(12) $n^{n+1} > (n+1)^n \quad , \quad n \geq 3$

(БИ) $n = 3$

$3^4 > 4^3$

(ИК) $n^{n+1} > (n+1)^n \overset{?}{\Rightarrow} (n+1)^{n+2} > (n+2)^{n+1} \; ? \quad / : (n+2)^{n+1}$

$/ : (n+1)^n$

$\frac{n^{n+1}}{(n+1)^n} > 1 \qquad \Leftrightarrow \qquad \frac{(n+1)^{n+2}}{(n+2)^{n+1}} > 1 \; ?$

$n \cdot \left(\frac{n}{n+1}\right)^n > 1 \qquad \qquad (n+1) \cdot \left(1 - \frac{1}{n+2}\right)^{n+1} > 1 \; ?$

$n\left(1 - \frac{1}{n+1}\right)^n > 1$

$\left(1 - \frac{1}{n+1}\right)^n > \frac{1}{n}$

$n+1 < n+2$

$\frac{1}{n+1} > \frac{1}{n+2}$

$1 - \frac{1}{n+1} < 1 - \frac{1}{n+2}$

$(n+1) \cdot \left(1 - \frac{1}{n+2}\right)^{n+1} > (n+1) \cdot \left(1 - \frac{1}{n+1}\right)^{n+1} =$

$= (n+1) \cdot \left(1 - \frac{1}{n+1}\right) \cdot \left(1 - \frac{1}{n+1}\right)^n >$

$\underset{ИХ}{>} (n+1) \frac{n}{n+1} \cdot \frac{1}{n} = 1$

(14) $a, b, c, d > 0 \qquad a+b+c+d = 1$

$\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right)\left(\frac{1}{d} - 1\right) \geq 81 \; ?$

$\frac{1-a}{a} \cdot \frac{1-b}{b} \cdot \frac{1-c}{c} \cdot \frac{1-d}{d} \overset{a+b+c+d=1}{=} \frac{b+c+d}{a} \cdot \frac{a+c+d}{b} \cdot \frac{a+b+d}{c} \cdot \frac{a+b+c}{d}$

($b+c+d \geq 3\sqrt[3]{bcd}$, $a+c+d \geq 3\sqrt[3]{acd}$, $a+b+d \geq 3\sqrt[3]{abd}$, $a+b+c \geq 3\sqrt[3]{abc}$)

$\underset{АГ}{\geq} 3^4 \frac{\sqrt[3]{a^3 \cdot b^3 c^3 d^3}}{abcd} = 81 \cdot \frac{abcd}{abcd} = 81.$

(16) $A_4 = \underbrace{\left\{ \frac{nm^2 + 2nm - n - 4m^2 - 8m + 4}{nm^2 + 2nm} \,\middle|\, n, m \in \mathbb{N} \right\}}_{A} \cup \underbrace{\left\{ \frac{10n^2}{m^2 + m + 7n^2} \,\middle|\, m, n \in \mathbb{N} \right\}}_{B}$

$\inf A_4 = ?$

$\sup A_4 = ?$

$A_4 = A \cup B$

$\Rightarrow \sup A_4 = \max\{\sup A, \sup B\}$

$\inf A_4 = \min\{\inf A, \inf B\}$

$$A: \quad \frac{nm \cdot \left(m+2-\frac{n}{m}-\frac{4m^2}{n}-8\frac{m}{n}+\frac{4}{nm}\right)}{nm\,(m+2)} = 1 - \underbrace{\frac{n+4m^2+8m-4}{nm\,(m+2)}}\,, \quad n,m \in \mathbb{N}$$

$n = m = 1$

$$\frac{1+4+8-4}{1\cdot 1\cdot 3} = \frac{9}{3} = 3$$

$\sup A = 1$

$\inf A = \qquad \leq -2$

$n,m = ?$: $\dfrac{n+4m^2+8m-4}{nm(m+2)} > 3$?

$$n+4m^2+8m-4 > 3nm^2+6nm$$

$$4m^2+8m-4 > n(3m^2+6m-1) \qquad < 1$$

$$\Rightarrow \quad n < \frac{4m^2+8m-4}{3m^2+6m-1} = 1 + \underbrace{\frac{m^2+2m-3}{3m^2+6m-1}} < 2$$

$$m^2+2m-3 < 3m^2+6m-1$$

$$0 < 2m^2+4m+2 \quad \checkmark$$

$\Rightarrow$ ga δu $\dfrac{n+4m^2+8m-4}{n\cdot m(m+2)} > 3$, $n = 1$.

Dakle, $\dfrac{1+4m^2+8m-4}{m\cdot(m+2)} > 3 \longrightarrow \dfrac{4m^2+8m-3}{m^2+2m} = 4 - \dfrac{3}{m^2+2m}$

$n = 1 \quad \Rightarrow \quad 1 - 4 + \dfrac{3}{m^2+2m} = -3 + \dfrac{3}{m^2+2m}$

-3

$\Rightarrow \inf A = -3$

$$\forall n,m \qquad 1 - \frac{n+4m^2+8m-4}{nm\,(m+2)} \geq \underline{\underline{-3}}$$

$$\frac{n+4m^2+8m-4}{nm\,(m+2)} \leq 4$$

$$n+4m^2+8m-4 \leq 4nm(m+2)$$

$$4m^2+8m-4 \leq n(4m(m+2)-1)$$

$$n \geq \frac{4m^2+8m-4}{4m^2+8m-1} = 1 - \frac{3}{4m^2+8m-1} \qquad \checkmark\checkmark \text{ jer } n \in \mathbb{N}$$

$\varepsilon > 0$ произвољно

$n, m \in \mathbb{N}$ $\quad 1 - \frac{n+4m^2+8m-4}{nm(m+2)} < -3+\varepsilon$

$n = 1$
$m = ?$ $\Rightarrow 4 - \frac{1+4m^2+8m-4}{m(m+2)} < \varepsilon$

$$\frac{4m^2+8m - 4m^2-8m+3}{m^2+2m} < \varepsilon$$

$$3 < \varepsilon \cdot (m+2) \cdot m$$

$(m+2) \cdot m > \frac{3}{\varepsilon}$ $\quad$ довољно је да је $\underline{\underline{m > \frac{3}{\varepsilon}}}$

за $m = \left[\frac{3}{\varepsilon}\right] + 1$

$\sup A = 1$ ...

$\inf B \geq 0$

$\sup B = \frac{10}{7}$

$\underline{\underline{m = 1}}$ $\leadsto$ $\frac{10n^2}{m^2+m+7n^2} \leq \underline{\underline{\frac{10}{7}}}$

$\varepsilon > 0$ $\quad m, n = ?$ $\quad \frac{10n^2}{m^2+m+7n^2} > \frac{10}{7} - \varepsilon$

$m = 1$
$n = ?$ $\quad \varepsilon > \frac{10}{7} - \frac{10n^2}{2+7n^2} = \frac{\cancel{70n^2}+20-\cancel{70n^2}}{7\cdot(2+7n^2)}$

$n > \ldots$

заврши за домаћи ...

$\sup A_4 = \frac{10}{7}$

$\inf A_4 = -3$.