(1) $A = \bigcup\limits_{n=1}^{\infty} A_n$

$\sup A = \sup\{ \sup A_n \mid n\in\mathbb{N} \}$

$\inf A = \inf\{ \inf A_n \mid n\in\mathbb{N} \}$

$\sup\{ \underbrace{\sup A_n}_{a_n} \mid n\in\mathbb{N} \} = a \implies \forall n\in\mathbb{N} \quad a_n \le a$

? $a = \sup A$ ?  1) $\forall x\in A \quad x\le a$

$x\in A = \bigcup\limits_{n=1}^{\infty} A_n$ , $\exists n\in\mathbb{N} \quad x\in A_n$

$\sup A_n = a_n \implies x \le a_n \le a$.

2) $\forall \varepsilon>0 \quad \exists x\in A : \quad x > a-\varepsilon$ :

$\varepsilon>0$ произвољно $\implies \exists n\in\mathbb{N} \quad a_n > a - \varepsilon/2$ јер $a = \sup\{a_n \mid n\in\mathbb{N}\}$

$a_n = \sup A_n \implies \exists x\in A_n \quad x > a_n - \varepsilon/2$

$x > a - \varepsilon/2 - \varepsilon/2 = a - \varepsilon$

$x\in A_n \implies x \in \bigcup\limits_{n=1}^{\infty} A_n = A$ ✓

за inf: $\inf\{ \inf A_n \} = a$

$\inf \bigcup\limits_{n=1}^{\infty} A_n = a$ ? $\longrightarrow$ за вежбу

$A = \left\{ \dfrac{m^2+m}{m^2-n^2} - \dfrac{1}{n} \;\middle|\; m>n,\ m,n\in\mathbb{N} \right\}$

$A_n = \left\{ \dfrac{m^2+m}{m^2-n^2} - \dfrac{1}{n} \;\middle|\; m>n,\ m\in\mathbb{N} \right\}$

$\sup A_n = \max A_n \underset{m=n+1}{\downarrow}= \dfrac{(n+1)^2+n+1}{(n+1)^2-n^2} - \dfrac{1}{n}$

$\overline{\mathbb{R}} = \mathbb{R}\cup\{+\infty,-\infty\}$

$\sup A = \sup\left\{ \dfrac{n^2+3n+2}{2n+1} - \dfrac{1}{n} \;\middle|\; n\in\mathbb{N} \right\}$ "$= +\infty$"

$\implies \sup A$ не постоји у $\mathbb{R}$

min $A$ не постоји

$\inf A_n = 1 - \dfrac{1}{n}$

$\inf A = \inf\{ \underline{\underline{1-\tfrac{1}{n}}} \mid n\in\mathbb{N} \} = \min\{ 1-\tfrac{1}{n} \mid n\in\mathbb{N} \} = 0$ (*)

не значи да је $0 = \min A$

# Низови

- $a : \mathbb{N} \to X$ , $a(n) = a_n$ je низ.

- a je монотон: $a_n \uparrow$ $a_{n+1} > a_n$ , $\forall n \in \mathbb{N}$ строго растући

  $a_n \nearrow$ $a_{n+1} \geq a_n$ $\forall n \in \mathbb{N}$ растући

  $a_n \downarrow$ $a_{n+1} < a_n$ $\forall n \in \mathbb{N}$ строго опадајући

  $a_n \searrow$ $a_{n+1} \leq a_n$ $\forall n \in \mathbb{N}$ опадајући

- ограничен одозго: $\exists M \in \mathbb{R}$ $\forall n \in \mathbb{N}$ $a_n \leq M$

  одоздо: $\exists m \in \mathbb{R}$ $\forall n \in \mathbb{N}$ $a_n \geq m$

- ограничен $\exists M \geq 0$ $\forall n \in \mathbb{N}$ $|a_n| \leq M$

$a_n \nearrow \Rightarrow$ ограничен одоздо са $a_1$

$a_n \searrow \Rightarrow$ ограничен одозго са $a_1$

(1) $f : \mathbb{R} \to \mathbb{R}$ конвексна. Испитати монотоност низова

$a_n = f(n) + f(-n)$ и $b_n = a_{n+1} - a_n$ , $n \in \mathbb{N} \cup \{0\}$.

- - - - -

$f$ конвексна фја $\forall x, y \in \mathbb{R}$ $\forall \lambda \in (0,1)$

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y).$$

$f$ конвексна $\Rightarrow \lambda = \frac{1}{2}$ $\quad f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$

$x = 1, y = -1 \Rightarrow f(0) \leq \frac{f(1)+f(-1)}{2} \Rightarrow \underbrace{2f(0)}_{a_0} \leq \underbrace{f(1)+f(-1)}_{a_1}$

Да ли je $a_n \nearrow$ ?

$\forall n \in \mathbb{N}$ $a_{n+1} \geq a_n$

$$f(n+1) + f(-(n+1)) \geq f(n) + f(-n) \quad ?$$

$$f(n+1) - f(n) \geq f(-n) - f(-n-1) \quad ?$$

$x = m+1$
$y = m-1$
$\lambda = \frac{1}{2}$

$\underset{f \text{ конвексна}}{\Rightarrow}$ $f\left(\frac{m+1+m-1}{2}\right) \leq \frac{f(m+1)+f(m-1)}{2}$

$\Rightarrow$ $2f(m) \leq f(m+1) + f(m-1)$

$f(m+1) - f(m) \geq f(m) - f(m-1)$ $\quad m \in \mathbb{Z}$

$n \in \mathbb{N}$

$$f(n+1)-f(n) \geq f(n)-f(n-1) \geq f(n-1)-f(n-2) \geq \dots \geq f(1)-f(0) \geq f(0)-f(-1)$$
$$\geq f(-1)-f(-2) \geq \dots \geq f(-n)-f(-n-1) \geq f(-n-1)-f(-n-2)$$

$$\Rightarrow f(n+1)-f(n) \geq f(-n)-f(-n-1).$$

$$b_n := a_{n+1}-a_n = f(n+1)+f(-(n+1))-f(n)-f(-n)$$

$b_{n+1} \stackrel{?}{} b_n$ $\qquad b_1, b_0$

$$b_1 = f(2)+f(-2)-f(1)-f(-1), \quad b_0 = f(1)+f(-1)-2f(0)$$

$b_1 \stackrel{?}{} b_0$ $\qquad b_1 - b_0 = f(2)+f(-2)+2f(0)-2(f(1)+f(-1)) \geq 0$ ✓

$$f(2)-f(1) \geq f(1)-f(0)$$
$$\Rightarrow f(2)+f(0) \geq 2f(1)$$
$$f(-2)+f(0) \geq 2f(-1) \ ✓$$

$b_1 \geq b_0$

$b_{n+1} \geq b_n$ $\qquad ? \quad f(n+2)+f(-(n+2))+f(n)+f(-n)-2(f(n+1)+f(-(n+1))) \geq 0 \ ?$ ✓

$$f(n+2)-f(n+1) \geq f(n+1)-f(n)$$
$$\Rightarrow f(n+2)+f(n)-2f(n+1) \geq 0$$
$$f(-(n+2))+f(-n)-2f(-(n+1)) \geq 0$$

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· $a_n$ je periodičan $\exists T \in \mathbb{N} \quad a_{n+T} = a_n$.

① Ispitati periodičnost niza

$$x_{n+2} = \frac{1+x_{n+1}}{x_n}, \quad x_1 = a, \; x_2 = b, \quad a, b \neq 0, -1, \quad a+b \neq -1.$$

$T \in \mathbb{N}$ $x_{n+T} = x_n$ $\qquad x_{n+2+T} = \frac{1+x_{n+1+T}}{x_{n+T}} = x_{n+2}$

dovoljno je naći $\underline{\underline{T}}$: $x_{T+1} = x_1$ i $x_{T+2} = x_2$ $\Rightarrow$ niz je period. jer

$x_{T+3} = x_3$

$\vdots$

$x_{nT+k} = x_k, \quad 1 \leq k \leq T$

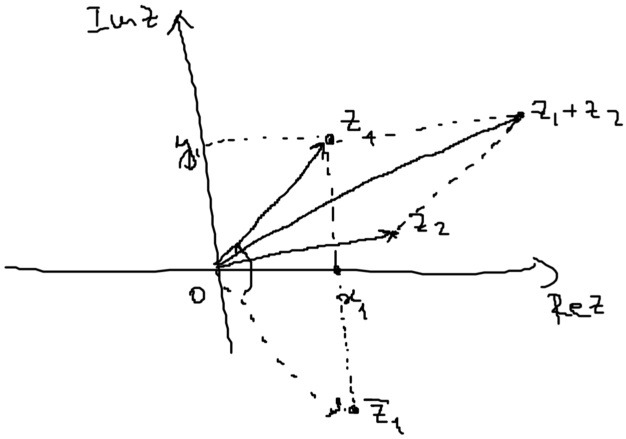
$$x_3 = \frac{1+b}{a}, \quad x_4 = \frac{1+x_3}{x_2} = \frac{1+\frac{1+b}{a}}{b} = \frac{1+a+b}{ab}$$

$$x_5 = \frac{1+x_4}{x_3} = \frac{1+\frac{1+a+b}{ab}}{\frac{1+b}{a}} = \frac{1+a+b+ab}{b\cdot(1+b)} = \frac{(1+a)\cdot(1+b)}{b\cdot(1+b)} = \frac{1+a}{b}$$

$$x_6 = \frac{1+x_5}{x_4} = \frac{1+\frac{1+a}{b}}{\frac{1+a+b}{ab}} = a \quad , \quad x_7 = \frac{1+a}{\frac{1+a}{b}} = b \quad \checkmark$$

$$\Rightarrow \boxed{T=5}$$

## Комплексни бројеви




$x^2=-1$ → има решење $x=i$
$x=-i$

$i$ - имагинарна јединица, $i^2=-1$

$$\mathbb{C} = \{ z = x+iy \mid x,y\in\mathbb{R} \}$$

$z = x+iy$ $\quad$ $\mathrm{Re}\, z = x$
$\mathrm{Im}\, z = y$

$$z_1+z_2 = (x_1+x_2)+i(y_1+y_2)$$

$z_1 = x_1+iy_1$
$z_2 = x_2+iy_2$

$$z_1\cdot z_2 = (x_1+iy_1)\cdot(x_2+iy_2) = x_1x_2 + ix_1y_2 + iy_1x_2 + \overset{-1}{i^2}\, y_1y_2$$
$$= (x_1\cdot x_2 - y_1y_2) + i(x_1y_2+y_1x_2)$$

$z = x+iy \Rightarrow \bar{z} := x-iy$ конјугат броја $z$

$$\overline{z_1+z_2} = (x_1+x_2)-i(y_1+y_2) = (x_1-iy_1)+(x_2-iy_2) = \bar{z}_1+\bar{z}_2$$

$$\overline{z_1\cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$$

$a,b,c\in\mathbb{R}$

$$ax^2+bx+c=0 \quad , \quad x_{1/2} = \frac{-b\pm\sqrt{D}}{2a} \in \mathbb{C}$$

$$D = b^2-4ac \in \mathbb{R}$$

$$D<0 \Rightarrow \sqrt{b^2-4ac} = i\sqrt{|D|} \rightsquigarrow x_1 = \frac{-b+i\sqrt{|D|}}{2a} \quad , \quad x_2 = \frac{-b-i\sqrt{|D|}}{2a}$$

свеједно да ли бирамо $+$ или $-$ $i\sqrt{|D|}$.

$$\underline{\underline{x_2 = \bar{x}_1}}$$

Уколико имамо што комплексну нулу $z_1\in\mathbb{C}\setminus\mathbb{R}$ онда је и $\bar{z}_1$ нула.

Уколико је $z=2$, $\bar{z}=2$ (јер је $\mathrm{Im}\, z = y = 0$).

sup / inf  $([0,1]\cap\mathbb{Q}) = \mathbb{Q}' \longrightarrow \mathbb{Q}$

$\mathbb{N}$  $\mathbb{Q}$  $f: \mathbb{Q}' \longrightarrow \mathbb{Q}$

$\mathbb{N} \leftrightarrow \mathbb{Z} \leftrightarrow \mathbb{Q}$  $[0,1]\cap\mathbb{Q} = \{a_n\}$

$q \in \mathbb{Q}$  $q = \underbrace{[q]}_{\in\mathbb{Z}} + \overbrace{\{q\}}^{\in[0,1]\cap\mathbb{Q}}$

$a_1 - 1$

---

$a_1$  $a_2$ ... $a_n$ ...

$1+a_1$  $1+a_2$ ....

$a_1 - 1$

$2+a_1$

$a_1 - 2$

$\vdots$