40. $F, G : \mathbb{R} \to \mathbb{R}$ непрекидне

$$\inf_{0 \le x \le 1} F(x) = \inf_{0 \le x \le 1} G(x)$$

? $\exists \xi \in (0,1) \quad F(\xi) = G(\xi)$?

$F, G : [0,1] \to \mathbb{R}$ непрекидне

Вајерштрас $\Rightarrow$ F, G ограничене и $\exists \min\limits_{[0,1]} F = F(x_1)$ и $\exists \min\limits_{[0,1]} G = G(x_2)$ $\quad x_1, x_2 \in [0,1]$

$$F(x_1) = \min_{[0,1]} F = \inf_{[0,1]} F = \inf_{[0,1]} G = \min_{[0,1]} G = G(x_2)$$

$$F(x_2) \ge \underbrace{F(x_1)}_{\min F} = G(x_2)$$

$$G(x_1) \ge \underbrace{G(x_2)}_{\min G} = F(x_2)$$

$\Rightarrow$ $F(x_2) \ge G(x_2)$, $F(x_1) \le G(x_1)$

1° $x_1 = x_2 \Rightarrow F(x_1) = G(x_1) = G(x_2) \Rightarrow \xi = x_1$

2° $x_1 \ne x_2$: $H = F - G$ непр на $[0,1]$

$$H(x_2) = F(x_2) - G(x_2) \ge 0$$

$$H(x_1) = F(x_1) - G(x_1) \le 0$$

Т. Болцано-Коши $\Rightarrow$ $\forall C \in [H(x_1), H(x_2)] \quad \exists c \in [x_1, x_2] \quad H(c) = C$

$H(x_1) \le 0$, $H(x_2) \ge 0$

$C = 0 \Rightarrow \xi \in [x_1, x_2] \; H(\xi) = 0$, $\quad \xi \in [x_1, x_2] \subseteq [0,1]$.

$$F(\xi) = G(\xi).$$

47. $f(x) + f\left(\frac{2}{3}x\right) = x \quad \forall x \in \mathbb{R}$

$f : \mathbb{R} \to \mathbb{R}$ непрекидна

$f = ?$

$$f(0) + f\left(\tfrac{2}{3} \cdot 0\right) = 0$$

$$f(x) = x - f\left(\tfrac{2}{3}x\right) = x - \left(\tfrac{2}{3}x - f\left(\left(\tfrac{2}{3}\right)^2 x\right)\right) = x - \tfrac{2}{3} \cdot x + f\left(\left(\tfrac{2}{3}\right)^2 x\right)$$

$$= x - \tfrac{2}{3}x + \left(\tfrac{2}{3}\right)^2 x - f\left(\left(\tfrac{2}{3}\right)^3 x\right)$$

$$f(x) = \underbrace{x - \frac{2}{3}x + \left(\frac{2}{3}\right)^2 x - \left(\frac{2}{3}\right)^3 x + \dots + (-1)^n \left(\frac{2}{3}\right)^n x + (-1)^{n+1} f\left(\left(\frac{2}{3}\right)^{n+1} x\right)}$$

ПМИ ... за сваку

$$f(x) = \lim_{n\to\infty} f(x) = \lim_{n\to\infty} x\underbrace{\left(1 - \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots + (-1)^n \cdot \left(\frac{2}{3}\right)^n\right)}_{A_n \to \frac{3}{5}} + \underbrace{(-1)^{n+1} f\left(\left(\frac{2}{3}\right)^{n+1} x\right)}_{\xrightarrow{n\to\infty} 0} = \frac{3}{5}x$$

$$\lim_{n\to\infty} \left(\frac{2}{3}\right)^{n+1} x = 0 \qquad \left(\frac{2}{3} < 1\right)$$

f непр $\Rightarrow \lim_{n\to\infty} f\left(\left(\frac{2}{3}\right)^{n+1} x\right) = f(0) = 0.$

$$A_n = 1 - \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots + (-1)^n \left(\frac{2}{3}\right)^n = \frac{1 - q^{n+1}}{1 - q} = \frac{1 - \left(-\frac{2}{3}\right)^{n+1}}{5/3} \xrightarrow{n\to\infty} \frac{1}{5/3} = \frac{3}{5}$$

$$q = -\frac{2}{3}$$

49. б) $f(x) = \sqrt{|x|}\, e^{-x}$ на $(0,+\infty) \ni x, y$

$$|f(x) - f(y)| = |\sqrt{x}\, e^{-x} - \sqrt{y}\, e^{-y}| = |\sqrt{x}\, e^{-x} - \sqrt{y}\, e^{-x} + \sqrt{y}\, e^{-x} - \sqrt{y}\, e^{-y}|$$

$$\leq |\sqrt{x} - \sqrt{y}| \cdot \underbrace{e^{-x}}_{<1 \text{ јер } x\in(0,+\infty)} + \sqrt{y}\, |e^{-x} - e^{-y}| = |\sqrt{x} - \sqrt{y}| \cdot e^{-x} + \sqrt{y}\, e^{-y} |e^{-x+y} - 1|$$

$$\lim_{t\to 0} \frac{e^t - 1}{t} = 1 \qquad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad |t| < \delta \Rightarrow \left|\frac{e^t - 1}{t} - 1\right| < \varepsilon$$

$$|e^t - 1 - t| < \varepsilon |t|$$

$$|e^t - 1| < |t| + \varepsilon \cdot |t|$$

$$\varepsilon = 1 \quad \exists \delta > 0 \quad |t| < \delta \Rightarrow |e^t - 1| < 2|t|$$

$$|x - y| < \delta \Rightarrow |e^{-x+y} - 1| < 2|x - y|.$$

$$\lim_{y\to+\infty} \sqrt{y} \cdot e^{-y} = \lim_{y\to+\infty} \frac{\sqrt{y}}{e^y} = 0 \Rightarrow \exists M > 0 \quad \forall y > M \quad \frac{\sqrt{y}}{e^y} \leq 1$$

$$\Rightarrow \sqrt{y}\, e^{-y} |e^{-x+y} - 1| \leq 1 \cdot 2|x - y|, \quad y > M, \quad x > M, \quad |x - y| < \delta$$

$$e^{-x} \leq 1, \quad x \geq 0$$

$$|\sqrt{x} - \sqrt{y}| \leq \frac{1}{2}|x - y|, \quad x, y \geq 1$$

$\Rightarrow \quad |f(x) - f(y)| \leq \frac{1}{2}|x-y| + 2|x-y|$, $\quad |x-y|<\delta$ и $x,y > \max\{M, 1\}$

$|x-y|<\delta \quad \Rightarrow \quad |f(x)-f(y)| \leq \frac{3}{2}\underbrace{|x-y|}_{<\delta_1} < \varepsilon$

$\delta_1 = \min\{ \underset{\uparrow}{\delta}, \frac{2}{3}\varepsilon \}$

f равн нейр на $(M, +\infty)$
Канѿор f равн нейр на $[0, M+1]$ $\Big\} \Rightarrow f$ равн нейр на $\mathbb{R}$.

$\hookrightarrow$ имамо нейр. ӣродужење ф-је f у 0.

- -

$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \sqrt{x}\cdot e^{-x} = 0 \quad \Rightarrow f$ равн нейр на $[M, +\infty)$ за $M>0$.

Канѿор $\Rightarrow$ f има нейр. ӣрод. у $\underline{\underline{0}}$ $\Rightarrow$ f равн нейр на $[0, M]$

$\underset{\uparrow}{(0, +\infty)}$ $\Downarrow$

f равн нейр на $(0, +\infty)$

$x_n = \frac{a_n}{b_n}$, $x_1 = 1$

$x_{n+1} = \frac{4x_n}{2x_n + 3}$

$\frac{a_{n+1}}{b_{n+1}} = \frac{4\frac{a_n}{b_n}}{2\frac{a_n}{b_n} + 3} = \frac{4a_n}{2a_n + 3b_n}$, $\quad a_1 = 1$, $b_1 = 1$

$a_{n+1} = 4a_n$, $a_1 = 1 \longrightarrow$ $a_2 = 4$

$a_3 = 4^2 = 16$

$a_4 = 4^3 \dashrightarrow \underline{\underline{a_n = 4^{n-1}}}, n\in\mathbb{N}$

$b_{n+1} = 2a_n + 3b_n$

$\Downarrow$

$2a_n = b_{n+1} - 3b_n$

$\underline{\underline{b_{n+2} - 3b_{n+1} = 4b_{n+1} - 12b_n}} \rightarrow b_n = \ldots$

$$b_n = a_n + \alpha a_{n-1} + \dots + \alpha^{n-1} a_1 \quad , \quad \alpha \in (0,1)$$

$$\alpha = \frac{1}{\beta} \quad , \quad \beta > 1$$

$$b_n = \frac{x_1 y_n + \dots +}{n}$$

$$b_n = 1 \cdot a_n + \frac{1}{\beta} a_{n-1} + \dots + \frac{1}{\beta^{n-1}} a_1 = \frac{\beta^{n-1} a_n + \dots + a_1}{\beta^{n-1}} \uparrow +\infty$$

$$\overset{\text{ШТ}}{=} \frac{\beta^n a_{n+1}}{\beta^n - \beta^{n-1}} \dots -$$

$$n\sqrt{2} - 1 \le [n\sqrt{2}] < n\sqrt{2}$$

e

36. ППС. $q_n \not\to +\infty$

$$\neg\left(q_n \to +\infty \quad \forall M > 0 \ \exists n_0 \in \mathbb{N} \ \forall n \ge n_0 \quad q_n > M \right)$$

$$\Rightarrow \exists M > 0 \ \forall n_0 \in \mathbb{N} \ \exists n \ge n_0 \quad q_n \le M$$

$q_n \in \mathbb{N} \qquad q_n \ge 1$

$n_k \uparrow \qquad q_{n_k} \le M$

$$q_n \in \mathbb{N} \qquad q_{n_k} \in \{1, 2, \dots, M\} \Rightarrow \exists q_{n_{k_e}} : \lim q_{n_{k_e}} = a$$

$\frac{p_n}{q_n} \to \alpha \qquad \varepsilon < 1 \qquad (a-\varepsilon, a+\varepsilon)$

$p_{n_{k_e}} \to \alpha \qquad a \in \mathbb{N} \qquad q_{n_{k_e}} = a$

$q_{n_{k_e}} = a \in \mathbb{N} \qquad \notin \mathbb{Q}$

$$p_{n_{k_e}} \to a\alpha \Rightarrow p_{n_{k_e}} \notin \mathbb{N}.$$

n – фикс. $x \to 0$

$$(x+o(x))^n = \sum_{k=0}^{n} \binom{n}{k} x^k \cdot (o(x))^{n-k}$$

$$= x^n + \binom{n}{n-1} x^{n-1} \cdot o(x)$$

$$+ \binom{n}{n-2} x^{n-2} \, o(x)^2$$

$$+ \ldots + o(x)^n = x^n + o(x^n).$$

$o(x)^n = o(x^n)$?

$o(x)^n = o(x^n)$

$o(x) \cdot o(x) = o(x^2)$

$\overset{||}{\alpha x}$ $\overset{||}{\beta x}$ $\underset{\to 0}{\alpha \cdot \beta} \cdot x^2$

$$\binom{n}{k} x^k \, o(x)^{n-k} = \binom{n}{k} x^k \, o(x^{n-k})$$

$f \; o(g) = o(fg)$

$$= \binom{n}{k} o(x^k \cdot x^{n-k}) = \binom{n}{k} o(x^n) =$$

$$= o(x^n)$$