(13) $a, b, c > 0$

$$A = \frac{a^3}{a^2+ab+b^2} + \underbrace{\frac{b^3}{b^2+bc+c^2}}_{a_2^2} + \underbrace{\frac{c^3}{a^2+ac+c^2}}_{a_3^2} \overset{?}{\geq} \frac{a+b+c}{3}$$

$$\overset{\shortparallel}{\frac{a^3-b^3}{a-b}} \quad \dots$$

Коши-Шварц: $(a_1^2+a_2^2+\dots+a_n^2)(b_1^2+\dots b_n^2) \geq (\overset{\overset{a}{\shortparallel}}{a_1 \cdot b_1}+a_2 \cdot b_2+\dots+a_n \cdot b_n)^2$

$$a_1^2 = \frac{a^3}{a^2+ab+b^2} \qquad b_1^2 = \frac{a^2+ab+b^2}{a} \qquad a_1^2 \cdot b_1^2 = a^2$$

$$a_1 = \frac{a^{3/2}}{\sqrt{a^2+ab+b^2}}$$

имамо

$$A \cdot 3(a+b+c) \overset{?}{\geq} A \cdot \left(\frac{a^2+ab+b^2}{a} + \frac{b^2+bc+c^2}{b} + \frac{a^2+ac+c^2}{c} \right) \geq (a+b+c)^2$$

$$3(a+b+c) \overset{?}{\geq} \frac{a^2+ab+b^2}{a} + \frac{b^2+bc+c^2}{b} + \frac{a^2+ac+c^2}{c} = a+b+\frac{b^2}{a}+b+c+\frac{c^2}{b}+a+c+\frac{a^2}{c}$$

? $a+b+c \geq \frac{b^2}{a}+\frac{c^2}{b}+\frac{a^2}{c}$? $\to$ да ли ово доводи до решења?

Пробајте на други начин! Зашто не?

(16)

$$A_1 = \left\{ \frac{m+n}{mn+1} \;\middle|\; m, n \in \mathbb{N} \right\}$$

n фиксирано $\frac{1 \cdot m+n}{mn+1} \downarrow$ по m

$$\frac{m+n}{mn+1} \geq \frac{m+1+n}{(m+1)n+1} \; ?$$

$$(m+n) \cdot ((m+1) \cdot n+1) \geq (m+n+1) \cdot (mn+1)$$

$$m \cdot (\cancel{m}+\cancel{1}) \cdot n + n^2(\cancel{m}+1) + \cancel{m}+\cancel{n} \overset{?}{\geq} \cancel{m^2 n} + \cancel{n^2 m} + \cancel{mn} + \cancel{m}+\cancel{n}+1$$

$$n^2 \geq 1 \; \checkmark$$

$$A_1 = \bigcup_{n \in \mathbb{N}} \underbrace{\left\{ \frac{m+n}{mn+1} \;\middle|\; m \in \mathbb{N} \right\}}_{A_n} \to \inf A_n = \frac{1}{n}$$

$$\sup A_n = 1 = \max_{m=1} A_n$$

$\bullet\ \inf A_1 = \inf\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \inf\{\inf A_n \mid n\in\mathbb{N}\} = \inf\{\tfrac{1}{n} \mid n\in\mathbb{N}\} = 0$

$\sup A_1 = \underbrace{1}_{m=1} = \max A_1$

ово треба формално показати:

1° $\forall m,n\in\mathbb{N}\quad \dfrac{m+n}{mn+1} \geq 0$ ✓

2° $\varepsilon>0$ произвољно $\exists m,n\in\mathbb{N}$ тд $\dfrac{m+n}{mn+1} < \varepsilon + 0 = \varepsilon$

$$m+n < \varepsilon mn + \varepsilon$$

$$m(1-\varepsilon n) < \varepsilon - n \quad /\cdot(-1)$$

$$m(\varepsilon n - 1) > n - \varepsilon$$

$$m > \frac{n-\varepsilon}{\varepsilon n - 1}$$ уколико $\varepsilon n - 1 > 0$

$\varepsilon n > 1$ $\quad n > \dfrac{1}{\varepsilon}$

$n = \left[\dfrac{1}{\varepsilon}\right] + 1 > \dfrac{1}{\varepsilon}$

$$m = \left[\frac{n-\varepsilon}{\varepsilon n - 1}\right] + 1$$

$$A_2 = \left\{ \sin\Big(\underbrace{\frac{5n-10}{4n+1}\pi}_{\alpha_n}\Big) \;\Big|\; n\in\mathbb{N}\right\} \subseteq [-1,1]$$

$\in [-1,1]$

$\sup A_2 = ?$

$\inf A_2 = ?$

Да ли $\exists n\in\mathbb{N}$ $\sin\left(\frac{5n-10}{4n+1}\pi\right) = 1$?

$$\frac{\overset{=4n+1+n-11}{5n-10}}{4n+1}\pi \overset{?}{=} 2k\pi + \frac{\pi}{2}$$

$$\pi + \frac{n-11}{4n+1}\pi \overset{?}{=} 2k\pi + \frac{\pi}{2}$$

$$5n-10 = \frac{5}{4}\cdot(4n+1) + \dots$$

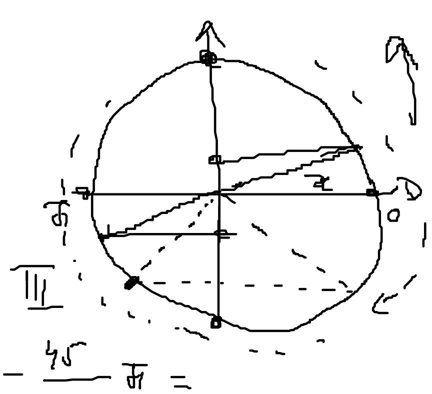
$$-10 = \frac{5}{4} + \dots$$

$$\dots = -10 - 5/4 = -\frac{45}{4}$$

$$\alpha_n = \frac{5}{4}\pi - \frac{45}{4(4n+1)}\pi$$

$\alpha_n \uparrow$ по n

$\sin\frac{5}{4}\pi$



за $n=1$ $\quad \alpha_n = \frac{5}{4}\pi - \frac{45}{4\cdot 5}\pi =$

$= \frac{5}{4}\pi - 5\pi < -\pi$

$$\frac{5}{4}\pi - \frac{45}{4\cdot(4n+1)}\pi = 2k\pi + \frac{\pi}{2}$$

$$-\frac{45\pi}{16n+4} = 2k\pi + \underset{\underset{\frac{2}{4}\pi}{\shortparallel}}{\frac{\pi}{2}} - \frac{5}{4}\pi = 2k\pi - \frac{3}{4}\pi \quad / \ \frac{16n+4}{\pi}$$

$$-45 = 2k\cdot(16n+4) - 3\cdot(4n+1) = n(32k-12) + 8k - 3$$

$n \in \mathbb{N}$ $\quad n=2$ $\quad k=1$

$$n\cdot(12-32k) = 8k+42$$

$\underline{\underline{k<0}}$ $\quad k=-1$

$$12-32k \geq 12+32 = 44$$

$$8k+42 < 42-8 < 44$$

$$n\cdot(\quad) \geq 44\cdot n = 8k+42 < 44 \ \lightning$$

$\Rightarrow$ овакво k не постоји

$\Rightarrow \sup A_1 \neq 1$

Нисмо успели да нађемо n и k тд $\sin\alpha_n = 1$.

Друга идеја: $\quad \sin(\pi + x) = -\sin x$

уместо A_2 посматрамо $B_2 = \{ \sin(\underbrace{\tfrac{n-11}{4n+1}\pi}_{\beta_n}) \mid n \in \mathbb{N} \}$

Тада је $B_2 = -A_2$, па важи

$\sin A_2 = -\inf B_2$ и $\inf A_2 = -\sup B_2$.

Приметимо за $n \geq 11$ $\quad 0 \leq \beta_n < \frac{\pi}{4}$. $\qquad \frac{n-11}{4n+1} = \frac{1}{4} - \frac{45}{4\cdot(4n+1)}$

$\sin \uparrow$ на $[0, \frac{\pi}{4})$ и $\beta_n \uparrow$ по n

$\Rightarrow 0 \leq \sin\beta_n < \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ за $n \geq 11$

Треба проверити шта се дешава ако $n < 11$.

$n=1 \Rightarrow \sin\beta_1 = \sin -2\pi = 0$

$n=2 \Rightarrow \sin\beta_2 = \sin -\pi = 0$

$n=3 \Rightarrow \sin\beta_3 = \sin\frac{-8}{13}\pi < 0$

$n=4 \Rightarrow \sin\beta_4 = \sin\frac{-7}{17}\pi$

$n=5 \Rightarrow \sin\beta_5 = \sin\frac{-6}{18}\pi = \sin -\frac{\pi}{3}$

$n=6 \Rightarrow$ $\quad \vdots$

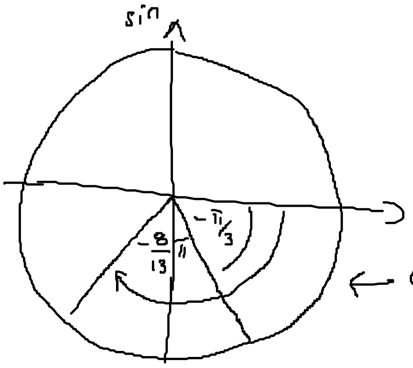
$n=7$

$n=9$

$n=10$

како $\beta_n \uparrow$ по n

$\Rightarrow -\frac{\pi}{3} \leq \beta_n \leq 0$ за $5 \leq n \leq 10$



← овде се налазе β_n за $5 \le n \le 10$

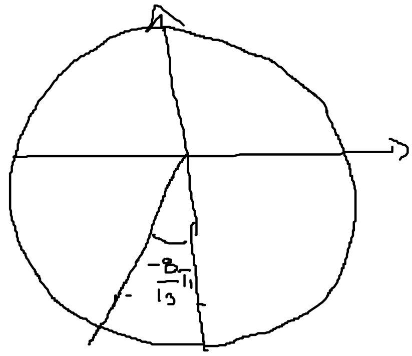
$\Rightarrow -\sin\frac{\pi}{3} \le \sin\beta_n \le 0$ и знамо $-\frac{\pi}{2} \le -\frac{7}{17}\pi \le -\frac{\pi}{3}$

Треба проверити који је мањи број

$\sin -\frac{8}{13}\pi$ или $\sin -\frac{7}{17}\pi$,

довољно је проверити који

од аргумената је ближи $-\frac{\pi}{2}$.



$$\left| -\frac{\pi}{2} + \frac{8}{13}\pi \right| = \left| -\frac{13}{26} + \frac{16}{26} \right| \pi =$$

$$= \frac{3}{26}\pi \quad >$$

$$\left| -\frac{\pi}{2} + \frac{7}{17}\pi \right| = \left| \frac{-17}{34} + \frac{14}{34} \right| \pi = \frac{3}{34}\pi$$

$\Rightarrow -\frac{7}{17}\pi$ је ближе $-\frac{\pi}{2}$ од $-\frac{8}{13}\pi$

$\Rightarrow \boxed{\sin -\frac{7}{17}\pi} < \sin -\frac{8}{13}\pi$

$\Rightarrow \min B_2 = \sin -\frac{7}{17}\pi = \sin\beta_4$

Како за све $n \le 11$ $\sin\beta_n \le 0$, довољно је

посматрати за sup $n > 11$.

Доказаћемо $\sup B_2 = \sin\frac{\pi}{4}$.

Јасно $\forall n \in \mathbb{N}$ $\sin\beta_n \le \sin\frac{\pi}{4}$.

$\varepsilon > 0$ произвољно ? $\exists n_0$: $\sin\beta_{n_0} > \sin\frac{\pi}{4} - \varepsilon$.

? $\exists n_0$: $\varepsilon > \sin\frac{\pi}{4} - \sin\beta_{n_0} = 2\sin\left(\frac{\pi}{4} - \beta_{n_0}\right) \cdot \cos\left(\frac{\pi}{4} + \beta_{n_0}\right) \ge 2\sin\left(\frac{\pi}{4} - \beta_{n_0}\right)$

јер $\cos \le 1$.

$n_0 = ?$: $\sin\left(\frac{\pi}{4} - \beta_{n_0}\right) < \varepsilon/2$

$\|$

$\sin\frac{45}{16n_0+4}\pi < \varepsilon/2$ → такво n_0 постоји

јер $\sin x \le x$ за $x \ge 0$.