20) a) $6x_{n+2} = 5x_{n+1} - x_n$, $\quad x_0 = 5 \quad x_1 = \frac{13}{6}$

$6t^2 = 5t - 1 \Rightarrow 6t^2 - 5t + 1 = 0 \Rightarrow t_1 = \frac{1}{2}$, $\quad t_2 = 1/3$

$x_n = c_1 \cdot \left(\frac{1}{2}\right)^n + c_2\left(\frac{1}{3}\right)^n$, $n \in \mathbb{N}$

$5 = c_1 + c_2$

$\frac{13}{6} = \frac{c_1}{2} + \frac{c_2}{3} \quad / \cdot 6 \Rightarrow 13 = 3c_1 + 2c_2 = 10 + c_1 \Rightarrow c_1 = 3, \ c_2 = 2$

$x_n = 3\left(\frac{1}{2}\right)^n + 2\left(\frac{1}{3}\right)^n$

пми? ....

б)
$$\lim_{n\to\infty} \frac{x_n}{5\cdot\left(\frac12\right)^n + 3\cdot\left(\frac14\right)^n} = \lim_{n\to\infty} \frac{3\cdot\left(\frac12\right)^n + 2\cdot\left(\frac13\right)^n}{5\cdot\left(\frac12\right)^n + 3\cdot\left(\frac14\right)^n}\cdot\frac{12^n}{12^n} =$$
$$= \lim_{n\to\infty}\frac{3\cdot 6^n + 2\cdot 4^n}{5\cdot 6^n + 3\cdot 3^n}\cdot\frac{\frac{1}{6^n}}{\frac{1}{6^n}} = \lim_{n\to\infty}\frac{3 + \underbrace{2\left(\frac23\right)^n}_{\to 0}}{5 + \underbrace{3\cdot\left(\frac12\right)^n}_{\to 0}} = \frac35$$

в)
$$\lim_{n\to\infty}\sqrt[n]{\underbrace{\frac{x_n}{5\cdot\left(\frac12\right)^n + 3\left(\frac14\right)^n}}_{y_n}} = \lim_{n\to\infty}\sqrt[n]{\underbrace{\frac{3\cdot\left(\frac12\right)^n + 2\left(\frac13\right)^n}{5\cdot\left(\frac12\right)^n + 3\left(\frac14\right)^n}}_{\to\frac35}} = 1$$

$\lim\limits_{n\to\infty}\sqrt[n]{a} = 1$, $\quad a > 0$

(*) $\varepsilon = \min\left(1 - \frac35, \frac35 - \frac12\right)$

$\exists n_0\ \forall n \geq n_0 \quad y_n \in \left(\frac35 - \varepsilon, \frac35 + \varepsilon\right) \subseteq \left(\frac12, 1\right)$

(*) $\Rightarrow n \geq n_0 \quad \frac12 < y_n < 1 \rightarrow \sqrt[n]{\frac12} < \sqrt[n]{y_n} < \sqrt[n]{1}$

$\sqrt[n]{\frac12} \to 1$, $\quad \sqrt[n]{y_n} \xrightarrow{\text{зато}} 1$, $\quad \sqrt[n]{1} \to 1$

24. $a_1 = a \in \mathbb{R}$

$a_{n+1} = 2a_n - a_n^2 = a_n(2 - a_n)$

Испитати конв. низа a_n.

1° $a = 0$ или $a = 2 \Rightarrow a_n = 0 \quad n \geq 2. \Rightarrow a_n$ конв.

2° $a > 2$ или $a < 0 \Rightarrow a_2 < 0$

$\Rightarrow \forall n \geq 2 \Rightarrow a_n < 0$

$\frac{a_n}{a_{n+1}} = \frac{a_n}{2a_n - a_n^2} = \frac{1}{2 - a_n} < 1 \quad / \cdot a_{n+1} < 0 \quad \Rightarrow a_n > a_{n+1} \Rightarrow a_n \downarrow, \quad a_n < 0$

$\Rightarrow \lim\limits_{n\to\infty} a_n = x \in \mathbb{R}$ или $\lim\limits_{n\to\infty} a_n = -\infty$

пп. $\lim\limits_{n\to\infty} a_n = x = \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} 2a_n - a_n^2 = 2x - x^2 \Rightarrow x = x^2 \Rightarrow x = 1$ или $x = 0$

$x \neq 1$ јер $a_n < 0$ $\forall n \in \mathbb{N}$

$x \neq 0$ $a_n < 0$ и $\underline{\underline{a_n \downarrow}}$

$\Rightarrow \lim\limits_{n\to\infty} a_n = -\infty$

3° $a \in (0,2)$ $[$ $($

$a_2 = a\cdot(2-a) > 0$

да ли $\forall n \in \mathbb{N}$ $a_n \in (0,2)$?

(БИ) $n=2$ $a_2 = a\cdot(2-a) > 0$, $a \in (0,2)$ ✓

$a_2 < 2$?

$2a - a^2 < 2 \rightsquigarrow 0 < 2 - 2a + a^2 = 1 + (1-a)^2$ ✓

$a_2 \in (0,2)$

(ИК) $n \in \mathbb{N}$ $a_n \in (0,2)$

$a_{n+1} = a_n(2 - a_n) > 0$

$2 - a_{n+1} = 1 + (1 - a_n)^2 > 0 \Rightarrow a_{n+1} \in (0,2)$.

$\Rightarrow a_n \in (0,2)$

$a_n - a_{n+1} = a_n^2 - a_n = a_n(a_n - 1) \underset{\uparrow\, a_n < 1}{<0} \rightsquigarrow$ зависи од односа a_n и 1.

$a \in (0,1) \Rightarrow a_n \in (0,1)$ $n \in \mathbb{N}$

$a_1 = a \in (0,1)$

$n \in \mathbb{N}$ $a_n \in (0,1)$

$1 - a_{n+1} = 1 + a_n^2 - 2a_n = (1 - a_n)^2 > 0$

$a_{n+1} \in (0,1)$

$\Rightarrow a_n \in (0,1)$ и $a_n \uparrow \Rightarrow$ конвергира $x = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1} = 2x - x^2$

$x^2 = x \Rightarrow x = 1$ или $x = 0$

$a = 1 \Rightarrow a_{n+1} = 1 \Rightarrow a_n$ константан

$\lim\limits_{n\to\infty} a_n = 1$

$a \in (0,1) \Rightarrow a_n \in (0,1)$ $a_n \uparrow \Rightarrow \lim\limits_{n\to\infty} a_n = 1$

$a \in (1,2) \Rightarrow a_n \in (1,2)$?

$\underline{a_n \in (0,2)}$, довольно показати $a_n > 1$

ПМИ ...

$a_n > 1 \Rightarrow a_n - a_{n+1} > 0$ $\checkmark$

$a_n \downarrow$, $a_n \in (1,2)$

$\Rightarrow x = 0$ или $x = 1$ ...

$a_{n+1} = 2a_n - a_n^2$

$a_1 = a \in (1,2)$

$$\underbrace{2a_n - a_n^2}_{a_{n+1}} - a_n = a_n - a_n^2 = a_n(1-a_n) \underset{a_n > 1}{<} 0 \Rightarrow a_n \downarrow$$

$$\Rightarrow \lim_{n\to\infty} a_n = 1.$$

$A, B \subseteq [1, +\infty)$

$$C = \frac{A}{B} \Rightarrow \sup C = \frac{\overset{= x}{\sup A}}{\underset{= y}{\inf B}} = \frac{x}{y}$$

$\varepsilon_1 > 0$

$\exists a \in A \quad a > x - \varepsilon_1$

$\varepsilon_2 > 0 \quad \exists b \in B \quad b < y + \varepsilon_2$

$\varepsilon > 0$

$\exists \underline{c = \frac{a}{b}}$? $\quad c > \frac{x}{y} - \varepsilon$

$$1 - \frac{\varepsilon_2}{y + \varepsilon_2}$$

$$\frac{a}{b} > \frac{x - \varepsilon_1}{y + \varepsilon_2} = \frac{x}{y + \varepsilon_2} - \frac{\varepsilon_1}{y + \varepsilon_2} = \frac{x}{y} \cdot \frac{y}{y + \varepsilon_2} - \frac{\varepsilon_1}{y + \varepsilon_2}$$

$$= \frac{x}{y} - \left(\frac{\varepsilon_2 \cdot x}{(y+\varepsilon_2)\, y} + \frac{\varepsilon_1}{y + \varepsilon_2} \right) \geq \frac{x}{y} - \varepsilon$$

$\varepsilon_1, \varepsilon_2 = ?$
> 0

$$\frac{\varepsilon_2 x + y\varepsilon_1}{y \cdot (y + \varepsilon_2)} \leq \frac{\varepsilon_2 x + y \varepsilon_1}{y^2} \leq \varepsilon$$

$$\varepsilon_2 x \le y^2\varepsilon - y\varepsilon_1$$

$$0 < \varepsilon_2 \le \frac{y^2\varepsilon - y\varepsilon_1}{x} = \frac{y}{x}(y\varepsilon - \varepsilon_1)$$

$$\varepsilon_1 > 0 \qquad \frac{y}{x}\cdot\underbrace{(y\varepsilon - \varepsilon_1)}_{>0} > 0$$

$$\boxed{\varepsilon_1 = \frac{1}{2}y\cdot\varepsilon} > 0 \longrightarrow a$$

$$\varepsilon_2 \le \underbrace{\frac{y}{x}\cdot\frac{1}{2}y\cdot\varepsilon}_{y_0} \Rightarrow \boxed{\varepsilon_2 = \frac{y^2\varepsilon}{2x} > 0} \longrightarrow b$$

$$\boxed{e = \frac{a}{b}}$$

$$\lim_{n\to\infty} \frac{1}{n} + \cdots + \frac{1}{2n} = \ln 2$$

k

$$\underset{(1+\frac{1}{k})^k}{\overset{}{a_k}} < e < \underset{(1+\frac{1}{k})^{k+1}}{b_k} \qquad k\in\mathbb{N}$$

$k > 1$

$$\frac{1}{k+1} < \ln\left(1+\frac{1}{k}\right)$$

$$\ln\left(1+\frac{1}{k}\right) < \frac{1}{k} < \ln\left(1+\frac{1}{k-1}\right), \quad k>1$$

$k\in\mathbb{N}$ $\qquad$ $k>1$

$$< 1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \ln(1+1) + \cdots + \ln\left(1+\frac{1}{n-1}\right)$$

$$e - x_n = e - \left(1 + \frac{1}{1!} + \cdots + \frac{1}{n!} \right) < \frac{1}{n \cdot n!} < \varepsilon$$

$\varepsilon > 0 \quad \exists n \in \mathbb{N}$

$$\frac{1}{n \cdot n!} < \varepsilon \qquad \frac{1}{2 \cdot 2!} > \varepsilon$$

$\downarrow$

0

$$\frac{1}{n \cdot n!} < \frac{1}{n} < \varepsilon \longrightarrow n = \left[\frac{1}{\varepsilon} \right] + 1$$