⑥ $H_n \le G_n$ $\quad x_1, \dots, x_n > 0$

$$H_n = \frac{n}{\frac{1}{x_1} + \dots + \frac{1}{x_n}}, \quad G_n = \sqrt[n]{x_1 \cdot x_2 \cdots x_n}$$

$f(x) = e^{-x}$ конвексна $\forall x \in \mathbb{R}$

Јенсен: $f\left(\frac{x_1 + \dots + x_n}{n}\right) \le \frac{f(x_1) + \dots + f(x_n)}{n}$

$$e^{\frac{-x_1 - \dots - x_n}{n}} = \sqrt[n]{\frac{1}{e^{x_1}} \cdot \frac{1}{e^{x_2}} \cdots \frac{1}{e^{x_n}}} \le \frac{e^{-x_1} + \dots + e^{-x_n}}{n} = \frac{\frac{1}{e^{x_1}} + \dots + \frac{1}{e^{x_n}}}{n}$$

$$\sqrt[n]{\frac{1}{e^{x_1}} \cdots \frac{1}{e^{x_n}}} = \frac{1}{\sqrt[n]{e^{x_1} \cdots e^{x_n}}}$$

$y_1, \dots, y_n > 0 \Rightarrow \exists x_1, \dots, x_n \in \mathbb{R} \quad y_k = e^{x_k} \quad 1 \le k \le n$

$$\frac{1}{\sqrt[n]{y_1 \cdots y_n}} \le \frac{\frac{1}{y_1} + \dots + \frac{1}{y_n}}{n} \quad \Uparrow -1$$

$$\sqrt[n]{y_1 \cdots y_n} \ge \frac{n}{\frac{1}{y_1} + \dots + \frac{1}{y_n}}.$$

⑦ $n \in \mathbb{N}$ $\quad \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} \ge \frac{1}{2}$?

(БИ) $n = 1$: $\frac{1}{2} \ge \frac{1}{2}$ ✓

(ИК) Ако $\frac{1}{n+1} + \dots + \frac{1}{2n} \ge 1/2$ за некое $n \in \mathbb{N}$

да ли оттуда $\frac{1}{n+2} + \dots \frac{1}{2(n+1)} \ge \frac{1}{2}$?

$$\underbrace{\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}}_{\ge \frac{1}{2}} + \frac{1}{2n+1} + \frac{1}{2(n+1)} \ge \frac{1}{2} + \frac{1}{n+1} \ ?$$

$$? \quad \frac{1}{2n+1} + \frac{1}{2(\cancel{n+1})} \ge \frac{1}{n+1} \ ?$$

$$\frac{1}{n+1} = \frac{\cancel{2}\ 1}{2(n+1)}$$

$$\frac{1}{2n+1} \ge \frac{1}{2n+2} \quad ✓✓$$

$\theta \in \mathbb{R}$ фиксирано $\forall n \in \mathbb{N}$

(8) $\exists\ a_0, a_1, \ldots, a_{n-1} \in \mathbb{Z}$

$$2\cos n\theta = 1 \cdot (2\cos\theta)^n + a_{n-1}(2\cos\theta)^{n-1} + \cdots + a_1(2\cos\theta) + a_0 \quad ?$$

(*)

$$\cos(n+1)\theta = \cos(n\theta+\theta) = \underbrace{\cos n\theta \cdot \cos\theta}\ \underbrace{-\sin n\theta \cdot \sin\theta}$$

$$\cos(n-1)\theta = \cos(n\theta-\theta) = \cos n\theta \cdot \cos\theta + \sin n\theta \cdot \sin\theta$$

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos n\theta \cdot \cos\theta$$

$$\cos(n+1)\theta = 2\cos n\theta \cdot \cos\theta - \cos(n-1)\theta \quad / \cdot 2$$

$$2\cos(n+1)\theta = \underbrace{(2\cos n\theta)}_{P_n(2\cos\theta)}(2\cos\theta) - \underbrace{2\cos(n-1)\theta}_{P_{n-1}(2\cos\theta)} = P_{n+1}(2\cos\theta)$$

$\Rightarrow$ Индукција са кораком 2:

(БИ) $n=1$ и $n=2$

$$2\cos 1\cdot\theta = 2\cos\theta + 0, \quad a_0 = 0 \in \mathbb{Z} \ \checkmark$$

$$2\cos(2\theta) = 2\cdot(\cos^2\theta - \underset{1-\cos^2\theta}{\sin^2\theta}) = 4\cos^2\theta - 1 = (2\cos\theta)^2 - 2 \rightarrow a_0 = -2, \ a_1 = 0$$

(ИК) Прет. да тврђење важи за неко n и $n-1$, тј. постоје полиноми $P_n(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

и $P_{n-1}(x) = x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0$

$a_{n-1}, \ldots, a_0, b_{n-2}, \ldots, b_0 \in \mathbb{Z}$ и $P_n(2\cos\theta) = 2\cos n\theta$ и $P_{n-1}(2\cos\theta) = 2\cos(n-1)\theta$.

$$2\cos(n+1)\theta = (2\cos n\theta)\cdot(\underset{=y}{2\cos\theta}) - 2\cos(n-1)\theta =$$

$$= (y^n + a_{n-1}y^{n-1} + \cdots + a_1 y + a_0)\cdot y - (y^{n-1} + b_{n-2}y^{n-2} + \cdots + b_1 y + b_0)$$

претпоставка $2\cos\theta = y$

$$= y^{n+1} + \overset{c_n}{a_{n-1}}\cdot y^n + \overset{c_{n-1}}{(a_{n-2}-1)}y^{n-1} + \overset{c_{n-2}}{(a_{n-3}-b_{n-2})}y^{n-2} + \cdots + \underset{c_2}{(a_1-b_2)}y^2 + \underset{c_1}{(a_0-b_1)}y \underset{c_0}{-b_0}$$

$$\Rightarrow 2\cos(n+1)\theta = (2\cos\theta)^{n+1} + c_n(2\cos\theta)^n + \cdots + c_1(2\cos\theta) + c_0.$$

(10) $S_n = q + 2q^2 + \dots + nq^n = ?$

$S_n = q(1 + 2q + \dots + nq^{n-1})$

$S_{n-1} = q(1 + 2q + \dots + nq^{n-1}) = q + 2q^2 + \dots + (n-1)q^{n-1} \quad / \cdot q$

$qS_{n-1} = q^2 + 2q^3 + \dots + (n-1)q^n$

$S_n = q + 2q^2 + \dots + nq^n$

$S_n - qS_{n-1} = q + q^2 + \dots + q^n = q(1 + q + \dots + q^{n-1}) = q \cdot \frac{1-q^n}{1-q}$

за $q \neq 1$

за $q = 1$ $\quad S_n = 1 + 2 + \dots + n = \frac{(n+1)n}{2}$

$q \neq 1 \quad S_n - qS_{n-1} = q \cdot \frac{1-q^n}{1-q} \Rightarrow S_n = ? \, ? \, ?$

. . . .

$S_n = q + 2q^2 + \dots nq^n \quad / \cdot q$

$qS_n = q^2 + 2q^3 + \dots + nq^{n+1}$

$S_n - qS_n = \underbrace{q + q^2 + \dots + q^n}_{q(1+q+\dots+q^{n-1})} - n \cdot q^{n+1} = q \cdot \frac{1-q^n}{1-q} - n \cdot q^{n+1}$

$S_n = \left(\frac{q - q^{n+1}}{1-q} - nq^{n+1} \right) \cdot \frac{1}{1-q} = \frac{q - (n+1)q^{n+1} + n\, q^{n+2}}{(1-q)^2}$

Индукцијом доказујемо да ово заиста важи:

(БИ) $\quad S_1 = q = \frac{q - 2q^2 + q^3}{(1-q)^2} = q \quad \checkmark$

(ИК) $\quad S_n = \frac{q - (n+1)q^{n+1} + nq^{n+2}}{(1-q)^2}$

$S_{n+1} = \underbrace{S_n}_{\frac{q-(n+1)q^{n+1}+nq^{n+2}}{(1-q)^2}} + (n+1)q^{n+1} \overset{?}{=} \frac{q - (n+2)q^{n+2} + (n+1)q^{n+3}}{(1-q)^2} \quad / \cdot \underbrace{(1-q)^2}_{1-2q+q^2}$

$\cancel{q} - \cancel{(n+1)q^{n+1}} + nq^{n+2} + \cancel{(n+1)q^{n+1}} - 2(n+1)q^{n+2} + \cancel{(n+1)q^{n+3}} \overset{?}{=} \cancel{q} - (n+2)q^{n+2} + \cancel{(n+1)q^{n+3}}$ ✓