$$H_n = \frac{n}{\frac{1}{x_1} + \dots + \frac{1}{x_n}} \qquad x_1, \dots, x_n > 0$$

$$G_n = \sqrt[n]{x_1 \cdot x_2 \cdot \dots \cdot x_n} \qquad x_1, \dots, x_n > 0$$

$H_n \leq G_n$

$$H_n = \frac{n}{\overbrace{\frac{x_2 \cdots x_n + x_1 x_3 \cdots x_n + \dots + x_1 \cdots x_{n-1}}{x_1 \cdots x_n}}^{\text{имамо их } n}} = \frac{n \cdot x_1 \cdots x_n}{x_2 \cdots x_n + x_1 \cdot x_3 \cdots x_n + \dots + x_1 \cdots x_{n-1}}$$

свака x_i се јавља $n-1$ пут

$$\frac{1}{H_n} = \frac{x_2 \cdots x_n + \dots + x_1 \cdots x_{n-1}}{n\, x_1 \cdots x_n} = \frac{1}{x_1 \cdots x_n} \left(\frac{\overbrace{x_2 \cdots x_n + \dots + x_1 \cdots x_{n-1}}^{n \text{ чланова}}}{n} \right)$$

$$\geq \frac{1}{x_1 \cdots x_n} \sqrt[n]{x_1^{n-1} x_2^{n-1} \cdots x_n^{n-1}}$$

$$= \sqrt[n]{\frac{1}{x_1 x_2 \cdots x_n}} = \frac{1}{\sqrt[n]{x_1 \cdots x_n}} = \frac{1}{G_n}$$

$\Rightarrow \quad H_n \leq G_n \quad \checkmark$

(8) $\theta \in \mathbb{R} \quad \forall n \in \mathbb{N} \ ?\ \exists\, p_n(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0, \quad a_{n-1}, \dots, a_0 \in \mathbb{Z}$

и $p_n(2\cos\theta) = 2\cos(n\theta)$?

$$\cos(n+1)\theta = \cos(n\theta + \theta) = \underbrace{\cos n\theta \cdot \cos\theta} - \sin n\theta \cdot \sin\theta$$

$$\cos(n-1)\theta = \cos(n\theta - \theta) = \cos n\theta \cdot \cos\theta + \sin n\theta \cdot \sin\theta \quad \Big) +$$

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos n\theta \cdot \cos\theta \quad / \cdot 2$$

$$2\cos(n+1)\theta = \underbrace{2\cos n\theta}_{p_n(\underbrace{2\cos\theta}_{x})} \cdot \underbrace{2\cos\theta}_{x} - \underbrace{2\cos(n-1)\theta}_{p_{n-1}(\underbrace{2\cos\theta}_{x})}.$$

Радимо индукцију са кораком 2:

(БИ) $n=1 \qquad p_1(2\cos\theta) = 2\cos 1\cdot\theta = 2\cos\theta \Rightarrow p_1(x) = x, \quad a_0 = 0 \in \mathbb{Z}$

$n=2 \qquad p_2(2\cos\theta) = 2\cos 2\theta = 2(\cos^2\theta - \underbrace{\sin^2\theta}_{1-\cos^2\theta}) = 2\cdot(2\cos^2\theta - 1) =$

$$= 4\cos^2\theta - 2 = (2\cos\theta)^2 - 2 \rightarrow \begin{array}{l} a_1 = 0 \\ a_0 = -2 \end{array}$$

$p_2(x) = x^2 - 2$

(ИК) Нека $\exists\, p_n$ и p_{n-1} за $n \geq 2$ т.ч.

$p_n(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$ $\qquad a_{n-1}, \dots, a_0, b_{n-2}, \dots, b_0 \in \mathbb{Z}$

$p_{n-1}(x) = x^{n-1} + b_{n-2}x^{n-2} + \dots + b_0$

$p_n(2\cos\theta) = 2\cos n\theta$, $p_{n-1}(2\cos\theta) = 2\cos(n-1)\theta$

Да ли $\exists\, p_{n+1}$; $p_{n+1}(x) = x^{n+1} + c_n x^n + \dots + c_1 x + c_0$, $c_n, \dots, c_0 \in \mathbb{Z}$

и $p_{n+1}(2\cos\theta) = 2\cos(n+1)\theta$?

$$2\cos(n+1)\theta = (2\cos n\theta)\cdot(2\cos\theta) - 2\cos(n-1)\theta =$$

$$= p_n(\underbrace{2\cos\theta}_{x})\cdot(\underbrace{2\cos\theta}_{x}) - p_{n-1}(\underbrace{2\cos\theta}_{x}) = p_{n+1}(2\cos\theta)$$

$$p_{n+1}(x) = x \cdot p_n(x) - p_{n-1}(x) =$$

$$= x\cdot(x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0) - (x^{n-1} + b_{n-2}x^{n-2} + \dots + b_0)$$

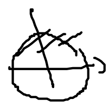
$$= x^{n+1} + \underbrace{a_{n-1}}_{c_n} x^n + \underbrace{(a_{n-2} - 1)}_{c_{n-1}} x^{n-1} + \underbrace{(a_{n-3} - b_{n-2})}_{c_{n-2}} x^{n-2} + \dots + \underbrace{(a_0 - b_1)}_{c_1} x \underbrace{- b_0}_{c_0}$$

$\Rightarrow c_n, c_{n-1}, \dots, c_1, c_0 \in \mathbb{Z}$

(11) $\quad \left|\sin\left(\sum_{i=1}^{n} x_i\right)\right| \leq \sum_{i=1}^{n} \sin x_i \qquad 1 \leq i \leq n \qquad x_i = [0, \pi]$

$\Downarrow$

$\sin x_i$



(БИ) $n = 1$: $|\sin x_1| \leq \sin x_1$, $x_1 \in [0, \pi]$.

(ИК) $\left|\sin\left(\sum_{i=1}^{n+1} x_i\right)\right| = \left|\sin\left(\sum_{i=1}^{n} x_i + x_{n+1}\right)\right| = \left|\sin\left(\sum_{i=1}^{n} x_i\right)\cdot\cos x_{n+1} + \sin x_{n+1}\cdot\cos\left(\sum_1^n x_i\right)\right|$

$n \Rightarrow n+1 \Rightarrow y_1, \dots, y_{n+1} \in [0, \pi]$

$\forall x_1, \dots, x_n \in [0, \pi]$

$$\leq \left|\sin\left(\sum_1^n x_i\right)\right| \cdot \underbrace{|\cos x_{n+1}|}_{\leq 1} + \underbrace{|\sin x_{n+1}|}_{\geq 0} \cdot \underbrace{\left|\cos\left(\sum_1^n x_i\right)\right|}_{\leq 1}$$

$$\leq \sum_{n=1}^{n} \sin x_i \cdot 1 + \sin x_{n+1} \cdot 1 = \sum_{n=1}^{n+1} \sin x_i \checkmark$$