① $f: X \to Y$ НА. f 1-1 акко $\forall A \subseteq X \ \ f(A)^c = f(A^c)$.

$\boxed{\Rightarrow:}$ f 1-1 : $A \subseteq X$ произвољан

Показујемо $f(A)^c = f(A^c)$.

$\boxed{\subseteq:}$ $y \in f(A)^c$ произвољно $\Rightarrow y \in \underbrace{Y}_{f(X)} \setminus f(A) \overset{f \text{ НА}}{\Rightarrow} y \in f(X) \wedge y \notin f(A)$

$\Rightarrow \exists x \in X \ f(x) = y \ \wedge \ \forall a \in A \ f(a) \neq y$

$\overset{f \text{ ф-ја}}{\Rightarrow} \exists x \in \underbrace{X \setminus A}_{A^c} \ f(x) = y \Rightarrow y \in f(A^c)$

$\boxed{\supseteq:}$ $y \in f(A^c)$ произвољно $\Rightarrow \exists x \in A^c \ f(x) = y$

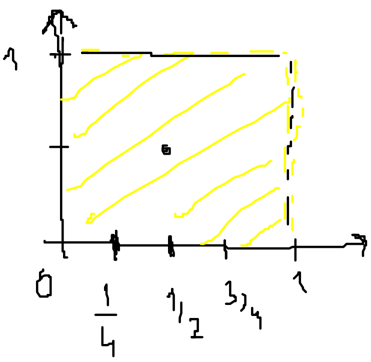
$\overset{f \ 1\text{-}1}{\Rightarrow} \forall a \in A \ f(a) \neq f(x) = y \Rightarrow y \notin f(A) \Rightarrow y \in f(A)^c$

$\boxed{\Leftarrow}$ f није 1-1 $\Rightarrow \exists x_1, x_2 \in X : f(x_1) = f(x_2) = y$ и $x_1 \neq x_2$

$$\left.\begin{array}{l} A = \{x_1\} \Rightarrow y = f(x_1) \in f(A) \Rightarrow y \notin f(A)^c \\ x_2 \in A^c \Rightarrow y = f(x_2) \in f(A^c) \end{array}\right\} \Rightarrow f(A)^c \neq f(A^c)$$

② $f: (0,1) \to (0,1)$ НА.

Доказати да $\exists x, y \in (0,1) \ |f(x) - f(y)| \geq |x - y|$.



Пп. $\forall x, y \in (0,1) \ |f(x) - f(y)| < |x - y|$

$y = \frac{1}{2}$ $\quad |x - \frac{1}{2}| < \frac{1}{2}$

$x \in (0,1)$

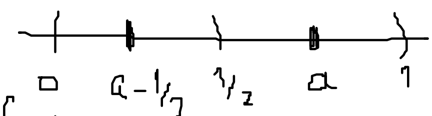
$f(\frac{1}{2}) = a \Rightarrow |f(x) - a| < \frac{1}{2}$

$f(x) \in (a - \frac{1}{2}, a + \frac{1}{2})$

$(0,1) \underset{f \text{ НА}}{=} f((0,1)) \subseteq (a - \frac{1}{2}, a + \frac{1}{2}) \cap (0,1)$

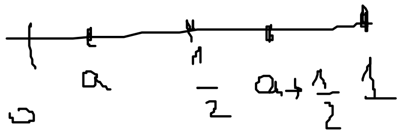
$\Rightarrow (0,1) = (a - \frac{1}{2}, a + \frac{1}{2}) \cap (0,1)$

$1^\circ \ a > \frac{1}{2} \Rightarrow a - \frac{1}{2} > 0$



$(a - \frac{1}{2}, a + \frac{1}{2}) \cap (0,1) \neq (0,1)$ ↯ f НА

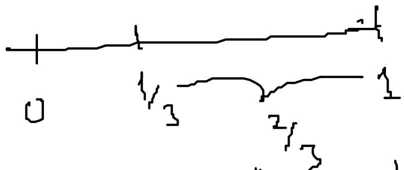
2° $a < \frac{1}{2} \Rightarrow a + \frac{1}{2} < 1$



$(a - \frac{1}{2}, a + \frac{1}{2}) \cap [0,1) \neq [0,1)$ ↯

Да смо узели уместо $y = 1/2$ нпр. $y = \frac{1}{3}$

$x \in [0,1) \Rightarrow \left| f(x) - \underbrace{f(\tfrac{1}{3})}_{"a"} \right| \le |x - 1/3| < \frac{2}{3}$



$|f(x) - "a"| < \frac{2}{3}$ $\quad f(x) \in ("a" - 2/3, "a" + 2/3)$

$\Rightarrow$ ми не знамо шта се дешава са $("a" - 2/3, "a" + 2/3) \cap [0,1)$

$\Rightarrow a = \frac{1}{2}$

$f(\frac{1}{2}) = \frac{1}{2}$

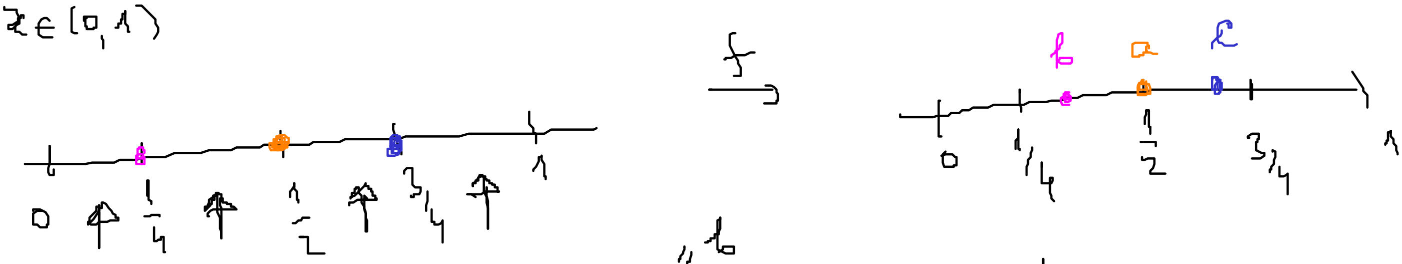
$\left| f(x) - \underbrace{\tfrac{1}{2}}_{f(\frac{1}{2})} \right| < \left| x - \frac{1}{2} \right|$

$x = \frac{1}{4}$ $\quad \left| f(\tfrac{1}{4}) - \tfrac{1}{2} \right| < \frac{1}{4} \Rightarrow \underbrace{b}_{} = f(\tfrac{1}{4}) \in (\tfrac{1}{4}, \tfrac{3}{4})$

$x = \frac{3}{4}$ $\quad \left| f(\tfrac{3}{4}) - \tfrac{1}{2} \right| < \frac{1}{4} \Rightarrow c = f(\tfrac{3}{4}) \in (\tfrac{1}{4}, \tfrac{3}{4})$

x - произвољно $\quad f(x) \in$?

$x \in [0,1)$



1° $x \in [0, \frac{1}{2}) \Rightarrow \left| f(x) - \underbrace{f(\tfrac{1}{4})}_{b} \right| < \left| x - \tfrac{1}{4} \right| < \frac{1}{4}$

$f(x) \in (b - \frac{1}{4}, b + \frac{1}{4})$

2° $x \in [\frac{1}{2}, 1) \Rightarrow \left| f(x) - \underbrace{f(\tfrac{3}{4})}_{c} \right| < \left| x - \tfrac{3}{4} \right| < 1/4$

$$\Rightarrow \quad f(x) \in \left(c - \frac{1}{4}, c + \frac{1}{4} \right)$$

$x \in [0,1\rangle$

$$\Rightarrow \quad f(x) \in \left(b - \frac{1}{4}, b + \frac{1}{4} \right) \cup \left(c - \frac{1}{4}, c + \frac{1}{4} \right)$$

$$= \left[\min\left(b - \frac{1}{4}, c - \frac{1}{4} \right), \max\left(b + \frac{1}{4}, c + \frac{1}{4} \right) \right]$$

$$b, c \in \left[\frac{1}{4}, \frac{3}{4} \right\rangle \Rightarrow \quad m := \min\left(b - \frac{1}{4}, c - \frac{1}{4} \right) > 0$$

$$M = \max\left(b + \frac{1}{4}, c + \frac{1}{4} \right) < 1$$

$$f([0,1]) \subseteq [m, M] \subsetneq \langle 0, 1 \rangle$$

$$\Rightarrow \quad f \text{ nije HA} \quad \lightning \quad f \text{ HA}$$