2019. I колоквијум

5. $f \in C^1[a,b]$, $\exists c \in (a,b)$ $f'(c)=0$

$\overset{?}{\Rightarrow} \exists \xi \in (a,b)$ $\quad f(\xi) = f(a) + f'(\xi)(b-a)$.

- - - -

Лагранж: $[a,b]$ $\quad \frac{f(b)-f(a)}{b-a} = f'(m)$, $m \in (a,b)$

$$f(b) = f(a) + f'(m)(b-a)$$
$$f(\xi) - f(a) = f'(\xi)(b-a)$$
...

ако: $\forall x \in (a,b)$ $f'(x)=0 \Rightarrow$ $f = const$ $\Rightarrow \xi = c$ ✓

$g(x) = f(x) - f(a) - f'(x)(b-a)$, $\quad g \in C[a,b]$

? $\exists \xi \in (a,b)$ $\quad g(\xi) = 0$?

$g(a) = f(a) - f(a) - f'(a)(b-a) = -f'(a)(b-a)$

$g(c) = f(c) - f(a) - \underbrace{f'(c)}_{=0} \cdot (b-a) = f(c) - f(a)$

ако $f(c) = f(a)$ $\quad \xi = c$.

ако $f'(a) = 0$?

ако $f'(a) \neq 0$:

1° $f'(a) > 0 \Rightarrow g(a) < 0$

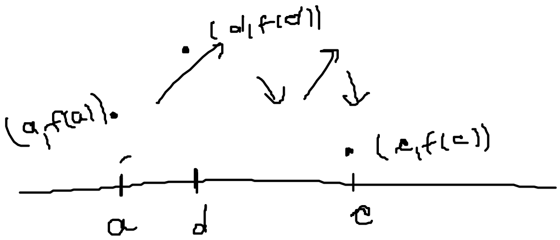
$f(c) > f(a) \Rightarrow g(c) > 0$

$\Big\}$ Коши-Болцано $\Rightarrow \xi \in (a,c)$: $g(\xi) = 0$

2° $f'(a) > 0 \Rightarrow g(a) < 0$ $\quad \Rightarrow \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} > 0$

$f(c) < f(a) \Rightarrow g(c) < 0$

$h < \varepsilon \Rightarrow f(\underbrace{a+h}_{=d}) > f(a)$ за h довољно мало



f' непр, $f'(d) > 0$

$g(d) = \underbrace{f(d) - f(a)}_{>0} - \underbrace{f'(d)(b-a)}_{>0}$

$\exists e \in (a,c) \Rightarrow e$ локални max на (a,c) где f $\Rightarrow f(e) \geq f(d) > f(a) > f(c)$

f диф у e. $\quad \overset{\text{Ферма}}{\Rightarrow} f'(e) = 0$

$g(e) = f(e) - f(a) - f'(e)(b-a) > 0$

$g(c) < 0$

Коши-Болцано на g $\Rightarrow \exists \xi \in (e,c)$ $\quad g(\xi) = 0$.

3° $f'(a) < 0 \Rightarrow g(a) > 0$ $\overset{\text{Коши-Болцано}}{\Rightarrow}$ ...

$f(c) < f(a) \Rightarrow g(c) < 0$

4° $f'(a) < 0 \Rightarrow g(a) > 0$ $\Rightarrow$ аналогно као 2°

$f(c) > f(a) \Rightarrow g(c) > 0$ ...

(4) a) $f(x) = x^p$, $0 \le p \le 1$

f равн непр на $(0, +\infty)$?

$p = 0$ $f(x) = x^0 = 1$, $x \in (0, +\infty) \Rightarrow f$ равн непр.

$$f'(x) = \begin{cases} p \cdot x^{p-1}, & p \ne 0 \\ 0, & p = 0 \end{cases}$$

$$\lim_{x \to +\infty} x^{p-1} = \begin{cases} 1, & p = 1 \\ 0, & 0 \le p < 1 \end{cases} \Rightarrow f' \text{ огран на } [M, +\infty)$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^p = 0, \quad p > 0$$

$\Rightarrow f$ може да се додефинише до $\tilde{f}$ непр у 0 $\overset{\text{Кантор}}{\Rightarrow} f$ равн непр на $(0, M]$

$\tilde{f}$ непр на $[0, +\infty)$

$\Rightarrow f$ равн непр на $(0, +\infty)$

б) $g(x) = x^{1/3}(\ln(1 + |x|) + 1)$ равн непр на $\mathbb{R}$?

g непр на $\mathbb{R}$ $\overset{\text{Кантор}}{\Rightarrow}$ g равн непр на $[-M, M]$, $\forall M > 0$

$$g'(x) = \frac{1}{3} x^{-2/3}(\ln(1 + |x|) + 1) + x^{1/3} \frac{1}{1 + |x|} \cdot \operatorname{sgn} x, \quad x \ne 0$$

$$\lim_{x \to +\infty} g'(x) = \lim_{x \to +\infty} \underbrace{\frac{1}{3} \frac{\ln(1 + x)}{x^{2/3}}}_{\to 0} + \underbrace{\frac{1}{3} x^{-2/3}}_{\to 0} + \underbrace{\frac{x^{1/3}}{1 + x}}_{\to 0} = 0$$

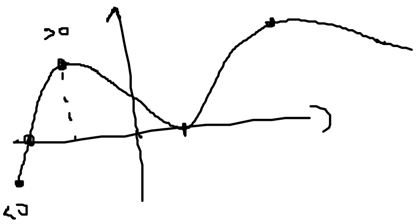
$$\lim_{x \to +\infty} \frac{\ln(1 + x)}{x^{2/3}} \overset{\text{ЛП}}{=} \lim_{x \to +\infty} \frac{\frac{1}{1+x}}{\frac{2}{3} \cdot x^{-1/3}} = \lim_{x \to +\infty} \frac{3x^{1/3}}{1 + x} = 0.$$

$$\lim_{x \to -\infty} g'(x) = \lim_{x \to -\infty} \frac{1}{3} x^{-2/3}(\ln(1 - x) + 1) - \frac{x^{1/3}}{1 - x} = 0$$

$\Rightarrow g'$ ограничен у околинама $\pm\infty$ $\overset{\exists M > 0}{\Rightarrow}$ g' огран. $(-\infty, -M] \cup [M, +\infty)$

$\Rightarrow g$ равн непр на $(-\infty, -M] \cup [M, +\infty)$

$e^x + nx = 2019$



$f(x) = e^x + nx$

$f'(x) = e^x + n > 0 \Rightarrow f \uparrow$ на $\mathbb{R}$

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \underbrace{e^x}_{0} + \underbrace{nx}_{-\infty} = -\infty$

Коши-Болцано $\Rightarrow f$ има нулу у $\mathbb{R}$ и нека је то α

$f(0) = e^0 + n \cdot 0 = 1$

$\alpha \in (-\infty, 0)$

$x < \alpha \quad f(x) < \overset{0}{f(\alpha)} < f(y)$

$\alpha < y \quad \alpha$ јединствена нула.

$f_n(a_n) = 0 \Rightarrow \lim\limits_{n \to \infty} a_n$

$\frac{\quad}{n} < a_n < \frac{\quad}{n}$