = ? \qquad \sup C = \frac{?}{\inf B}$

$\frac{a}{b} \leq \frac{a}{\inf B} \quad / \ b, \inf B < 0$

$\frac{a}{b} \leq \frac{a}{\inf B} \leq \frac{\inf A}{\inf B}$ јер $b, \inf B < 0$, $a, \inf A > 0$.

$\inf A = 4$?

$(n+1)^{4n^2} > 4$ ↑ $n^2/2$

$(n+1) > 4^{n^2/2} = 2^{n^2}$ ↯

$(n+1)^{2/n^2} = \left((n+1)^{\frac{1}{n}}\right)^{2/n} = \left(\underbrace{\sqrt[n]{n+1}}_{\to 1}\right)^{\boxed{2/n}\to 0} \xrightarrow[n\to\infty]{} 1$

$(n+1)^{2/n^2 \to 0} > 1$

$\Rightarrow \underline{\underline{\inf A = 1}}$

$\inf B = ?$

$x\cdot(\alpha - x) \geq \inf B$

$\alpha \leq 0$

$-x\cdot(x-\alpha) \geq \inf B \quad / \cdot(-1)$

$\underbrace{x\cdot(x-\alpha)} \leq -\inf B$

$-\alpha = \beta > 0$

$\underset{\vee 0}{x}\cdot\underset{\vee 0}{(x+\beta)}, \quad x \in (0,1)$

$x \to 1$ $\quad x < 1$

$x+\beta < \beta$

$x\cdot(x+\beta) < 1\cdot(\beta+1) = \beta+1 = 1-\alpha$

$x \to 1$

$x_n = 1 - \frac{1}{n} \Rightarrow x_n\cdot(x_n+\beta) \xrightarrow[n\to\infty]{} 1+\beta$

$-\inf B = 1-\alpha$

$\inf B = \alpha - 1$

$\Rightarrow \sup C = \dfrac{\inf A}{\inf B} = \dfrac{1}{\alpha-1}$.

21. $x_{n+1} = \dfrac{4x_n}{2x_n+3}$, $x_1 = 1$,

$x_n \;\square\; x_{n+1}$ $\quad x_n > 0 \quad \forall n\in\mathbb{N}$ ПМИ...

$x_n \;\square\; \dfrac{4x_n}{2x_n+3} \quad /: x_n$

$1 \;\square\; \dfrac{4}{2x_n+3} \rightsquigarrow x_n \gtrless 1/2$

$$x_{n+1} = \frac{4x_n}{2x_n+3}$$

$$x_n = \frac{a_n}{b_n} \rightsquigarrow \frac{a_{n+1}}{b_{n+1}} = \frac{4\frac{a_n}{b_n}}{2\frac{a_n}{b_n}+3} = \frac{4a_n}{2a_n+3b_n}$$

$a_{n+1} = 4a_n$ $\qquad a_1 = 1$

$b_{n+1} = 2a_n + 3b_n$ $\qquad b_1 = 1$

$$a_n = \frac{b_{n+1}-3b_n}{2}$$

$$a_{n+1}=4a_n \Rightarrow \frac{b_{n+2}-3b_{n+1}}{2} = 2(b_{n+1}-3b_n)$$

$$b_{n+2} = 4b_{n+1} - 12b_n + 3b_{n+1} = 7b_{n+1} - 12b_n$$

$a_{n+1} = 4^n$

$a_2 = 4\cdot 1$

$a_3 = 4\cdot a_2 = 4\cdot 4 = 4^2$

$a_4 = 4\cdot a_3 = 4^3$

ПМИ: $\forall n\in\mathbb{N}$ $\quad a_n = 4^{n-1}$

(БИ) $a_1 = 4^0 = 1$ ✓

(ИК) $a_n = 4^{n-1}$ $\qquad a_{n+1} = 4\cdot a_n = 4^n$ ✓

$b_{n+2} = 7b_{n+1} - 12b_n$

$x^2 = 7x - 12 \rightsquigarrow x^2 - 7x + 12 = 0 \rightsquigarrow x_{1,2} = \frac{7\pm\sqrt{49-48}}{2}$

$x_1 = \frac{7+1}{2}$ $\qquad x_2 = \frac{7-1}{2}$

$x_1 = 4$, $x_2 = 3$

$b_n = c_1\cdot 4^n + c_2\cdot 3^n$

$b_1 = 1$

$b_2 = 2a_1 + 3b_1 = 5$

$n=1 \quad c_1\cdot 4 + c_2\cdot 3 = 1 \quad / \cdot 4$

$n=2 \quad c_1\cdot 16 + 9c_2 = 5 \quad -$

$\Rightarrow -3c_2 = 1$

$c_2 = -1/3$

$c_1 = \frac{1 - c_2\cdot 3}{4} = \frac{1+1}{4} = 1/2$

$$b_n = \frac{1}{2}4^n - \frac{1}{3}3^n$$

$$x_n = \frac{4^{n-1}}{\frac{1}{2}\cdot 4^n - \frac{1}{3}3^n} = \frac{4^{n-1}}{2\cdot 4^{n-1} - 3^{n-1}}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{4^{n-1}}{2\cdot 4^{n-1} - 3^{n-1}} \cdot \frac{1/4^{n-1}}{1/4^{n-1}} = \lim_{n\to\infty} \frac{1}{2 - \left(\frac{3}{4}\right)^{n-1}} = \frac{1}{2}.$$