44. $f \in C[0,2]$

$? \exists c \in [0,1] \qquad f(c+1) - f(c) = \frac{1}{2}\left(f(2) - f(0)\right) \;?$

$F(x) = f(x+1) - f(x) - \frac{1}{2}\left(f(2) - f(0)\right), \quad F : [0,1] \to \mathbb{R}$ непр.

$F(0) = f(1) - f(0) - \frac{1}{2}\left(f(2) - f(0)\right) =$

$= -\frac{1}{2} f(2) - \frac{1}{2} f(0) + f(1)$

$F(1) = f(2) - f(1) - \frac{1}{2}\left(f(2) - f(0)\right)$

$= \frac{1}{2} f(2) + \frac{1}{2} f(0) - f(1) = -F(0)$

1° $F(1) = F(0) = 0 \Rightarrow c = 0$ или $c = 1$

2° $F(1) \neq F(0) \Rightarrow$ нису истог знака

$0 \in [F(0), F(1)] \xRightarrow{\text{Коши-Болцано}} \exists c \in [0,1] \quad F(c) = 0$

$f(c+1) - f(c) = \frac{1}{2}\left(f(2) - f(0)\right)$

46. $f : \mathbb{R} \to \mathbb{R}$ непрекидна и $\forall x, y \in \mathbb{R} \quad f(x+y) = f(x) + f(y)$. $f = ?$

$x = y \qquad f(2x) = 2f(x)$

$x = y = 0 \Rightarrow f(0) = 2f(0) \Rightarrow f(0) = 0$

$x = y = 1 \Rightarrow f(2) = 2f(1)$

$x = 1, y = n \Rightarrow f(n+1) = f(n) + f(1)$

$\Big\}$ $f(3) = f(2) + f(1) = 2f(1) + f(1) = 3f(1)$

$f(4) = 4f(1) \ldots$

ПМИ: $f(n) = nf(1), \quad n \in \mathbb{N}$

за вежбу ...

$x = y = -1 \Rightarrow f(-2) = f(-1) + f(-1) = 2f(-1)$

$x = -1, y = -n \Rightarrow f(-n-1) = f(-1) + f(-n)$

$\Big\} \xRightarrow{\text{ПМИ}} f(-n) = nf(-1)$

$f(0) = 0 = f(1) + f(-1) \Rightarrow f(-1) = -f(1)$

$f(-n) = -nf(1), \quad n \in \mathbb{N}$

$n \in \mathbb{Z} \Rightarrow f(n) = nf(1)$.

$m \in \mathbb{Z}$, $k \in \mathbb{N}$ $\quad \boxed{\frac{m}{k}}$

$$\underbrace{f\left(\frac{m}{k}\right) + f\left(\frac{m}{k}\right) + \cdots + f\left(\frac{m}{k}\right)}_{k} = f\Big(\underbrace{\frac{m}{k} + \cdots + \frac{m}{k}}_{k}\Big) = f(m) = m \cdot f(1)$$

$f\left(\frac{m}{k}\right) = \frac{m}{k} \cdot f(1)$.

$\forall q \in \mathbb{Q} \quad f(q) = q \cdot f(1)$.

$r \in \mathbb{R} \quad f(r) = ?$ $\quad q_n \in \mathbb{Q} \ \forall n \in \mathbb{N} \quad \lim_{n\to\infty} q_n = r$

f непрекидна $\longrightarrow f(r) = \lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} q_n \cdot f(1) = r \cdot f(1)$

$\forall x \in \mathbb{R} \quad f(x) = x \cdot f(1), \quad f(1) \in \mathbb{R}$

$f(x+y) = (x+y)\cdot f(1) = x\cdot f(1) + y \cdot f(1) = f(x) + f(y).$

49.7) $f(x) = \sin x^2, \quad x \in (0, +\infty)$

$x, y \in (0, +\infty)$

$|\sin x^2 - \sin y^2| = |2 \sin \frac{x^2-y^2}{2} \cdot \underbrace{\cos \frac{x^2+y^2}{2}}_{\leq 1}| \leq 2 |\sin \frac{x^2-y^2}{2}| = 2 |\sin \frac{(x-y)(x+y)}{2}|$

$\leq 2 \left| \frac{(x-y)(x+y)}{2} \right|$

$\downarrow$

$\sin t \leq t$

$y_n = \sqrt{2n\pi} \qquad x_n, y_n$

$\sin x_n^2$

$x_n = y_n + a_n, \quad a_n = x_n - y_n \to 0$

$\sin x_n^2 = \sin(y_n^2 + 2a_n y_n + a_n^2) = \sin(2n\pi + 2a_n \cdot \sqrt{2n\pi} + a_n^2)$

$= \sin(2a_n \cdot \sqrt{2n\pi} + a_n^2) = \sin(1 + \frac{1}{4\cdot(2n\pi)})$

$a_n = \frac{1}{2\sqrt{2n\pi}}$ $\qquad \downarrow$

$\sin 1 \neq 0$

$x_n = \sqrt{2n\pi} + \frac{1}{2\sqrt{2n\pi}}$

$x_n - y_n = \frac{1}{2\sqrt{2n\pi}} \to 0, \ n \to +\infty$

$\sin x_n^2 - \sin y_n^2 \longrightarrow \sin 1, \ n \to +\infty$

$\Big\} \Rightarrow f$ није равн. непр.