2017. II кол.

1

$$I_\alpha = \int_0^1 \frac{\ln(1-x^2)}{(\alpha-x)^2}\,dx$$, испитати конв и за $\alpha > 1$ израчунати.

$\alpha > 1$:

$$\int_0^1 \frac{\ln(1-x^2)}{(\alpha-x)^2}\,dx = \int_0^1 \frac{\ln(1+x)+\ln(1-x)}{(\alpha-x)^2}\,dx \overset{(*)}{=} \underbrace{\int_0^1 \frac{\ln(1+x)}{(\alpha-x)^2}dx}_{\text{конв.}} + \underbrace{\int_0^1 \frac{\ln(1-x)}{(\alpha-x)^2}dx}_{t=1-x}$$

$(1-x^2) = (1-x)(1+x)$

(*) ако не див. оба инт. са десне стр.

$$\int_0^1 \frac{\ln(1-x)}{(\alpha-x)^2}dx = \begin{bmatrix} t = 1-x \\ dt = -dx \end{bmatrix} = \int_1^0 \frac{\ln t}{(\alpha-1+t)^2}(-dt) = \underbrace{\int_0^1 \frac{\ln t}{(\alpha-1+t)^2}\,dt}_{\text{конв.}}$$

$$\frac{\ln t}{(\alpha-1+t)^2} \sim \frac{\ln t}{(\alpha-1)^2}, \quad t \to 0$$

$$\underbrace{\int_0^1 \ln t\,dt}_{\text{конв.}} = \begin{bmatrix} u = \ln t & dv = dt \\ du = \frac{dt}{t} & v = t \end{bmatrix} = t\cdot \ln t\Big|_0^1 - \int_0^1 dt = -1$$

$$t \cdot \ln t\Big|_0^1 = \underbrace{1\cdot\ln 1}_{=0} - \underbrace{\lim_{t\to 0^+} t\cdot \ln t}_{=0}$$

II н.ч. $\Rightarrow$

$$\int_0^1 \frac{\ln(1-x^2)}{(\alpha-x)^2}dx = \begin{bmatrix} u = \ln(1-x^2) \\ du = -\frac{2x}{1-x^2}dx \\ dv = \frac{dx}{(\alpha-x)^2} = \frac{dx}{(x-\alpha)^2}, \quad v = -\frac{1}{x-\alpha} \end{bmatrix}$$

$$= -\frac{\ln(1-x^2)}{x-\alpha}\Big|_0^1 - \underbrace{\int_0^1 \frac{2x\,dx}{(1-x^2)(x-\alpha)}}_{\text{за вежбу ...}}$$

$\alpha = 1$:

$$I_1 = \int_0^1 \frac{\ln(1-x^2)}{(1-x)^2}dx = \begin{bmatrix} t = 1-x \\ dt = -dx \end{bmatrix} = \int_0^1 \frac{\ln(2-t)+\ln t}{t^2}\,dt$$

$\ln(1-x^2) = \ln(1+x)+\ln(1-x)$, $\frac{dt}{t^2} = -du$

$$= \begin{bmatrix} u = \frac{1}{t}, \ln u = -\ln t \\ du = -\frac{1}{t^2}dt \end{bmatrix} = \int_{+\infty}^1 \left(\ln\left(2-\frac{1}{u}\right) - \ln u\right)(-du) = \int_1^{+\infty} \left(\ln\left(2-\frac{1}{u}\right) - \ln u\right) du$$

$$\ln\left(2-\frac{1}{u}\right) - \ln u \sim \underbrace{-\ln u + \ln 2}_{\to -\infty}, \quad u \to +\infty, \quad \int_2^{+\infty} \ln u\,du \geq \int_2^{+\infty} \ln 2\,du = +\infty$$ дивергира

$0<\alpha<1$ $\quad \displaystyle\int_0^1 \frac{\ln(1-x^2)}{(\alpha-x)^2} := \underbrace{\int_0^{\alpha} \frac{\ln(1-x^2)}{(\alpha-x)^2}dx}+\underbrace{\int_{\alpha}^{1} \dots}$

$\alpha, 1$ – синг.

да би конв. инт. са леве стр. треба да конв. оба инт. са десне.

$$\int_0^{\alpha} \frac{\ln(1-x^2)}{(\alpha-x)^2}dx = \left[\begin{array}{l} t=\alpha-x \\ dt=-dx \end{array}\right] = \int_0^{\alpha} \frac{\ln(1-(\alpha-t)^2)}{t^2}dt$$

$$\frac{\ln(1-(\alpha-t)^2)}{t^2} \sim \frac{\ln(1-\alpha^2)}{t^2}, \quad t\to 0, \quad \int_0^{\alpha} \frac{dt}{t^2} \text{ див.}$$

II п.к. $\Rightarrow \displaystyle\int_0^{\alpha} \frac{\ln(1-(\alpha-t)^2)}{t^2}dt$ див.

$$\Rightarrow \int_0^1 \frac{\ln(1-x^2)}{(\alpha-x)^2}dx \text{ див.}$$

2021. Септ.

$\{a_n\}$ $\quad a_n\uparrow, \quad a_n>0, \quad n\geq 1$

$f: \mathbb{R}\to(0,+\infty) \quad \uparrow, \quad \underbrace{\displaystyle\int_{a_1}^{+\infty} \frac{1}{x\cdot f(x)}dx}$ конв.

$$? \sum_{n=1}^{\infty}\left(1-\frac{a_n}{a_{n+1}}\right)\cdot\frac{1}{f(a_n)} \text{ конв. } ?$$

1° $\quad \underline{\underline{\lim_{n\to+\infty} a_n = a\in\mathbb{R}}}, \ a_n\uparrow \Rightarrow a_n\leq a, \ f\uparrow \Rightarrow f(a_n)\leq f(a) \quad \frac{1}{f(a_1)}\geq\frac{1}{f(a_n)}\geq\frac{1}{f(a)}$

2° $\quad \lim_{n\to+\infty} a_n = +\infty$

1° $\lim\limits_{n\to+\infty} a_n = a \in \mathbb{R}$, $a_n \uparrow \implies a_1 \le a_n \le a$

$f \uparrow \implies f(a_1) \le f(a_n) \le f(a)$

$$\frac{1}{f(a)} \le \frac{1}{f(a_n)} \le \frac{1}{f(a_1)}$$

$$x \cdot f(x) \uparrow \implies \frac{1}{x f(x)} \downarrow$$

$$\frac{1}{a f(a)} \le \frac{1}{a_n f(a_n)} \le \frac{1}{f(a_1)\cdot a_1}$$

$$\sum_{n=1}^{\infty} \underbrace{\left(1 - \frac{a_n}{a_{n+1}}\right)\frac{1}{f(a_n)}}_{\geq 0} = \sum_{n=1}^{\infty} \underbrace{\frac{a_{n+1}-a_n}{a_{n+1}\cdot f(a_n)}}_{\le \frac{a_{n+1}-a_n}{a_n f(a_n)}} \le \sum_{n=1}^{\infty} \frac{a_{n+1}-a_n}{a_n f(a_n)} \le \sum_{n=1}^{\infty} \frac{(a_{n+1}-a_n)}{a_1 f(a_1)}$$

јер $\dfrac{a_n}{a_{n+1}} \le 1$

$$\frac{1}{a_{n+1}} \le \frac{1}{a_n}$$

$$\sum_{n=1}^{\infty} \frac{a_{n+1}-a_n}{a_1 f(a_1)} = \frac{1}{a_1 f(a_1)} \sum_{n=1}^{\infty} (a_{n+1}-a_n) = \frac{1}{a_1 f(a_1)} \lim_{n\to\infty} \sum_{k=1}^{n} (a_{k+1}-a_k) =$$

$$= \frac{1}{a_1 f(a_1)} \cdot \left(\lim_{n\to+\infty} a_{n+1} - a_1\right) = \frac{a-a_1}{a_1 \cdot f(a_1)} < +\infty$$

I поредб. крит.

$$\implies \sum_{n=1}^{\infty} \left(1 - \frac{a_n}{a_{n+1}}\right)\frac{1}{f(a_n)} \quad \text{конв.}$$

2° $\lim\limits_{n\to+\infty} a_n = +\infty$

$$\left.\begin{array}{l} \displaystyle\int_{a_1}^{+\infty} \frac{dx}{x f(x)} \text{ конв.} \\ f \uparrow \end{array}\right\} \implies \lim_{x\to+\infty} f(x) = +\infty$$

$$\lim_{x\to+\infty} \frac{1}{f(x)} = 0$$

$$\frac{a_{n+1}-a_n}{a_{n+1} f(a_{n+1})} \le \frac{a_{n+1}-a_n}{a_{n+1} f(a_n)} \le \frac{a_{n+1}-a_n}{a_n f(a_n)}$$

$f(a_{n+1}) \geq f(a_n)$ $\qquad$ $a_{n+1} \geq a_n$

Не можемо као у 1° !

$$+\infty > \int_{a_1}^{+\infty} \frac{dx}{xf(x)} = \sum_{n=1}^{+\infty} \int_{a_n}^{a_{n+1}} \frac{dx}{xf(x)} \underset{\frac{1}{xf(x)}\downarrow}{\geq} \sum_{n=1}^{+\infty} \frac{1}{a_{n+1} f(a_{n+1})} \int_{a_n}^{a_{n+1}} dx$$

$$= \sum_{n=1}^{+\infty} \frac{a_{n+1}-a_n}{a_{n+1} f(a_{n+1})}$$

$$\Rightarrow \underbrace{\sum_{n=1}^{+\infty} \frac{a_{n+1}-a_n}{a_{n+1} f(a_{n+1})}}_{\text{овај ред конв.}} \leq \sum_{n=1}^{\infty} \frac{a_{n+1}-a_n}{a_{n+1} f(a_n)} \leq \underbrace{\sum_{n=1}^{\infty} \frac{a_{n+1}-a_n}{a_n f(a_n)}}_{\text{о овом не знамо ништа}}$$

за сада на основу овога не можемо ништа закључити о средњем реду.
Он је позитиван, дакле или конв. или је $+\infty$. Посматрамо разлику њега и конв. реда и надамо се да добијамо ред за који знамо да ли конв.

$$\sum_{n=1}^{+\infty} \frac{a_{n+1}-a_n}{a_{n+1} f(a_n)} - \sum_{n=1}^{\infty} \frac{a_{n+1}-a_n}{a_{n+1} f(a_{n+1})} = \sum_{n=1}^{\infty} \underbrace{\frac{a_{n+1}-a_n}{a_{n+1}} \cdot \left(\frac{1}{f(a_n)} - \frac{1}{f(a_{n+1})} \right)}_{\geq 0}$$

ово можемо јер знамо да један конвергира

$$\frac{a_{n+1}-a_n}{a_{n+1}} < 1 \Rightarrow \sum_{n=1}^{\infty} \frac{a_{n+1}-a_n}{a_{n+1}} \left(\frac{1}{f(a_n)} - \frac{1}{f(a_{n+1})} \right) < \sum_{n=1}^{\infty} \left(\frac{1}{f(a_n)} - \frac{1}{f(a_{n+1})} \right)$$

$$= \lim_{n\to+\infty} \sum_{k=1}^{n} \left(\frac{1}{f(a_k)} - \frac{1}{f(a_{k+1})} \right) = \underbrace{\frac{1}{f(a_1)}}_{<+\infty} - \underbrace{\lim_{n\to+\infty} \frac{1}{f(a_n)}}_{=0} < +\infty$$

I п.к. $\Rightarrow \sum_{n=1}^{\infty} \frac{a_{n+1}-a_n}{a_{n+1}} \left(\frac{1}{f(a_n)} - \frac{1}{f(a_{n+1})} \right)$ конв.

$$\Rightarrow \underbrace{\sum_{n=1}^{\infty}\left(1-\frac{a_n}{a_{n+1}}\right)\frac{1}{f(a_n)}}_{\text{конв}} = \sum_{n=1}^{+\infty}\left(1-\frac{a_n}{a_{n+1}}\right)\frac{1}{f(a_{n+1})} + \sum_{n=1}^{\infty}\left(1-\frac{a_n}{a_{n+1}}\right)\left(\frac{1}{f(a_n)}-\frac{1}{f(a_{n+1})}\right)$$

конв
као збир
два конв. реда!

$\boxed{2}$ (исто са неког рока ...)

$$\sum_{n=1}^{\infty}\underbrace{\left(\int_n^{+\infty}\frac{\sin x}{x}dx\right)}_{a_n} \text{ конв ?}$$

$\int_0^{+\infty}\frac{\sin x}{x}dx$ конв $\Rightarrow a_n \to 0,\ n\to+\infty$

покушаћемо парцијалну инт.

$$\int_n^{+\infty}\frac{\sin x}{x}dx \overset{*}{\underset{①}{=}} \left[\begin{array}{l} u=\frac{1}{x} \Rightarrow du=-\frac{1}{x^2}dx \\ dv=\sin x\,dx \Rightarrow v=-\cos x \end{array}\right]$$

$$= \underbrace{-\frac{\cos x}{x}\Big|_n^{+\infty}}_{-\underbrace{\lim_{x\to+\infty}\frac{\cos x}{x}}_{=0}+\frac{\cos n}{n}} - \underbrace{\int_n^{+\infty}\frac{\cos x}{x^2}dx}_{\text{конв апсолутно}} \overset{②}{=} \left[\begin{array}{l} u=\frac{1}{x^2} \Rightarrow du=-\frac{2}{x^3}dx \\ dv=\cos x\,dx \Rightarrow v=\sin x \end{array}\right]$$

$$= \frac{\cos n}{n} - \frac{\sin x}{x^2}\Big|_n^{+\infty} - \int_n^{+\infty}\frac{2\sin x}{x^3}dx \quad \Big/ \sum_{n=1}^{\infty}$$

$$\sum_{n=1}^{\infty}\int_n^{+\infty}\frac{\sin x}{x}dx = \sum_{n=1}^{\infty}\left(\frac{\cos n}{n}+\frac{\sin n}{n^2}-\int_n^{+\infty}\frac{2\sin x}{x^3}dx\right)$$

$$\overset{*}{=} \underbrace{\sum_{n=1}^{\infty}\frac{\cos n}{n}}_{\text{конв. по Дирихлеу}} + \underbrace{\sum_{n=1}^{\infty}\frac{\sin n}{n^2}}_{\text{конв. aпс.}} - \underbrace{\sum_{n=1}^{\infty}\left(\int_n^{+\infty}\frac{2\sin x}{x^3}dx\right)}$$

ако сва
3 конв. (или само један див.)

$$\left|\int_n^{+\infty}\frac{2\sin x}{x^3}\,dx\right| \le \int_n^{+\infty}\frac{2|\sin x|}{x^3}\,dx \underset{|\sin x|\le 1}{\le} \int_n^{+\infty}\frac{2}{x^3}\,dx = \frac{1}{n^2}$$

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ конв. $\xRightarrow{\text{I п.к.}}$ $\sum_{n=1}^{\infty}\int_n^{+\infty}\frac{2\sin x}{x^3}\,dx$ конв. апсолутно.

$\Rightarrow$ $\sum_{n=1}^{\infty}\int_n^{+\infty}\frac{\sin x}{x}\,dx$ конв као збир 3 конв. реда.

да смо исто урадили након прве парц. инт. не бисмо добили одмах реш. било би потребно одредити конв. реда $\sum_{n=1}^{\infty}\int_n^{+\infty}\frac{\cos x}{x^2}\,dx$, што ако би радили на претходни начин имали бисмо: $\sum\left|\int_n^{+\infty}\frac{\cos x}{x^2}\,dx\right| \le C\cdot\underbrace{\sum\frac{1}{n}}_{\text{дивергира}}$

$\Rightarrow$ не знамо ништа о реду с леве стране